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PAPER *Special Section on Discrete Mathematics and Its Applications*

Various Diamond Properties in Combinatorial Game Theory

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SUMMARY We investigate conditions under which positions in combinatorial games admit simple values. We introduce a unified diamond framework, the $\diamond_{\mathcal{A}}$-property ($\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$), for sets of positions closed under options. Under certain conditions, this framework guarantees that all values are integers, dyadic rationals, or pairs $\{m \mid n\}$ (on $\mathbb{Z}$ or $\mathbb{D}$). As an application, we establish that every position in Yashima game on bipartite graphs has an integer pair value.

key words: *Combinatorial Game Theory, Diamond Property, Yashima game*

1. Introduction

A combinatorial game is a two-player game with no chance elements and no hidden information, and the collection of results concerning such games forms the field of combinatorial game theory [1], [6]. In this theory, each game position is assigned a position value, and these values can be handled algebraically to support various kinds of analysis. Position values include the set $\mathbb{D}$ of numbers (the dyadic rationals), as well as non-numeric values such as $*$ ($\cong \{0 \mid 0\}$) and $\uparrow$ ($\cong \{0 \mid *\}$). However, since the game tree often grows very fast, the analysis tends to be hard and the resulting values can become quite complicated.

As an example, we introduce Yashima game [2]. At the beginning, each of the two players places one token on a vertex of an undirected graph. Following the usual convention, the players are referred to as Left and Right. On each turn, a player moves their token along an edge to an adjacent vertex and deletes the edge just traversed. The players then continue alternately in this manner. The two tokens may never occupy the same vertex simultaneously. The player who has no legal move loses.

An example of Yashima game is shown in Fig.1. In this figure, Ⓛ and Ⓡ denote the tokens of Left and Right, respectively. In this example, Right eventually has no legal move and therefore loses.

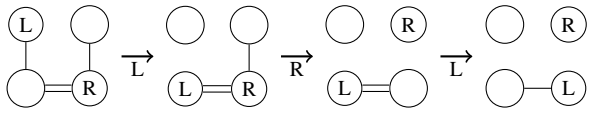


Fig. 1 Yashima

In our analysis of Yashima game, we find that in many cases the position value takes a remarkably simple form, namely an integer pair $\{m \mid n\}$. Although this is not valid in full generality, we discover that it holds for bipartite graphs. As a key to this phenomenon, we introduce the notion of the diamond property, illustrated in Fig.2. The name derives from the diamond-shaped diagram. This property typically arises when the intended move is not blocked by the opponent, a situation that frequently occurs in the early stages of the game.

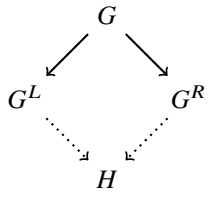


Fig.2 Diamond property

A related work considers a set of positions $\mathcal{G}$ that is closed under options. For such a set, authors in [4] proposed two properties: (i) the F_2-property, which guarantees $\mathcal{G} \subset \mathbb{Z}$, and (ii) the F_1-property, which guarantees $\mathcal{G} \subset \mathbb{D}$. To maintain consistent terminology in this paper, we refer to the F_2-property as the $\diamond^{\leq}$-property, and the F_1-property as the $\diamond^{\leq}_L$ and $\diamond^{\leq}_R$-properties. These results are very closely related to our property: (iii) the $\diamond$-property, which guarantees $\mathcal{G} \subset \{\mathbb{Z} \mid \mathbb{Z}\}$ (the integer pairs); in particular, the F_2-property subsumes the $\diamond$-property as a special case. These observations suggest the existence of a more general framework that unifies all of them.

In Section 3, we introduce such a framework called the $\diamond_{\mathcal{A}}$-property for $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. This provides a unified method guaranteeing that, for any set $\mathcal{G}$ closed under options, all positions in $\mathcal{G}$ have values that are integer pairs, integers, number pairs, or numbers. In Section 4, we demonstrate that both our $\diamond$-based framework and the results of [4] emerge as particular instances of this approach. We further present several variants that conform to the same structure. Finally, Section 5 applies our results to Yashima game on bipartite graphs, establishing that every position value is an integer pair.

2. Preliminaries: Combinatorial Game Theory

We assume basic familiarity with combinatorial game theory and recall only the notions and results needed for this paper; for background, see [1], [6]. By convention, the two players are called Left and Right.

2.1 Conway Algebra

We introduce the Conway algebra, which provides a theoret-

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ical basis for combinatorial game theory.

