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Groups of order 64 and non-homeomorphic double Kodaira fibrations with the same biregular invariants

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Abstract
Let Σ_b be a compact Riemann surface of genus b. We investigate finite quotients G of the pure braid group on two strands $\mathrm{P}_2(\Sigma_b)$ which do not factor through $\pi_1(\Sigma_b \times \Sigma_b)$. Building on our previous work on some special systems of generators on finite groups that we called *diagonal double Kodaira structures*, we prove that, if G does not have order 32, then $|G| \geq 64$, and we completely classify the cases where $b = 2$ and equality holds. In the last section, as a geometric application of our algebraic results, we construct two 3-dimensional families of double Kodaira fibrations having the same biregular invariants and the same Betti numbers but different fundamental group.

Keywords Surface braid groups · Finite groups · Kodaira fibrations

Mathematics Subject Classification 14J29 · 14J25 · 20D15

1 Introduction

A *Kodaira fibration* is a smooth, connected holomorphic fibration $f_1 : S \longrightarrow B_1$, where S is a compact complex surface and B_1 is a compact closed curve, which is *non-isotrivial*, meaning that the fibres are not all biholomorphic to one another. The genus of the base curve B_1, denoted by $b_1 := g(B_1)$, is called the *base genus*, while the genus of any fibre F, denoted $g := g(F)$, is known as the *fibre genus*. A surface S occurring as the total space of a Kodaira fibration is called a *Kodaira fibred surface*. For any Kodaira fibration, the inequalities $b_1 \geq 2$ and $g \geq 3$ are always satisfied, as established in [15, Theorem 1.1]. Because of

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the restriction on the base genus, the surface S cannot contain rational or elliptic curves, implying that S is minimal. Furthermore, by the sub-additivity of the Kodaira dimension, S is a surface of general type and is thus algebraic.

Kodaira fibred surfaces raise both fascinating and challenging questions that lie at the intersection of the algebro-geometric properties of compact complex surfaces and the topological aspects of the underlying closed, oriented 4-manifolds. These surfaces can be investigated not only through traditional algebraic geometry tools but also by applying techniques from geometric topology. Notable methods include the Meyer signature formula, Birman-Hilden relations in the mapping class group, and the decomposition of Lefschetz fibrations, as outlined in [8, 9, 17, 23]. For an extensive overview and additional references, we suggest consulting [5].

The classical example introduced by Kodaira (see [2, Chapter V, Section 14]), along with its variants described in [1, 14], are constructed via ramified coverings of a product of curves, thus providing a pair of Kodaira fibrations in the same surface. This leads to the concept of a *double Kodaira fibration*, which is formally defined as follows [3, 4, 6, 16, 18, 22, 24]:

Definition 1.1 A *double Kodaira surface* is a compact, complex surface S, endowed with a *double Kodaira fibration*, namely a surjective, holomorphic map $f : S \longrightarrow B_1 \times B_2$ yielding, by composition with the natural projections, two Kodaira fibrations $f_i : S \longrightarrow B_i$, $i = 1, 2$.

In the sequel, we will detail our strategy for constructing double Kodaira fibrations by utilizing the techniques introduced in [7, 20], along with presenting our new results. Our primary approach involves the "detopologization" of the problem, namely, transforming it into a purely algebraic one. This will be applied in the context of *diagonal* double Kodaira fibrations: these are defined as Stein factorizations $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ of finite Galois covers of the form

$$\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b, \tag{1}$$

branched with order $n \geq 2$ over the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$, where Σ_b stands for a compact Riemann surface of genus b. According to the Grauert-Remmert extension theorem and Serre's GAGA principle, the existence of a G-cover as in (1) corresponds to the existence of a group epimorphism

$$\varphi : \pi_1(\Sigma_b \times \Sigma_b - \Delta) \longrightarrow G,$$

and the condition that branches with order n along Δ can be expressed by requiring that $\varphi(\gamma_\Delta)$ has order n in G, where γ_Δ represents a loop in $\Sigma_b \times \Sigma_b - \Delta$ that encircles the diagonal. The condition $n \geq 2$ means that φ does not factor through $\pi_1(\Sigma_b \times \Sigma_b)$ and, crucially, it also implies that G must be non-abelian, since γ_Δ is a non-trivial commutator in $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$. We refer to such epimorphisms, for which $\varphi(\gamma_\Delta)$ is non-trivial, as *admissible*.

Recall that the fundamental group $\pi_1(\Sigma_b \times \Sigma_b - \Delta)$ is isomorphic to $\mathrm{P}_2(\Sigma_b)$, the pure braid group with two strands on Σ_b. This group has a finite geometric presentation with

$4b+1$ generators, as proved in [13, Theorem 7]. By applying an admissible epimorphism to these generators, we get an ordered sequence

$$\mathfrak{S} = (\mathrm{r}_{11}, \mathrm{t}_{11}, \ldots, \mathrm{r}_{1b}, \mathrm{t}_{1b}, \mathrm{r}_{21}, \mathrm{t}_{21}, \ldots, \mathrm{r}_{2b}, \mathrm{t}_{2b}, \mathrm{z}),$$

consisting of $4b+1$ elements that generate G. Among these, z is a commutator satisfying $o(\mathrm{z}) = n$, and the whole sequence is constrained by a certain set of relations coming from the relations of $\mathrm{P}_2(\Sigma_b)$. This configuration is referred to as a *diagonal double Kodaira structure* of type (b, n) on the group G. Hence, what begins as the geometric problem of constructing an admissible epimorphism onto G is ultimately converted into the combinatorial and algebraic task of finding a diagonal double Kodaira structure of type (b, n) on G.

Via the composition with the two natural projections, and up to a simultaneous Stein factorization, the G-cover $\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b$ provides a double Kodaira structure on S, called a *diagonal double Kodaira fibration*. In the light of the previous considerations, classifying diagonal double Kodaira fibrations is equivalent to describing finite groups which admit a diagonal double Kodaira structure.

In [20] we solved this problem for groups of order at most 32, showing that the only groups in this range having diagonal double Kodaira structures are the two extra-special groups $G(32, 49)$ and $G(32, 50)$, and we computed the number of structures of type $(b, n) = (2, 2)$ in each case (see Proposition 4.6).

In this work, which is intended as a sequel to [20], we analize groups of order at most 64. It turns out that there are no groups G with order $33 \leq |G| \leq 63$ admitting double Kodaira structures, see Proposition 4.7. Our first result concerns the complete classification of the cases where $|G| = 64$, see Theorem 5.2:

Theorem A *Let G be a group of order 64 admitting a diagonal double Kodaira structure. The following holds :*

- if G is non-monolithic then it is isomorphic to $G(64, t)$, with $t \in \{199, 200, 201, 264, 265\}$;
- if G is monolithic, then it is isomorphic to $G(64, t)$, with $t \in \{249, 266\}$.

We are also able to provide the number of diagonal double Kodaira structures with $b = 2$ in each of these situations.[1] For all of them we have $n = 2$, because we always get $[G, G] = \mathbb{Z}_2$. Furthermore, in the non-monolithic cases all diagonal double Kodaira structures can be derived as lifts from structures on an extra-special quotient of order 32. In contrast, the structures appearing in the monolithic case are genuinely "new", making the monolithic cases particularly interesting from this perspective.

As a geometric application of our algebraic results, we can describe the biregular and topological invariants of all diagonal double Kodaira fibrations associated with these structures, see Theorem 6.2:

Theorem B *Let $\mathrm{f} : S \longrightarrow \Sigma_2 \times \Sigma_2$ be the G-cover associated with a diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on a finite group G of order 64, and let $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be the corresponding diagonal double Kodaira fibration. Then the*

[1] The existence and enumeration of diagonal double Kodaira structures in genus $b \geq 3$ is an interesting problem, which we hope to address in future work.

occurrences for $H_1(S, \mathbb{Z})$, $q(S) = \frac{1}{2}\operatorname{rank} H_1(S, \mathbb{Z})$, K_S^2, $c_2(S)$, $\sigma(S)$, b_1, b_2, g_1, g_2 are as in the following table.

IdSmallGroup(G)	$H_1(S, \mathbb{Z})$	$q(S)$	K_S^2	$c_2(S)$	$\sigma(S)$	b_1	b_2	g_1	g_2
$G(64, 199)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 200)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 201)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 264)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 265)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 249)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 266)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 266)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	6	736	320	32	3	3	41	41

Thus, in the case $G = (64, 266)$ there are two possibilities for the torsion part of $H_1(S, \mathbb{Z})$. Moreover, *every* curve of genus $b = 2$ can be used as a starting point for our construction, and so such a construction depends on 3 parameters. This implies the following result, see Corollary 6.3:

Corollary C *There exists two 3-dimensional families $\mathscr{F}_1$, $\mathscr{F}_2$ of Kodaira doubly fibred surfaces with $b_1 = b_2 = 3$, $g_1 = g_2 = 41$, $\sigma(S) = 32$ such that*

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have the same biregular invariants

$$p_g(S) = 93, \quad q(S) = 6, \quad K_S^2 = 736$$

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have the same Betti numbers

$$\mathrm{b}_0 = \mathrm{b}_4 = 1, \quad \mathrm{b}_1 = \mathrm{b}_3 = 12, \quad \mathrm{b}_2 = 342$$

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have different torsion part of $H_1(S, \mathbb{Z})$. In particular, they are not homotopically equivalent.

As far as we know, this yields the first explicit example of positive-dimensional families of (doubly) fibred Kodaira surfaces having the same base and fibre genera, the same biregular invariants, the same Betti numbers and different fundamental group, see Remark 6.4.

Let us briefly describe the computational tools employed in this work. We implemented a function in the computational algebra system `GAP4` (see [11]), named `CheckStructures`, which we used to perform several of our calculations. For the convenience of the reader, we provide all relevant scripts in a repository on GitHub, see [12]. In addition, selected excerpts of the `GAP4` code are included in the paper itself, in order to illustrate the nature of the computational methods involved.

This work is organized as follows. Sections 2 and 3 cover the algebraic preliminaries necessary for the rest of the paper, focusing in particular on CCT groups and monolithic groups.

In Sect. 4 we define diagonal double Kodaira structures and explain their relationship to their geometric topology counterparts, namely admissible epimorphisms from pure surface braid groups to finite groups. The main point is that CCT groups cannot admit diagonal double Kodaira structures (Remark 4.3), whereas all structures on a non-monolithic group are liftings of structures on a proper quotient (Proposition 4.4). In our research we can therefore exclude both non-CCT groups and non-monolithic groups without quotients that admit diagonal double Kodaira structures. This rules out all groups of order less than 64, except for two of them, and all groups of order 64, except for 68 of them.

Finally, in Sect. 5 we present Theorem A, whereas Sect. 6 deals with Theorem B and Corollary C.

Notation and conventions.

If S is a smooth projective surface over $\mathbb{C}$, then $c_1(S)$, $c_2(S)$ denote the first and second Chern class of its tangent bundle T_S, respectively. Moreover, K_S is the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\sigma(S)$ is the $signature$.

Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$ is the cyclic group of order n.
- $\mathrm{Aut}(G)$ is the automorphism group of G.
- S_n is the symmetricgroup on n symbols. The Klein subgroup of S_4 is denoted by V_4.
- The order of a finite group G is denoted by $|G|$. If $x \in G$, the order of x is denoted by $o(x)$ and its centralizer in G by $C_G(x)$.
- If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$.
- The commutator subgroup of G is denoted by $[G, G]$, the center of G by $Z(G)$.
- If $S = \{s_1, \ldots, s_n\} \subset G$, the subgroup generated by S is denoted by $\langle S\rangle = \langle s_1, \ldots, s_n\rangle$.
- $\mathrm{IdSmallGroup}(G)$ indicates the label of the group G in the `GAP4` database of small groups. For instance $\mathrm{IdSmallGroup}(\mathrm{S}_4) = G(24, 12)$ means that S_4 is the 12th in the list of groups of order 24.
- If N is a normal subgroup of G and $g \in G$, we denote by $\bar{g}$ the image of g in the quotient group G/N.

2 Group-theoretical preliminaries I: CCT groups

2.1 CCT-groups

This subsection relies on [20, Section 1], to which we refer the reader for more details.

Definition 2.1 A finite non-abelian group G is said to be a *center commutative-transitive group* (or a CCT-group, for short) if commutativity is a transitive relation on the set of non-central elements of G. In other words, if $x, y, z \in G - Z(G)$ and $[x, y] = [y, z] = 1$, then $[x, z] = 1$.

Proposition 2.2 *For a finite non-abelian group G, the following properties are equivalent.*

(1) *G is a CCT-group.*

(2) For every pair x, y of non-central elements in G, the relation $[x, y] = 1$ implies $C_G(x) = C_G(y)$.
(3) For every non-central element $x \in G$, the centralizer $C_G(x)$ is abelian.
Moreover, the following holds.

Proposition 2.3 *Let G be a finite non-abelian group.*

(1) If $|G|$ is the product of at most three prime factors (not necessarily distinct), then G is a *CCT*-group.
(2) If $|G| = p^4$, with p prime, then G is a *CCT*-group.
(3) If G contains an abelian normal subgroup of prime index, then G is a *CCT*-group.

Groups of order at most 32 that are not CCT-groups were classified in [20, Section 2]: they are S_4 and the two extra-special groups of order $2^5 = 32$, denoted by $\mathrm{H}_5(\mathbb{Z}_2)$ and $\mathrm{G}_5(\mathbb{Z}_2)$. Here we want to extend this classification to groups of order at most 64. First of all, looking at Proposition 2.3, we get

Proposition 2.4 *Let G be a finite group such that $33 \leq |G| \leq 64$. If G is not a CCT-group, then*

$$|G| \in \{36, 40, 48, 54, 56, 60, 64\}.$$

We now examine the first six cases; the case $|G| = 64$ will be considered in Sect. 5. Our analysis makes essential use of the computer algebra system GAP4, see [11].

Proposition 2.5 *The non-abelian groups G of order $33 \leq |G| \leq 63$ which are not* CCT *-groups are the following.*

- The group $G(36, 10)$.
- The groups of the form $G(48, t)$, where

$$t \in \{15, 16, 17, 18, 30, 38, 39, 40, 41, 48\}.$$

- The groups $G(54, 5)$ and $G(54, 6)$.
- The groups $G(60, 7)$ and $G(60, 8)$.

Proof We include the GAP4 code used in the case $|G| = 36$. The remaining cases can be treated in the same way, by changing the value of the variable *order*.

```
order:=36;;
#creation of list_nonab
list_nonab:=IdsOfAllSmallGroups(Size, order, IsAbelian, false);;
#creation of list_nonab_non_cct
list_nonab_non_cct:=[];;
for t in list_nonab do
G:=SmallGroup(t);
   #non-central elements in G
    H:=[];;
    for g in G do
    boole:=g in Center(G);
    if boole=false then
    Add(H, g);
    fi; od;
    #checking CCT property via centralizers
    ab:=0;; non_ab:=0;;
    for h in H do
    C:=Centralizer(G, h);
    boole:=IsAbelian(C);
    if boole=true then ab:=ab+1;
    else non_ab:=non_ab+1;
    fi; od;
if ab < Size(H) then
Add(list_nonab_non_cct, t);
fi; od;
Size(list_nonab); list_nonab_non_cct;
```

The output is

```
10
[ [ 36, 10 ] ]
```

hence, up to automorphisms, there are 10 non-abelian groups of order 36, and the unique non-CCT group among them is $G(36,\ 10)$. □

3 Group-theoretical preliminaries II: monolithic groups

For a finite group G, the *monolith* of G, denoted $\mathrm{mon}(G)$, is the intersection of all non-trivial normal subgroups of G. We say that G is *monolithic* if $\mathrm{mon}(G) \neq 1$. Equivalently, G is monolithic precisely when it has a unique minimal, non-trivial normal subgroup.[2] For instance:

- G is simple if and only if $\mathrm{mon}(G) = G$;
- a non-trivial product of the form $G = H \times K$ is not monolithic, since $H \times \{1\}$ and $\{1\} \times K$ are two non-trivial normal subgroups intersecting only at the identity.

[2] In [20, Section 4] we called the intersection of all non-trivial normal subgroups of a finite group G the *socle* of G, and we denoted it by $\mathrm{soc}(G)$. However, we later realized that this is a non-standard terminology: indeed, in the literature, $\mathrm{soc}(G)$ generally denotes the subgroup of G generated by the minimal, non-trivial normal subgroups, so we prefer to use here the notation $\mathrm{mon}(G)$. By definition, if G is monolithic then $\mathrm{mon}(G)$ is non-trivial and coincides with $\mathrm{soc}(G)$. Instead, if G is non-monolithic, then $\mathrm{mon}(G) = \{1\}$ but $\mathrm{soc}(G)$ is always non-trivial. For instance, if $G = G(64,\ 199)$ then $\mathrm{mon}(G) = \{1\}$ and $\mathrm{soc}(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 3.1 *If G is monolithic and $Z(G) \neq \{1\}$, then $\mathrm{mon}(G)$ is cyclic of prime order.*

Proof By definition $\mathrm{mon}(G) \subseteq Z(G)$, hence $\mathrm{mon}(G)$ is abelian and all its subgroups are normal in G. Thus the claim follows by the minimality of $\mathrm{mon}(G)$. □

Since a p-group has non-trivial center, Proposition 3.1 immediately implies

Corollary 3.2 *If G is a monolithic p-group, then $\mathrm{mon}(G) \simeq \mathbb{Z}_p$.*

We end this section by classifying, up to order 63, the non-abelian groups that are both non-CCT and monolithic. The corresponding analysis for groups of order 64 is deferred to Section 5.

Proposition 3.3 *Let G be a non-abelian, monolithic group of order $|G| \leq 63$. If G is not a* CCT*-group, then G is isomorphic to one of the following groups :*

$$S_4,\ H_5(\mathbb{Z}_2),\ G_5(\mathbb{Z}_2),\ G(54,\,5),\ G(54,\,6).$$

Proof Clearly $\mathrm{mon}(\mathrm{S}_4) = \mathrm{V}_4$, whereas for the two extra-special groups of order 32 we have $\mathrm{mon}(G) = Z(G) \simeq \mathbb{Z}_2$. We may therefore assume that $|G| \geq 33$. Proposition 2.5 lists the 15 non-abelian, non-CCT groups G with $33 \leq |G| \leq 63$. Among these, the monolithic ones are identified using the following GAP4 script:

```
list:=[[36, 10], [48, 15], [48,16],[48, 17], [48, 18], [48, 30],
[48, 38], [48, 39], [48, 40], [48, 41], [48, 48],
[54, 5], [54, 6], [60, 7], [60, 8]];;
list_mon:=[];;
for u in list do
G:=SmallGroup(u);
norm:=NormalSubgroups(G);;
Np:=[];;
for N in norm do
if Order(N)>1 then
Add(Np, N);
fi; od;
M:=G;
for N in Np do
M:=Intersection(M, N);
od;
if Order(M)>1 then
Add(list_mon, IdSmallGroup(G));
fi; od;
list_mon;
```

The output is

```
[ [ 54, 5 ], [ 54, 6 ] ]
```

and this completes the proof. □

4 Diagonal double Kodaira structures and diagonal double Kodaira fibrations

For more details about the content of this section, we refer the reader to [19, 20] and to the survey paper [21]. Let G be a finite group and let $b,\ n \geq 2$ be two positive integers. A *diagonal double Kodaira structure* of type $(b,\ n)$ on G is an ordered set of $4b+1$ generators

$$\mathfrak{S} = (\mathrm{r}_{11},\ \mathrm{t}_{11}, \ldots, \mathrm{r}_{1b},\ \mathrm{t}_{1b},\ \mathrm{r}_{21},\ \mathrm{t}_{21}, \ldots, \mathrm{r}_{2b},\ \mathrm{t}_{2b},\ \mathrm{z}),$$

with $o(\mathrm{z}) = n$, such that the following relations are satisfied. We use the commutator notation in order to indicate relations of conjugacy type, writing for instance $[x,\ y] = zy^{-1}$ instead of $xyx^{-1} = z$.

- Surface relations

$$[\mathrm{r}_{1b}^{-1},\ \mathrm{t}_{1b}^{-1}]\,\mathrm{t}_{1b}^{-1}\,[\mathrm{r}_{1\,b-1}^{-1},\ \mathrm{t}_{1\,b-1}^{-1}]\,\mathrm{t}_{1\,b-1}^{-1}\cdots[\mathrm{r}_{11}^{-1},\ \mathrm{t}_{11}^{-1}]\,\mathrm{t}_{11}^{-1}\,(\mathrm{t}_{11}\,\mathrm{t}_{12}\cdots\mathrm{t}_{1b}) = \mathrm{z}$$
$$[\mathrm{r}_{21}^{-1},\ \mathrm{t}_{21}]\,\mathrm{t}_{21}\,[\mathrm{r}_{22}^{-1},\ \mathrm{t}_{22}]\,\mathrm{t}_{22}\cdots[\mathrm{r}_{2b}^{-1},\ \mathrm{t}_{2b}]\,\mathrm{t}_{2b}\,(\mathrm{t}_{2b}^{-1}\,\mathrm{t}_{2\,b-1}^{-1}\cdots\mathrm{t}_{21}^{-1}) = \mathrm{z}^{-1}$$

- Conjugacy action of r_{1j}

$$\begin{aligned}
&[\mathrm{r}_{1j},\ \mathrm{r}_{2k}] = 1 && \text{if}\quad j<k\\
&[\mathrm{r}_{1j},\ \mathrm{r}_{2j}] = 1 &&\\
&[\mathrm{r}_{1j},\ \mathrm{r}_{2k}] = \mathrm{z}^{-1}\,\mathrm{r}_{2k}\,\mathrm{r}_{2j}^{-1}\,\mathrm{z}\,\mathrm{r}_{2j}\,\mathrm{r}_{2k}^{-1} && \text{if}\quad j>k\\
&&&\\
&[\mathrm{r}_{1j},\ \mathrm{t}_{2k}] = 1 && \text{if}\quad j<k\\
&[\mathrm{r}_{1j},\ \mathrm{t}_{2j}] = \mathrm{z}^{-1} &&\\
&[\mathrm{r}_{1j},\ \mathrm{t}_{2k}] = [\mathrm{z}^{-1},\ \mathrm{t}_{2k}] && \text{if}\quad j>k\\
&&&\\
&[\mathrm{r}_{1j},\ \mathrm{z}] = [\mathrm{r}_{2j}^{-1},\ \mathrm{z}] &&
\end{aligned}$$

- Conjugacy action of t_{1j}

$$\begin{aligned}
&[\mathrm{t}_{1j},\ \mathrm{r}_{2k}] = 1 && \text{if}\quad j<k\\
&[\mathrm{t}_{1j},\ \mathrm{r}_{2j}] = \mathrm{t}_{2j}^{-1}\,\mathrm{z}\,\mathrm{t}_{2j} &&\\
&[\mathrm{t}_{1j},\ \mathrm{r}_{2k}] = [\mathrm{t}_{2j}^{-1},\ \mathrm{z}] && \text{if}\quad j>k\\
&&&\\
&[\mathrm{t}_{1j},\ \mathrm{t}_{2k}] = 1 && \text{if}\quad j<k\\
&[\mathrm{t}_{1j},\ \mathrm{t}_{2j}] = [\mathrm{t}_{2j}^{-1},\ \mathrm{z}] &&\\
&[\mathrm{t}_{1j},\ \mathrm{t}_{2k}] = \mathrm{t}_{2j}^{-1}\,\mathrm{z}\,\mathrm{t}_{2j}\,\mathrm{z}^{-1}\,\mathrm{t}_{2k}\,\mathrm{z}\,\mathrm{t}_{2j}^{-1}\,\mathrm{z}^{-1}\,\mathrm{t}_{2j}\,\mathrm{t}_{2k}^{-1} && \text{if}\quad j>k\\
&&&\\
&[\mathrm{t}_{1j},\ \mathrm{z}] = [\mathrm{t}_{2j}^{-1},\ \mathrm{z}] &&
\end{aligned}$$

Note that z is a commutator of order $n \geq 2$ in G, hence G is necessarily *non-abelian*.

For later use, let us write down the special case consisting of a diagonal double Kodaira structure of type $(2,\ n)$. It is an ordered set of nine generators of G

$$\mathfrak{S} = (\mathrm{r}_{11},\ \mathrm{t}_{11},\ \mathrm{r}_{12},\ \mathrm{t}_{12},\ \mathrm{r}_{21},\ \mathrm{t}_{21},\ \mathrm{r}_{22},\ \mathrm{t}_{22},\ \mathrm{z}),$$

with $o(\mathrm{z}) = n \geq 2$, subject to the following relations:

$$
\begin{array}{ll}
\multicolumn{2}{l}{\text{(S1) } [\mathrm{r}_{12}^{-1}, \mathrm{t}_{12}^{-1}]\,\mathrm{t}_{12}^{-1}\,[\mathrm{r}_{11}^{-1}, \mathrm{t}_{11}^{-1}]\,\mathrm{t}_{11}^{-1}\,(\mathrm{t}_{11}\,\mathrm{t}_{12}) = \mathrm{z}} \\
\multicolumn{2}{l}{\text{(S2) } [\mathrm{r}_{21}^{-1}, \mathrm{t}_{21}]\,\mathrm{t}_{21}\,[\mathrm{r}_{22}^{-1}, \mathrm{t}_{22}]\,\mathrm{t}_{22}\,(\mathrm{t}_{22}^{-1}\,\mathrm{t}_{21}^{-1}) = \mathrm{z}^{-1}} \\
\text{(R1) } [\mathrm{r}_{11}, \mathrm{r}_{22}] = 1 & \text{(R6) } [\mathrm{r}_{12}, \mathrm{r}_{22}] = 1 \\
\text{(R2) } [\mathrm{r}_{11}, \mathrm{r}_{21}] = 1 & \text{(R7) } [\mathrm{r}_{12}, \mathrm{r}_{21}] = \mathrm{z}^{-1}\,\mathrm{r}_{21}\,\mathrm{r}_{22}^{-1}\,\mathrm{z}\,\mathrm{r}_{22}\,\mathrm{r}_{21}^{-1} \\
\text{(R3) } [\mathrm{r}_{11}, \mathrm{t}_{22}] = 1 & \text{(R8) } [\mathrm{r}_{12}, \mathrm{t}_{22}] = \mathrm{z}^{-1} \\
\text{(R4) } [\mathrm{r}_{11}, \mathrm{t}_{21}] = \mathrm{z}^{-1} & \text{(R9) } [\mathrm{r}_{12}, \mathrm{t}_{21}] = [\mathrm{z}^{-1}, \mathrm{t}_{21}] \\
\text{(R5) } [\mathrm{r}_{11}, \mathrm{z}] = [\mathrm{r}_{21}^{-1}, \mathrm{z}] & \text{(R10) } [\mathrm{r}_{12}, \mathrm{z}] = [\mathrm{r}_{22}^{-1}, \mathrm{z}] \\
\text{(T1) } [\mathrm{t}_{11}, \mathrm{r}_{22}] = 1 & \text{(T6) } [\mathrm{t}_{12}, \mathrm{r}_{22}] = \mathrm{t}_{22}^{-1}\,\mathrm{z}\,\mathrm{t}_{22} \\
\text{(T2) } [\mathrm{t}_{11}, \mathrm{r}_{21}] = \mathrm{t}_{21}^{-1}\,\mathrm{z}\,\mathrm{t}_{21} & \text{(T7) } [\mathrm{t}_{12}, \mathrm{r}_{21}] = [\mathrm{t}_{22}^{-1}, \mathrm{z}] \\
\text{(T3) } [\mathrm{t}_{11}, \mathrm{t}_{22}] = 1 & \text{(T8) } [\mathrm{t}_{12}, \mathrm{t}_{22}] = [\mathrm{t}_{22}^{-1}, \mathrm{z}] \\
\text{(T4) } [\mathrm{t}_{11}, \mathrm{t}_{21}] = [\mathrm{t}_{21}^{-1}, \mathrm{z}] & \text{(T9) } [\mathrm{t}_{12}, \mathrm{t}_{21}] = \mathrm{t}_{22}^{-1}\,\mathrm{z}\,\mathrm{t}_{22}\,\mathrm{z}^{-1}\,\mathrm{t}_{21}\,\mathrm{z}\,\mathrm{t}_{22}^{-1}\,\mathrm{z}^{-1}\,\mathrm{t}_{22}\,\mathrm{t}_{21}^{-1} \\
\text{(T5) } [\mathrm{t}_{11}, \mathrm{z}] = [\mathrm{t}_{21}^{-1}, \mathrm{z}] & \text{(T10) } [\mathrm{t}_{12}, \mathrm{z}] = [\mathrm{t}_{22}^{-1}, \mathrm{z}]
\end{array}
\tag{2}
$$

The definition of diagonal double Kodaira structure can be motivated by means of some concepts in geometric topology. Let Σ_b be a compact Riemann surface of genus b and let $\mathrm{P}_2(\Sigma_b)$ be the pure braid group with two strands on Σ_b, namely, the fundamental group of the ordered configuration space $\Sigma_b \times \Sigma_b - \Delta$, where $\Delta \subset \Sigma_b \times \Sigma_b$ is the diagonal. Let $A_{12} \in \mathrm{P}_2(\Sigma_b)$ be the braid corresponding to the homotopy class in $\Sigma_b \times \Sigma_b - \Delta$ of a loop in $\Sigma_b \times \Sigma_b$ that winds once around the diagonal Δ. Therefore we can state the

Proposition 4.1 *A finite group G admits a diagonal double Kodaira structure of type $(b,\ n)$ if and only if there is a surjective group homomorphism*

$$\varphi : P_2(\Sigma_b) \longrightarrow G$$

such that $z = \varphi(A_{12})$ has order $n \geq 2$. Note that the last condition implies that φ does not factor through $\pi_1(\Sigma_b \times \Sigma_b)$.

A group epimorphism $\varphi : \mathrm{P}_2(\Sigma_b) \longrightarrow G$ as in Proposition 4.1 will be called *admissible*. If φ is admissible, then the two subgroups

$$
\begin{aligned}
K_1 &:= \langle \mathrm{r}_{11},\ \mathrm{t}_{11}, \ldots, \mathrm{r}_{1b},\ \mathrm{t}_{1b},\ \mathrm{z} \rangle \\
K_2 &:= \langle \mathrm{r}_{21},\ \mathrm{t}_{21}, \ldots, \mathrm{r}_{2b},\ \mathrm{t}_{2b},\ \mathrm{z} \rangle
\end{aligned}
$$

are both normal in G, and that there are two short exact sequences

$$
\begin{aligned}
1 &\longrightarrow K_1 \longrightarrow G \longrightarrow Q_2 \longrightarrow 1 \\
1 &\longrightarrow K_2 \longrightarrow G \longrightarrow Q_1 \longrightarrow 1,
\end{aligned}
\tag{3}
$$

such that the elements $\mathrm{r}_{21},\ \mathrm{t}_{21}, \ldots, \mathrm{r}_{2b},\ \mathrm{t}_{2b}$ yield a complete system of coset representatives for Q_2, whereas the elements $\mathrm{r}_{11},\ \mathrm{t}_{11}, \ldots, \mathrm{r}_{1b},\ \mathrm{t}_{1b}$ yield a complete system of coset representatives for Q_1. A diagonal double Kodaira structure on G is called *strong* if $K_1 = K_2 = G$.

A *prestructure* on G is an ordered set of nine elements

$$(\mathrm{r}_{11},\ \mathrm{t}_{11},\ \mathrm{r}_{12},\ \mathrm{t}_{12},\ \mathrm{r}_{21},\ \mathrm{t}_{21},\ \mathrm{r}_{22},\ \mathrm{t}_{22},\ \mathrm{z}),$$

with $o(\mathrm{z}) = n \geq 2$, subject to the relations $(\mathrm{R1}), \ldots, (\mathrm{R10})$, $(\mathrm{T1}), \ldots, (\mathrm{T10})$ in (2).

In other words, the nine elements must satisfy all the relations defining a diagonal double Kodaira structure of type $(2,\ n)$, except the surface relations. Here we are *not* requiring that the elements of the prestructure generate G. Note that, if G admits no prestructure, then it admits no diagonal double Kodaira structure of any type.

Remark 4.2 Let G be a finite group that admits a prestructure. Then z and all its conjugates are non-trivial elements of G and so, from relations $(\mathrm{R4})$, $(\mathrm{R8})$, $(\mathrm{T2})$, $(\mathrm{T6})$, it follows that $\mathrm{r}_{11},\ \mathrm{r}_{12},\ \mathrm{r}_{21},\ \mathrm{r}_{22}$ and $\mathrm{t}_{11},\ \mathrm{t}_{12},\ \mathrm{t}_{21}, \mathrm{t}_{22}$ are non-central elements of G.

A key point is now provided by the following

Remark 4.3 A CCT-group admits no prestructure, see [20, Proposition 4.4]. Thus, when looking for diagonal double Kodaira structures, we may restrict ourselves to the groups which are not CCT-groups.

Let

$$\mathfrak{S} = (\mathrm{r}_{11},\ \mathrm{t}_{11}, \ldots, \mathrm{r}_{1b},\ \mathrm{t}_{1b},\ \mathrm{r}_{21},\ \mathrm{t}_{21}, \ldots, \mathrm{r}_{2b},\ \mathrm{t}_{2b},\ \mathrm{z})$$

be a diagonal double Kodaira structure of type $(b,\ n)$ on a finite group G. If G is non-monolithic, we can find a non-trivial normal subgroup N of G such that $\mathrm{z} \notin N$. Thus, the ordered set

$$\bar{\mathfrak{S}} = (\bar{\mathrm{r}}_{11},\ \bar{\mathrm{t}}_{11}, \ldots, \bar{\mathrm{r}}_{1b},\ \bar{\mathrm{t}}_{1b},\ \bar{\mathrm{r}}_{21},\ \bar{\mathrm{t}}_{21}, \ldots, \bar{\mathrm{r}}_{2b},\ \bar{\mathrm{t}}_{2b},\ \bar{\mathrm{z}})$$

is a diagonal double Kodaira structure of type $(b,\ \bar{n})$ on the quotient group G/N, with $\bar{n}$ dividing n, and $\mathfrak{S}$ is obtained by lifting $\bar{\mathfrak{S}}$ via the quotient map $\pi : G \longrightarrow G/N$. Consequently we get the following result, cf. [20, Proposition 4.7].

Proposition 4.4 *Let G be a finite group admitting a diagonal double Kodaira structure $\mathfrak{S}$. If G is non-monolithic, then there exist a proper quotient H of G and a diagonal double Kodaira structure $\bar{\mathfrak{S}}$ on H such that $\mathfrak{S}$ is obtained as a lifting of $\bar{\mathfrak{S}}$ via the quotient homomorphism $\pi : G \longrightarrow H$.*

Corollary 4.5 *Assume that G admits a diagonal double Kodaira structure $\mathfrak{S}$, whereas no proper quotient of G does. Then G is monolithic and $z \in \mathrm{mon}(G)$.*

The following complete description of diagonal double Kodaira structures with $b = 2$ on groups of order at most 32 can be found in [20, Section 4].

Proposition 4.6 *Let G be a finite group.*

(1) If either $|G| < 32$ or $|G| = 32$ and G is not extra-special, then G admits no prestructures and, consequently, no diagonal double Kodaira structures of any type.

(2) Both extra-special groups of order 32 admit $2211840 = 1152 \cdot 1920$ distinct di-

agonal double Kodaira structures of type $(b,\, n) = (2,\, 2)$. Every such a structure is strong, and moreover

- if $G = G(32,\, 49) = \mathrm{H}_5(\mathbb{Z}_2)$, these structures form 1920 orbits under the natural action of $\mathrm{Aut}(G)$;

if $G = G(32,\, 50) = \mathrm{G}_5(\mathbb{Z}_2)$, these structures form 1152 orbits under the natural action of $\mathrm{Aut}(G)$.We can now exclude all the remaining groups of order less than 64.

Proposition 4.7 *No finite group G with $33 \leq |G| \leq 63$ admits diagonal double Kodaira structures.*

Proof If $33 \leq |G| \leq 63$ then G cannot have quotients of order 32 and so, by Corollary 4.5 and Proposition 4.6, if G admits a diagonal double Kodaira structure then G is monolithic and $\mathrm{z} \in \mathrm{mon}(G)$. Now, looking at Remark 4.3 and Proposition 3.3, we see that it suffices to check that $G(54,\, 5)$ and $G(54,\, 6)$ have no diagonal double Kodaira structures. This can be done using Part 1 and Part 2 of the `GAP4` script in the GitHub repository [12]. More precisely, the script works as follows. First, Part 1 creates a `GAP4` function called `Check-Structures`, which provides a complete list of double Kodaira structures of a given finite group G in the `GAP4` database, modulo the action of $\mathrm{Aut}(G)$. Then, Part 2 applies the function to $G(54,\, 5)$ and $G(54,\, 6)$. We illustrate the procedure for $G(54,\, 5)$; the case of $G(54,\, 6)$ is analogous.

```
Read("Function_Checkstructures.g");
G := SmallGroup(54,5);;
L := CheckStructures(G);;
```

The output is

```
Group ID: [ 54, 5 ] - Structure: (C3 x C3) : C6
Number of non-central elements of G = 53
Generating the list of five candidates ...
List of five candidates ready, size = 41796
...computing up to automorfism:
Elements modulo auto: 729
------------
Checking for prestructures now, 729 elemets to go
No prestructure found
**************************************
```

This shows that $G(54,\, 5)$ admits no prestructures, and hence no diagonal double Kodaira structures. □

If $|G| = 64$ then the conclusion of Proposition 4.7 does not hold anymore, but we have the following alternatives.

Proposition 4.8 *Let G be a finite group with $|G| = 64$ and admitting a diagonal double Kodaira structure $\mathfrak{S}$.*

(1) If G is non-monolithic, then there exist an extra-special quotient H of order 32 and a diagonal double Kodaira structure $\bar{\mathfrak{S}}$ on H such that $\mathfrak{S}$ is obtained as a lifting of $\bar{\mathfrak{S}}$ via the quotient homomorphism $\pi : G \longrightarrow H$.

(2) If we are in situation **(1)** and moreover $[G, G] \simeq \mathbb{Z}_2$, then there exists an extra-special quotient H of order 32 such that *every* diagonal double Kodaira structure on G is a lifting of a structure on H.

(3) If G does not have extra-special quotients of order 32, then G is monolithic and $\mathrm{z} \in \mathrm{mon}(G) \simeq \mathbb{Z}_2$.

Proof Part **(1)** follows directly from Propositions 4.4 and 4.6. Furthermore, if $[G, G]$ has order 2, then its generator provides the element z for *every* diagonal double Kodaira structure on G; taking $H = G/N$, where N is any non-trivial normal subgroup of G such that $\mathrm{z} \notin N$, we get **(2)**. Finally, part **(3)** is a consequence of Corollaries 4.5 and 3.2. □

Remark 4.9 The "universal" extra-special quotient H in part **(2)** of Proposition 4.8 needs not be unique. In fact, the diagonal double Kodaira structures on $G = G(64, 201)$ can be lifted from both extra-special groups of order 32, see Subsect. 5.3.

We end this section by recalling how existence of diagonal double Kodaira structures is equivalent to the existence of some special double Kodaira fibrations (see the Introduction for the definition), that we call *diagonal double Kodaira fibrations*. We closely follow the treatment given in [20, Section 5]. With a slight abuse of notation, in the sequel we will use the symbol Σ_b to indicate both a compact Riemann surface of genus b and its underlying real surface. By Grauert-Remmert's extension theorem and Serre's GAGA, the group epimorphism $\varphi : \mathrm{P}_2(\Sigma_b) \longrightarrow G$ induced by a diagonal double Kodaira structure yields the existence of a smooth, complex, projective surface S endowed with a Galois cover

$$\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b,$$

with Galois group G and branched precisely over Δ with branching order n, see [7, Proposition 4.4].

The braid group $\mathrm{P}_2(\Sigma_b)$ is the middle term of two split short exact sequences

$$1 \longrightarrow \pi_1(\Sigma_b - \{p_i\}, p_j) \longrightarrow \mathrm{P}_2(\Sigma_b) \longrightarrow \pi_1(\Sigma_b, p_i) \longrightarrow 1, \tag{4}$$

where $\{i, j\} = \{1, 2\}$, induced by the two natural projections of pointed topological spaces

$$(\Sigma_b \times \Sigma_b - \Delta, (p_1, p_2)) \longrightarrow (\Sigma_b, p_i),$$

see [13, Theorem 1]. Composing the left homomorphisms in (4) with $\varphi : \mathrm{P}_2(\Sigma_b) \longrightarrow G$, we get two homomorphisms

$$\varphi_1 : \pi_1(\Sigma_b - \{p_2\}, p_1) \longrightarrow G, \quad \varphi_2 : \pi_1(\Sigma_b - \{p_1\}, p_2) \longrightarrow G,$$

whose respective images coincide with the normal subgroups K_1 and K_2 defined in (3). By construction, these are the homomorphisms induced by the restrictions $\mathrm{f}_i : \Gamma_i \longrightarrow \Sigma_b$ of the Galois cover $\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b$ to the fibres of the two natural projections $\pi_i : \Sigma_b \times \Sigma_b \longrightarrow \Sigma_b$. Since Δ intersects transversally at a single point all the fibres of the natural projections, it follows that both such restrictions are branched at precisely one point,

and the number of connected components of the smooth curve $\Gamma_i \subset S$ equals the index $m_i := [G : K_i]$ of the normal subgroup K_i in G.

So, taking the Stein factorizations of the compositions $\pi_i \circ \mathrm{f} : S \longrightarrow \Sigma_b$, namely

$$\begin{array}{ccc} S & \xrightarrow{\pi_i \circ \mathbf{f}} & \Sigma_b \\ & {\scriptstyle f_i} \searrow \quad \nearrow {\scriptstyle \theta_i} & \\ & \Sigma_{b_i} & \end{array} \tag{5}$$

where $\theta_i : \Sigma_{b_i} \longrightarrow \Sigma_b$ is étale of degree m_i, we obtain two distinct Kodaira fibrations $f_i : S \longrightarrow \Sigma_{b_i}$, hence a double Kodaira fibration by considering the product morphism

$$f = f_1 \times f_2 : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}.$$

Definition 4.10 We call $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ the *diagonal double Kodaira fibration* associated with the diagonal double Kodaira structure $\mathfrak{S}$ on the finite group G. Conversely, we will say that a double Kodaira fibration $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is *of diagonal type* (b, n) if there exists a finite group G and a diagonal double Kodaira structure $\mathfrak{S}$ of type (b, n) on it such that f is associated with $\mathfrak{S}$.

Since the morphism $\theta_i : \Sigma_{b_i} \longrightarrow \Sigma_b$ is étale of degree m_i, by using the Hurwitz formula we obtain

$$b_1 - 1 = m_1(b-1), \quad b_2 - 1 = m_2(b-1).$$

Moreover, the fibre genera g_1, g_2 of the Kodaira fibrations $f_1 : S \longrightarrow \Sigma_{b_1}$, $f_2 : S \longrightarrow \Sigma_{b_2}$ are computed by the formulae

$$2g_1 - 2 = \frac{|G|}{m_1}(2b - 2 + \mathfrak{n}), \quad 2g_2 - 2 = \frac{|G|}{m_2}(2b - 2 + \mathfrak{n}),$$

where $\mathfrak{n} := 1 - 1/n$. Finally, the surface S fits into a diagram

$$\begin{array}{ccc} S & \xrightarrow{\mathbf{f}} & \Sigma_b \times \Sigma_b \\ & {\scriptstyle f} \searrow \quad \nearrow {\scriptstyle \theta_1 \times \theta_2} & \\ & \Sigma_{b_1} \times \Sigma_{b_2} & \end{array}$$

so that the diagonal double Kodaira fibration $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ is a finite cover of degree $\frac{|G|}{m_1 m_2}$, branched precisely over the curve

$$(\theta_1 \times \theta_2)^{-1}(\Delta) = \Sigma_{b_1} \times_{\Sigma_b} \Sigma_{b_2}.$$

Such a curve is always smooth, being the preimage of a smooth divisor via an étale morphism. However, it is reducible in general, see [7, Proposition 4.11]. The invariants of S can be now computed as follows, see [7, Proposition 4.8].

Proposition 4.11 *Let $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be a diagonal double Kodaira fibration, associated with a diagonal double Kodaira structure $\mathfrak{S}$ of type (b, n) on a finite group G. Then we have*

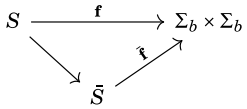
$$c_1^2(S) = |G|\,(2b-2)(4b-4+4\mathfrak{n}-\mathfrak{n}^2)$$
$$c_2(S) = |G|\,(2b-2)(2b-2+\mathfrak{n})$$

where $\mathfrak{n} = 1 - 1/n$. *Therefore the signature of S can be expressed as*

$$\sigma(S) = \frac{1}{3}\left(c_1^2(S) - 2c_2(S)\right) = \frac{1}{3}\,|G|\,(2b-2)\left(1-\frac{1}{n^2}\right).$$

Remark 4.12 By definition, the diagonal double Kodaira structure $\mathfrak{S}$ is strong if and only if $m_1 = m_2 = 1$, that in turn implies $b_1 = b_2 = b$, i.e., $f = \mathrm{f}$. In other words, $\mathfrak{S}$ is strong if and only if no Stein factorization as in (5) is needed or, equivalently, if and only if the Galois cover $\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b$ induced by the admissible epimorphism $\varphi : \mathrm{P}_2(\Sigma_b) \longrightarrow G$ is already a double Kodaira fibration, branched on the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$.

Remark 4.13 Proposition 4.4 can be rephrased in geometric terms as follows: if G is non-monolithic, there exists a proper quotient H of G and a factorization of the G-cover of the form



where $\bar{\mathrm{f}} : \bar{S} \longrightarrow \Sigma_b \times \Sigma_b$ is a H-cover branched on the diagonal $\Delta \subset \Sigma_b \times \Sigma_b$ with branching order $\bar{n} \geq 2$, where $\bar{n}$ divides n.

5 Diagonal double Kodaira structures on groups of order 64

By using `GAP4`, as in the proofs of Propositions 2.5 and 3.3, we show that there are precisely 124 non-abelian, non-CCT groups of order 64. Among these, 37 are monolithic and 87 are non-monolithic. By Proposition 4.8, non-monolithic groups of order 64 that do not admit extra-special quotients of order 32 cannot carry prestructures and may therefore be discarded. There are 56 such groups. Consequently, we are left with 37 monolithic groups and 31 non-monolithic groups, which are listed in the following statement .

Proposition 5.1 *Let G be a non-abelian, non-CCT group of order 64.*

(1) If G is monolithic, then G is of the form $G(64,\, t)$, with t in the set

$\{18,\ 19,\ 25,\ 28,\ 30,\ 32,\ 33,\ 34,\ 35,\ 36,\ 37,\ 41,\ 42,\ 43,\ 46,\ 91,\ 94,\ 102,\ 111,\ 125,\ 134,$
$135,\ 136,\ 137,\ 138,\ 139,\ 152,\ 153,\ 154,\ 190,\ 191,\ 249,\ 256,\ 257,\ 258,\ 259,\ 266\}.$

(2) If G is non-monolithic and admits an extra-special quotient of order 32, then G is of the form $G(64,\, t)$, with t in the set

$\{199,\ 200,\ 201,\ 215,\ 216,\ 217,\ 218,\ 219,\ 220,\ 221,\ 222,\ 223,\ 224,\ 225,\ 226,\ 227,\ 228,\ 229,$
$230,\ 231,\ 232,\ 233,\ 234,\ 235,\ 236,\ 237,\ 238,\ 239,\ 240,\ 264,\ 265\}.$

Proof We dedicated to this computation Part 3 of our GAP4 script in the GitHub repository [12]. The code is as follows:

```
# This script finds:
# - the set of monolithic, non-CCT groups of order 64 and
# - the set of non-monolithic groups of order 64 admitting extra-special
### quotients of order 32
# These are the only candidates for groups of order 64 with prestructures

List_filtered_monolithic := [];;
List_filtered_esquotient :=[];;
List_ids_groups_with_structures := [ [32,49], [32,50] ];;

#Order of G
order:=64;

  #list of nonabelian groups of given order
  List_id_non_abelian:= IdsOfAllSmallGroups(Size, order, IsAbelian, false);;

  for id in List_id_non_abelian do
    G := SmallGroup(id);;
    # Check CCT property
    H := [];; # List of non central elements of G
    ZG := Center(G);
    for g in G do
      if not g in ZG then
        Add(H, g);
      fi;
    od;

    size_ab := 0;;

    for h in H do
      C := Centralizer(G, h);
      if IsAbelian(C) then
        size_ab := size_ab + 1;
      fi;
    od;

    if size_ab = Size(H) then
      # The group is CCT
    else
      # The group is Non-CCT
      # Check wether G is monolitic
      List_Minimal_Normal := MinimalNormalSubgroups(G);
      if Size(List_Minimal_Normal) = 1 then
        Add(List_filtered_monolithic, id);
      else
        # The group is non monolithic
        List_Normal_Candidates := Filtered(NormalSubgroups(G),
        N -> Index(G,N) =  32);;
        for N in List_Normal_Candidates do
          if (IdSmallGroup(G/N) in List_ids_groups_with_structures) then
            Add(List_filtered_esquotient, id );
            break;
          fi;
        od;
      fi;
    fi;
  od;
List_filtered_monolithic;
List_filtered_esquotient;
```

The output provides precisely the two lists of groups of order 64 given in the statement. □

Although we considerably reduced the number of cases to check, these groups are still too many to be attacked one by one by hand. So we analyzed them by using our `GAP4` function `CheckStructures`, see Parts 3 and 4 of the `GAP4` script in the repository [12]. In this way we obtain

Theorem 5.2 *Let G be a group of order 64 admitting a diagonal double Kodaira structure.*

- If G is non-monolithic then it is isomorphic to $G(64,\ t)$, with $t \in \{199, 200, 201, 264, 265\}$.
- If G is monolithic, then it is isomorphic to $G(64,\ t)$, with $t \in \{249, 266\}$.

Moreover, in each case the number of diagonal double Kodaira structures with $b = 2$ is given in the table below. For all these groups we have $[G,\ G] = \mathbb{Z}_2$, so $n = o(z) = 2$. Moreover (and remarkably) the pair $(|K_1|,\ |K_2|)$ only depends on the group, and not on the structure.

IdSmallGroup(G)	*Monolithic*	*Number of structures with* $b = 2$	*Number of structures with* $b = 2$ *modulo* $\mathrm{Aut}(G)$	$(\lvert K_1\rvert,\ \lvert K_2\rvert)$
$G(64,\ 199)$	*no*	$138240 \cdot 40960 = 566231040$	138240	$(64,\ 64)$
$G(64,\ 200)$	*no*	$46080 \cdot 12288 = 566231040$	46080	$(64,\ 64)$
$G(64,\ 201)$	*no*	$184320 \cdot 3072 = 566231040$	184320	$(64,\ 64)$
$G(64,\ 264)$	*no*	$14400 \cdot 36864 = 530841600$	14400	$(32,\ 32)$
$G(64,\ 265)$	*no*	$8640 \cdot 61440 = 530841600$	8640	$(32,\ 32)$
$G(64,\ 249)$	*yes*	$368640 \cdot 1536 = 566231040$	368640	$(64,\ 64)$
$G(64,\ 266)$	*yes*	$23040 \cdot 23040 = 530841600$	23040	$(32,\ 32)$

We will now investigate in detail each of these groups separately. Part 6 of the `GAP4` script in [12] analyzes the group $G(64,\ 199)$; the scripts analyzing the other groups are similar. In order to check the liftings of the structures from a quotient, we need to apply our function `CheckStructures` to the two extra-special groups of order 32 and to generate the corresponding list of diagonal double Kodaira structures in each case; this is done by using Part 5 of the aforementioned `GAP4` script.

In the remainder of this section, for the sake of brevity we will simply write "structure" instead of "diagonal double Kodaira structure of type $(b,\ n) = (2,\ 2)$". The existence and enumeration of diagonal double Kodaira structures in genus $b \geq 3$ is an interesting problem, which we hope to address in future work.

5.1 The case $G = (64, 199)$

The group $G = G(64,\ 199)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$$
\begin{array}{llllll}
x_1^2 = x_6, & & & & & \\
x_2^2 = 1, & [x_1, x_2] = x_5, & & & & \\
x_3^2 = 1, & [x_1, x_3] = 1, & [x_2, x_3] = x_5, & & & \\
x_4^2 = 1, & [x_1, x_4] = x_5, & [x_2, x_4] = 1, & [x_3, x_4] = 1, & & \\
x_5^2 = 1, & [x_1, x_5] = 1, & [x_2, x_5] = 1, & [x_3, x_5] = 1, & [x_4, x_5] = 1, & \\
x_6^2 = 1, & [x_1, x_6] = 1, & [x_2, x_6] = 1, & [x_3, x_6] = 1, & [x_4, x_6] = 1, & [x_5, x_6] = 1.
\end{array}
$$

We have $Z(G) = \langle x_5, x_6\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $[G, G] = \langle x_5\rangle \simeq \mathbb{Z}_2$. Setting $x_6 = 1$ in the presentation above we get a presentation for the extra-special group $G(32, 49)$, so G is a central extension

$$1 \longrightarrow \langle x_6\rangle \longrightarrow G \longrightarrow G(32, 49) \longrightarrow 1. \tag{6}$$

The exact sequence (6) is non-split, otherwise G would be isomorphic to $G(32, 49) \times \mathbb{Z}_2$, a contradiction since the latter group is $G(64, 264)$. One easily checks that G is non-monolithic: in fact, $\langle x_5\rangle$ and $\langle x_6\rangle$ are two central, and hence normal, subgroups isomorphic to $\mathbb{Z}_2$ and having trivial intersection. Thus, by Proposition 4.8, every structure on G is obtained by lifting a structure from the extra-special quotient $G(32, 49)$.

All the liftings of a given structure

$$\mathfrak{S} = (\mathrm{r}_{11}, \mathrm{t}_{11}, \mathrm{r}_{12}, \mathrm{t}_{12}, \mathrm{r}_{21}, \mathrm{t}_{21}, \mathrm{r}_{22}, \mathrm{t}_{22}, \mathrm{z}),$$

in $G(32, 49)$ are obtained multiplying by x_6 any number of elements of the structure among the r_{ij}, t_{ij}. Multiplying to the right or to the left is irrelevant, since x_6 is central in G; so we have $2^8 = 256$ possible liftings, and one verifies that all of these liftings provide distinct strong structures on G.

The number of structures in $G(32, 49)$ being $1152 \cdot 1920$, we infer that the total number of structures on G is $1152 \cdot 1920 \cdot 256$.

Finally, since $|\operatorname{Aut}(G)| = 4096$, the number of structures on G modulo automorphisms is

$$(1152 \cdot 1920 \cdot 256)/4096 = 138240.$$

5.2 The case $G = (64, 200)$

The group $G = G(64, 200)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$$
\begin{array}{llllll}
x_1^2 = x_6, & & & & & \\
x_2^2 = x_5, & [x_1, x_2] = x_5, & & & & \\
x_3^2 = x_5, & [x_1, x_3] = 1, & [x_2, x_3] = x_5, & & & \\
x_4^2 = 1, & [x_1, x_4] = x_5, & [x_2, x_4] = 1, & [x_3, x_4] = 1, & & \\
x_5^2 = 1, & [x_1, x_5] = 1, & [x_2, x_5] = 1, & [x_3, x_5] = 1, & [x_4, x_5] = 1, & \\
x_6^2 = 1, & [x_1, x_6] = 1, & [x_2, x_6] = 1, & [x_3, x_6] = 1, & [x_4, x_6] = 1, & [x_5, x_6] = 1.
\end{array}
$$

We have $Z(G) = \langle x_5, x_6\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $[G, G] = \langle x_5\rangle \simeq \mathbb{Z}_2$. Setting $x_6 = 1$ in the presentation above we get a presentation for the extra-special group $G(32, 50)$, so G is a central extension

$$1 \longrightarrow \langle x_6\rangle \longrightarrow G \longrightarrow G(32, 50) \longrightarrow 1. \tag{7}$$

The exact sequence (7) is non-split, otherwise G would be isomorphic to $G(32, 50) \times \mathbb{Z}_2$, a contradiction since the latter group is $G(64, 265)$. One easily checks that G is non-monolithic: in fact, $\langle x_5\rangle$ and $\langle x_6\rangle$ are two central, and hence normal, subgroups isomorphic to $\mathbb{Z}_2$ and having trivial intersection. Thus, by Proposition 4.8, every structure on G is obtained by lifting a structure from the extra-special quotient $G(32, 50)$.

Since x_6 is central in G, all the liftings of a given structure in $G(32, 50)$ are obtained by multiplying by x_6 any number of elements among the r_{ij}, t_{ij}. As in the previous case we have $2^8 = 256$ possible liftings, and one verifies that all of these liftings provide distinct strong structures on G.

The number structures in $G(32, 50)$ being $1152 \cdot 1920$, we infer that the total number structures on G is $1152 \cdot 1920 \cdot 256$

Finally, since $|\operatorname{Aut}(G)| = 12288$, the number of structures on G modulo automorphisms is

$$(1152 \cdot 1920 \cdot 256)/12288 = 46080.$$

5.3 The case $G = (64, 201)$

The group $G = G(64, 201)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$x_1^2 = x_6$,					
$x_2^2 = 1$,	$[x_1, x_2] = x_5$,				
$x_3^2 = 1$,	$[x_1, x_3] = 1$,	$[x_2, x_3] = x_5$,			
$x_4^2 = x_5$,	$[x_1, x_4] = x_5$,	$[x_2, x_4] = 1$,	$[x_3, x_4] = 1$,		
$x_5^2 = 1$,	$[x_1, x_5] = 1$,	$[x_2, x_5] = 1$,	$[x_3, x_5] = 1$,	$[x_4, x_5] = 1$,	
$x_6^2 = 1$,	$[x_1, x_6] = 1$,	$[x_2, x_6] = 1$,	$[x_3, x_6] = 1$,	$[x_4, x_6] = 1$,	$[x_5, x_6] = 1$.

We have $Z(G) = \langle x_5, x_6\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $[G, G] = \langle x_5\rangle \simeq \mathbb{Z}_2$. In this case G is a (non-trivial) central extension of both the extra-special groups of order 32, in fact we have two short exact sequences

$$1 \longrightarrow \langle x_6\rangle \longrightarrow G \longrightarrow G(32, 49) \longrightarrow 1$$
$$1 \longrightarrow \langle x_5x_6\rangle \longrightarrow G \longrightarrow G(32, 50) \longrightarrow 1.$$

One easily checks that G is non-monolithic: in fact, $\langle x_5\rangle$ and $\langle x_6\rangle$ are two central, and hence normal, subgroups isomorphic to $\mathbb{Z}_2$ and having trivial intersection. Thus, by Proposition 4.8, every structure on G is obtained by lifting a structure either from the extra-special quotient $G(32, 49)$ or from the extra-special quotient $G(32, 50)$.

The number of structures on both $G(32,\,49)$ and $G(32,\,50)$ being $1152 \cdot 1920$, we infer as before that the total number of liftings on G is $1152 \cdot 1920 \cdot 256$; furthermore, one verifies that all these liftings provide distinct strong structures on G.

Finally, since $|\operatorname{Aut}(G)| = 3072$, the number of structures on G modulo automorphisms is

$$(1152 \cdot 1920 \cdot 256)/3072 = 184320.$$

5.4 The case $G = (64, 264)$

The group $G = G(64,\,264)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$x_1^2 = 1,$					
$x_2^2 = 1,$	$[x_1,\,x_2] = x_5,$				
$x_3^2 = 1,$	$[x_1,\,x_3] = 1,$	$[x_2,\,x_3] = x_5,$			
$x_4^2 = 1,$	$[x_1,\,x_4] = x_5,$	$[x_2,\,x_4] = 1,$	$[x_3,\,x_4] = 1,$		
$x_5^2 = 1,$	$[x_1,\,x_5] = 1,$	$[x_2,\,x_5] = 1,$	$[x_3,\,x_5] = 1,$	$[x_4,\,x_5] = 1,$	
$x_6^2 = 1,$	$[x_1,\,x_6] = 1,$	$[x_2,\,x_6] = 1,$	$[x_3,\,x_6] = 1,$	$[x_4,\,x_6] = 1,$	$[x_5,\,x_6] = 1.$

We have $Z(G) = \langle x_5,\,x_6\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $[G,\,G] = \langle x_5\rangle \simeq \mathbb{Z}_2$. The group G is a central extension of the extra-special group $G(32,\,49)$, given by a short exact sequence

$$1 \longrightarrow \langle x_6\rangle \longrightarrow G \longrightarrow G(32,\,49) \longrightarrow 1.$$

One checks that such a sequence is split, hence $G \simeq G(32,\,49) \times \mathbb{Z}_2$ and so G is non-monolithic. Thus, by Proposition 4.8, every structure on G is obtained by lifting a structure from the extra-special quotient $G(32,\,49)$.

As before, the total number of liftings on G is $1152 \cdot 1920 \cdot 256$. However, in this case not all liftings give structures: in fact, some of them do not generate G, but only a subgroup of order 32 inside it. More precisely, one verifies that for each structure on $G(32,\,49)$ there are precisely 16 liftings that do not generate G, and the correct number of distinct structures on G is $1152 \cdot 1920 \cdot (256 - 16)$. Moreover, all these structures are non-strong: more precisely, in all cases we have $|K_1| = |K_2| = 32$.

Finally, since $|\operatorname{Aut}(G)| = 36864$, the number of structures on G modulo automorphisms is

$$(1152 \cdot 1920 \cdot 240)/36864 = 14400.$$

5.5 The case $G = (64, 265)$

The group $G = G(64,\,265)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$x_1^2 = 1,$					
$x_2^2 = x_5,$	$[x_1, x_2] = x_5,$				
$x_3^2 = x_5,$	$[x_1, x_3] = 1,$	$[x_2, x_3] = x_5,$			
$x_4^2 = 1,$	$[x_1, x_4] = x_5,$	$[x_2, x_4] = 1,$	$[x_3, x_4] = 1,$		
$x_5^2 = 1,$	$[x_1, x_5] = 1,$	$[x_2, x_5] = 1,$	$[x_3, x_5] = 1,$	$[x_4, x_5] = 1,$	
$x_6^2 = 1,$	$[x_1, x_6] = 1,$	$[x_2, x_6] = 1,$	$[x_3, x_6] = 1,$	$[x_4, x_6] = 1,$	$[x_5, x_6] = 1.$

We have $Z(G) = \langle x_5, x_6\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $[G, G] = \langle x_5\rangle \simeq \mathbb{Z}_2$. The group G is a central extension of the extra-special group $G(32, 50)$, given by a short exact sequence

$$1 \longrightarrow \langle x_6\rangle \longrightarrow G \longrightarrow G(32, 50) \longrightarrow 1.$$

One checks that such a sequence is split, hence $G \simeq G(32, 50) \times \mathbb{Z}_2$ and so G is non-monolithic. Thus, by Proposition 4.8, every structure on G is obtained by lifting a structure from its extra-special quotient $G(32, 50)$.

As in the previous case, the total number of liftings on G is $1152 \cdot 1920 \cdot 256$ and, for each diagonal double Kodaira structure on $G(32, 50)$, there are precisely 16 liftings that do not generate G. The total number of distinct structures on G is again $1152 \cdot 1920 \cdot 240$. Moreover, all these structures are non-strong, since in all cases $|K_1| = |K_2| = 32$.

Finally, since $|\operatorname{Aut}(G)| = 61440$, the number of structures on G modulo automorphisms is

$$(1152 \cdot 1920 \cdot 240)/61440 = 8640.$$

5.6 The case $G = (64, 249)$

The group $G = G(64, 249)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$x_1^2 = x_5,$					
$x_2^2 = 1,$	$[x_1, x_2] = 1,$				
$x_3^2 = 1,$	$[x_1, x_3] = 1,$	$[x_2, x_3] = x_6,$			
$x_4^2 = 1,$	$[x_1, x_4] = x_6,$	$[x_2, x_4] = 1,$	$[x_3, x_4] = 1,$		
$x_5^2 = x_6,$	$[x_1, x_5] = 1,$	$[x_2, x_5] = 1,$	$[x_3, x_5] = 1,$	$[x_4, x_5] = 1,$	
$x_6^2 = 1,$	$[x_1, x_6] = 1,$	$[x_2, x_6] = 1,$	$[x_3, x_6] = 1,$	$[x_4, x_6] = 1,$	$[x_5, x_6] = 1.$

We have $Z(G) = \langle x_5\rangle \simeq \mathbb{Z}_4$ and $[G, G] = \langle x_6\rangle \simeq \mathbb{Z}_2$. Moreover, G is monolithic, with $\operatorname{mon}(G) = [G, G]$.

No quotient of G is isomorphic to either $G(32, 49)$ or $G(32, 50)$; therefore, no structure on G arises as a lifting from a quotient. In this sense, the structures on G are genuinely new. The computation with GAP4 show that there are precisely 368640 such structures on G, modulo the action of $\operatorname{Aut}(G)$. All these structures are strong.

5.7 The case $G = (64, 266)$

The group $G = G(64, 266)$ is generated by six elements $x_1, \ldots, x_6$ subject to the following set of relations:

$x_1^2 = 1,$					
$x_2^2 = 1,$	$[x_1, x_2] = x_6,$				
$x_3^2 = 1,$	$[x_1, x_3] = 1,$	$[x_2, x_3] = x_6,$			
$x_4^2 = 1,$	$[x_1, x_4] = x_6,$	$[x_2, x_4] = 1,$	$[x_3, x_4] = 1,$		
$x_5^2 = x_6,$	$[x_1, x_5] = 1,$	$[x_2, x_5] = 1,$	$[x_3, x_5] = 1,$	$[x_4, x_5] = 1,$	
$x_6^2 = 1,$	$[x_1, x_6] = 1,$	$[x_2, x_6] = 1,$	$[x_3, x_6] = 1,$	$[x_4, x_6] = 1,$	$[x_5, x_6] = 1.$

We have $Z(G) = \langle x_5 \rangle \simeq \mathbb{Z}_4$ and $[G, G] = \langle x_6 \rangle \simeq \mathbb{Z}_2$. Moreover, G is monolithic, with $\mathrm{mon}(G) = [G, G]$.

No quotient of G is isomorphic to either $G(32, 49)$ or $G(32, 50)$; hence no structure on G arises as a lifting from a quotient. As in the previous case, the structures on G are therefore genuinely new. The computation with `GAP4` shows that, modulo the action of $\mathrm{Aut}(G)$, there are precisely 23040 such structures on G. All these structures are non-strong, with $|K_1| = |K_2| = 32$.

6 The computation of the first homology group of *S*

We end this work by computing the first homology group $H_1(S, \mathbb{Z})$, where $\mathrm{f} : S \longrightarrow \Sigma_2 \times \Sigma_2$ is the G-cover associated with a diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on a finite group G of order 64. To this purpose, we will make use of the following result, which is a consequence of [10, Theorem p. 254].

Proposition 6.1 *Let G be a finite group, and $\varphi : P_2(\Sigma_b) \longrightarrow G$ be an admissible epimorphism such that $\varphi(A_{12})$ has order $n \geq 2$. If $\mathrm{f} : S \longrightarrow \Sigma_b \times \Sigma_b$ is the G-cover associated with φ, then $\pi_1(S)$ fits into a short exact sequence*

$$1 \longrightarrow \pi_1(S) \longrightarrow P_2(\Sigma_b)^{\mathrm{orb}} \xrightarrow{\bar{\varphi}} G \longrightarrow 1,$$

where the orbifold braid group $P_2(\Sigma_b)^{\mathrm{orb}}$ is defined as the quotient of $P_2(\Sigma_b)$ by the normal closure of the cyclic subgroup $\langle A_{12}^n \rangle$, and $\bar{\varphi} : P_2(\Sigma_b)^{\mathrm{orb}} \longrightarrow G$ is the group epimorphism naturally induced by φ.

Proposition 6.1 allows one to compute $\pi_1(S)$, and hence its abelianization $H_1(S, \mathbb{Z})$. However, carrying out these computations by hand appears impractical, so we again resort to `GAP4`. Part 7 of the `GAP4` script in the repository [12] contains the code used to analyse the group $G(64, 199)$; the scripts for the remaining groups are analogous.

For the reader's convenience, we briefly explain the procedure. First, we run the function `CheckStructures`, which computes the list of diagonal double Kodaira structures of type $(b, n) = (2, 2)$ on $G(64, 199)$. This list is saved in the file `List_structures_mod_64_199.g`. We then run the following script, which computes $\pi_1(S)$ for each structure in the list:

```
# Redefine commutator function according to our convention
comm := function(x, y) return x*y*x^-1*y^-1;  end;

# Construction of the orbifold braid group P_2(Sigma_2)^orb
F:=FreeGroup("a11", "b11", "a12", "b12",
"a21", "b21", "a22", "b22", "u");;
AssignGeneratorVariables(F);;

S1:=comm(a12^(-1), b12^(-1))*b12^(-1)*comm(a11^(-1), b11^(-1))*b12*u
    ^(-1);;
S2:=comm(a21^(-1), b21)*b21*comm(a22^(-1), b22)*b21^(-1)*u;;
R1:=comm(a11, a22);;
R2:=comm(a11, a21);;
R3:=comm(a11, b22);;
R4:=comm(a11, b21)*u;;
R5:=comm(a11, u)*comm(u, a21^(-1));;
R6:=comm(a12, a22);;
R7:=u^(-1)*a21*a22^(-1)*u*a22*a21^(-1)*comm(a21, a12);;
R8:=comm(a12, b22)*u;;
R9:=comm(a12, b21)*comm(b21, u^(-1));;
R10:=comm(a12, u)*comm(u, a22^(-1));;
T1:=comm(b11, a22);;
T2:=comm(b11, a21)*b21^(-1)*u^(-1)*b21;;
T3:=comm(b22, b11);;
T4:=comm(b11, b21)*comm(u, b21^(-1));;
T5:=comm(b11, u)*comm(u, b21^(-1));;
T6:=comm(b12, a22)*b22^(-1)*u^(-1)*b22;;
T7:=comm(b12, a21)*comm(u, b22^(-1));;
T8:=comm(b12, b22)*comm(u, b22^(-1));;
T9:=b22^(-1)*u*b22*u^(-1)*b21*u*b22^(-1)*u^(-1)*b22*b21^(-1)*comm(b21,
    b12);;
T10:=comm(b12, u)*comm(u, b22^(-1));;

P:=F/[R1, R2, R3, R4, R5, R6, R7, R8, R9, R10,
T1, T2, T3, T4, T5, T6, T7, T8, T9, T10,
S1, S2];;
AssignGeneratorVariables(P);;

P_orb:=P/[u^2];
AssignGeneratorVariables(P_orb);;

# Strong and non-strong structures on G
G:=SmallGroup(64, 199);
G:=Image(IsomorphismFpGroup(G));
RelatorsOfFpGroup(G);
AssignGeneratorVariables(G);
f1:=F1;; f2:=F2;; f3:=F3;; f4:=F4;; f5:=F5;; f6:=F6;;
Read("List_structures_mod_64_199.g");;
Size(List_structures_mod_64_199);

list_fund_groups_64_199:=[];;
strong:=0;; weak:=0;;

for y in [1..Size(List_structures_mod_64_199)] do
  l:=List_structures_mod_64_199[y];
  r11:=l[1];; t11:=l[2];; r12:=l[3];; t12:=l[4];;
  r21:=l[5];; t21:=l[6];; r22:=l[7];; t22:=l[8];;
  z:=comm(t21, r11);
  K1:=Subgroup(G, [r11, t11, r12, t12, z]);;
  K2:=Subgroup(G, [r21, t21, r22, t22, z]);;
  if Order(K1)=Order(G) and Order(K2)=Order(G) then
    sw:="strong"; strong:=strong+1;
  else
    sw:="weak"; weak:=weak+1;
  fi;
  Print(y, "\n");
  Print("Number of strong structures:", " "); Print(strong, "\n");
  Print("Number of non-strong structures:", " "); Print(weak, "\n");

# Construction of the group epimorphism \bar{\varphi}
  hom:=GroupHomomorphismByImages(P_orb,G,[a11,b11,a12,b12,a21,b21,a22,
    b22,u],
       [r11,t11,r12,t12,r21,t21,r22,t22,z]);

# Computation of the list of abelian invariants of the fundamental
    groups
       KK:=Kernel(hom);;
       iso:=IsomorphismFpGroup(KK);
       fpK:=Image(iso);
       fpKK:=Range(IsomorphismSimplifiedFpGroup(fpK));
       Print(Order(K1), " "); Print(Order(K2), " "); Print(
    AbelianInvariants(fpKK), "\n"); Print("\n");
       AddSet(list_fund_groups_64_199, [sw, Order(K1), Order(K2),
    AbelianInvariants(fpKK)]);
od;
       list_fund_groups_64_199;
```

The output is

```
Number of strong structures: 138240
Number of non-strong structures: 0
[ [ "strong", 64, 64, [ 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2 ] ] ]
```

This shows that all 138240 diagonal double Kodaira structures of type $(b, n) = (2, 2)$ on G are strong, and that for each of them the fundamental group $\pi_1(S)$ is isomorphic to $\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$.

We can now summarize our results.

Theorem 6.2 Let $\mathrm{f} : S \longrightarrow \Sigma_2 \times \Sigma_2$ be the G-cover associated with a diagonal double Kodaira structure of type $(b, n) = (2, 2)$ on a finite group G of order 64, and let $f : S \longrightarrow \Sigma_{b_1} \times \Sigma_{b_2}$ be the corresponding diagonal double Kodaira fibration. Then the occurrences for $H_1(S, \mathbb{Z})$, $q(S) = \frac{1}{2}\,\mathrm{rank}\ H_1(S, \mathbb{Z})$, K_S^2, $c_2(S)$, $\sigma(S)$, b_1, b_2, g_1, g_2 are as in the table below.

IdSmallGroup(G)	$H_1(S, \mathbb{Z})$	$q(S)$	K_S^2	$c_2(S)$	$\sigma(S)$	b_1	b_2	g_1	g_2
$G(64, 199)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 200)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 201)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 264)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 265)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 249)$	$\mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$	4	736	320	32	2	2	81	81
$G(64, 266)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$	6	736	320	32	3	3	41	41
$G(64, 266)$	$\mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	6	736	320	32	3	3	41	41

More precisely, in the case $G = G(64, 266)$, up to the natural action of $\mathrm{Aut}(G)$ *there exist 17280 diagonal double Kodaira structures such that $H_1(S, \mathbb{Z}) = \mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^3$ and 5760 diagonal double Kodaira structures such that $H_1(S, \mathbb{Z}) = \mathbb{Z}^{12} \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$.*

Thus, in the case $G = (64, 266)$ there are two occurrences for the torsion part of $H_1(S, \mathbb{Z})$. Moreover, *every* curve of genus $b = 2$ can be used as a starting point for our construction, and so such a construction depends on 3 parameters. We can therefore state the following

Corollary 6.3 *There exist two 3-dimensional families $\mathscr{F}_1$, $\mathscr{F}_2$ of Kodaira doubly fibred surfaces with $b_1 = b_2 = 3$, $g_1 = g_2 = 41$, $\sigma(S) = 32$ such that*

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have the same biregular invariants

$$p_g(S) = 93, \quad q(S) = 6, \quad K_S^2 = 736$$

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have the same Betti numbers

$$\mathrm{b}_0 = \mathrm{b}_4 = 1, \quad \mathrm{b}_1 = \mathrm{b}_3 = 12, \quad \mathrm{b}_2 = 342$$

- the surfaces in $\mathscr{F}_1$ and those in $\mathscr{F}_2$ have different torsion part of $H_1(S, \mathbb{Z})$. In particular, they are not homotopically equivalent.

Remark 6.4 To the best of our knowledge, the surfaces described in Corollary 6.3 provide the first explicit examples of positive-dimensional families of doubly fibred Kodaira surfaces with the same base and fibre genera, the same biregular invariants, the same Betti numbers, but with non-isomorphic fundamental groups.

We do not know whether the same phenomenon occurs for $|G| = 32$. Computations with GAP4 show that all surfaces in the two families constructed in [20, Section 5], corresponding to the extra-special groups $\mathrm{H}_5(\mathbb{Z}_2)$ and $\mathrm{G}_5(\mathbb{Z}_2)$, satisfy $H_1(S, \mathbb{Z}) = \mathbb{Z}^8 \oplus (\mathbb{Z}_2)^4$. Therefore, the torsion subgroup of $H_1(S, \mathbb{Z})$ does not distinguish these surfaces. It thus appears to be an interesting problem to determine whether the fundamental groups of surfaces belonging to different families are isomorphic or not.

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References

1. Atiyah, M.F.: The signature of fibre bundles, in Global Analysis (Papers in honor of K. Kodaira) 73–84, Univ. Tokyo Press (1969)
2. Barth, W., Hulek, K., Peters, C.A.M., Van de Ven, A.: *Compact Complex Surfaces*. Grundlehren der Mathematischen Wissenschaften, Vol **4**, Second enlarged edition, Springer-Verlag, Berlin, (2003)
3. Bryan, J., Donagi, R.: Surface bundles over surfaces of small genus, Geom. Topol. 6 , 59–67 (2002)
4. Bryan, J., Donagi, R., Stipsicz, A.I.: Surface bundles: some interesting examples, Turkish J. Math. 25 , 1, 61–68 (2001)
5. Catanese, F.: Kodaira fibrations and beyond: methods for moduli theory, Japan. J. Math. 12 , 2, 91–174 (2017)
6. Catanese, F., Rollenske, S.: Double Kodaira fibrations, J. Reine Angew. Math. 628 , 205–233 (2009)
7. Causin, A., Polizzi, F.: Surface braid groups, finite Heisenberg covers and double Kodaira fibrations, Ann. Sc. Norm. Super. Pisa Cl. Sci. Vol. **XXII** , 1309–1352 (2021)
8. Endo, H.: A construction of surface bundles over surfaces with non-zero signature, Osaka J. Math. 35 , 915–930 (1998)
9. Endo, H., Korkmaz, M., Kotschick, D., Ozbagci, B., Stipsicz, A.: Commutators, Lefschetz fibrations and the signature of surface bundles, Topology 41 5 , 961–977 (2002)
20. Polizzi, F., Sabatino, P.: Extra-special quotients of surface braid groups and double Kodaira fibrations with small signature, Geom. Dedicata **216**, 65 (2022)
10. Fox, R.H.: Covering spaces with singularities, A symposium in honor of S. Lefschetz, Princeton University Press , 243–257 (1957)
11. The GAP Group, GAP–Groups,Algorithms, and Programming, Version 4.11.1; (2021)
12. https://github.com/pietros16bit/gap4ddks
13. Gonçalves, D.L., Guaschi, J.: On the structure of surface pure braid groups, J. Pure Appl. Algebra 186 , 187–218 (2004)

14. Hirzebruch, F.: The signature of ramified covers, in Global Analysis (Papers in honor of K. Kodaira) 253–265, Univ. Tokyo Press (1969)
15. Kas, A.: On deformations of a certain type of irregular algebraic surface, Amer. J. Math. 90 , 789–804 (1968)
16. LeBrun, C.: Diffeomorphisms, symplectic forms and Kodaira fibrations, Geom. Topol. 4 , 451–456 (2000)
17. Lee, J.A.: Surface bundles over surfaces with a fixed signature, J. Korean Math. Soc. 54 , 2, 545–561 (2017)
18. Lee, J.A., Lönne, M., Rollenske, S.: Double Kodaira fibrations with small signature, Internat. J. Math. 31 , 7, 2050052, 42 (2020)
19. Polizzi, F.: Diagonal double Kodaira structures on finite groups, Current Trends in Analysis, its Applications and Computation, Trends in Mathematics, Birkäuser, Cham , 111-128 (2022)
21. Polizzi, F., Sabatino, P.: Finite quotients of surface braid groups and double Kodaira fibrations, pp. 339–361. Trends in Mathematics, Birkäuser, Cham, The Art of Doing Algebraic Geometry (2022)
22. Rollenske, S.: Compact moduli for certain Kodaira fibrations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. IX , 851-874 (2010)
23. Stipsicz, A.I.: Surface bundles with nonvanishing signature, Acta Math. Hungar. 95 , 4, 299–307 (2002)
24. Zaal, C.: Explicit complete curves in the moduli space of curves of genus three. Geom. Dedicata. **56**, 185–196 (1995)

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