The underlying set $\mathbb{G}$ consists of *positions*. A *position* G ($\in \mathbb{G}$) is inductively defined as an ordered pair $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$, where $G^{\mathcal{L}}$ (resp. $G^{\mathcal{R}}$) is a finite set of positions called the *Left* (resp. *Right*) *options* of G. We denote by 0 the simplest position $\{\,|\,\}$. When no confusion arises, we use $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ to denote these option sets, and we write G^L (resp. G^R) for a particular Left (resp. Right) option of G. When Left (resp. Right) moves from G to G' in one move, we write $G \xrightarrow[L]{} G'$ (resp. $G \xrightarrow[R]{} G'$), or simply write $G \to G'$. We say that two positions G and H are *isomorphic*, written $G \cong H$, if their game trees coincide.

We partition the set of positions into the four *outcome classes* $\mathcal{N}$, $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{R}$, which are inductively defined as follows:

$$\begin{array}{rcl} G \in \mathcal{L} & \Leftrightarrow & \exists G^L, G^L \in \mathcal{L} \cup \mathcal{P} \text{ and } \forall G^R, G^R \in \mathcal{L} \cup \mathcal{N} \\ G \in \mathcal{R} & \Leftrightarrow & \forall G^L, G^L \in \mathcal{R} \cup \mathcal{N} \text{ and } \exists G^R, G^R \in \mathcal{R} \cup \mathcal{P} \\ G \in \mathcal{P} & \Leftrightarrow & \forall G^L, G^L \in \mathcal{R} \cup \mathcal{N} \text{ and } \forall G^R, G^R \in \mathcal{L} \cup \mathcal{N} \\ G \in \mathcal{N} & \Leftrightarrow & \exists G^L, G^L \in \mathcal{L} \cup \mathcal{P} \text{ and } \exists G^R, G^R \in \mathcal{R} \cup \mathcal{P} \end{array}$$

Note that the cases $G^{\mathcal{L}} = \emptyset$ and $G^{\mathcal{R}} = \emptyset$ correspond to the base case of the inductive definition above. We denote by $o(G)$ the outcome class of G. Intuitively, $G \in \mathcal{L}$ (resp. $G \in \mathcal{R}$, $G \in \mathcal{P}$, and $G \in \mathcal{N}$) means that the Left (resp. Right, Previous, Next) player can win.

The *disjunctive sum* $G + H$ and the negative $-G$ are inductively defined as $\left\{G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \;\middle|\; G^{\mathcal{R}} + H, G + H^{\mathcal{R}}\right\}$ and $\left\{-G^{\mathcal{R}} \;\middle|\; -G^{\mathcal{L}}\right\}$. Note that, for $G^{\mathcal{L}} = \{G^{L_1}, \ldots, G^{L_k}\}$, $G^{\mathcal{L}} + H$ denotes $G^{L_1} + H, \ldots, G^{L_k} + H$, and $-G^{\mathcal{L}}$ denotes $-G^{L_1}, \ldots, -G^{L_k}$. Similar conventions apply to other expressions. The equivalence $G = H$ is defined as $\forall X, o(G + X) = o(H + X)$. The algebra is also equipped with a partial order $\leq$ on positions. Its induced equivalence relation $\leq \cap \geq$ coincides with $=$. The order is compatible with $+$, in the sense that $G_1 \leq G_2 \Leftrightarrow G_1 + H \leq G_2 + H$. Moreover, the order satisfies the following properties:

$$\begin{array}{rclcrcl} G > 0 & \Leftrightarrow & G \in \mathcal{L} & \qquad & G < 0 & \Leftrightarrow & G \in \mathcal{R} \\ G = 0 & \Leftrightarrow & G \in \mathcal{P} & & G \parallel 0 & \Leftrightarrow & G \in \mathcal{N} \end{array}$$

where $\parallel$ denotes the incomparability ("fuzzy") relation. We also use the derived relations $\geq$, $\leq$, $\parallel\!\rhd$, and $\lhd\!\parallel$ defined respectively as $> \cup =$, $< \cup =$, $> \cup \parallel$, and $< \cup \parallel$. We can extend these operations and relations to the quotient set $\mathbb{G}/=$: $[G] + [H] = [G + H]$, $-[G] = [-G]$, and $[G] \leq [H] \Leftrightarrow G \leq H$. The Conway algebra is the triple $\langle \mathbb{G}, +, \leq \rangle$, whose quotient algebra $\langle \mathbb{G}/=, +, \leq \rangle$ is known to be an ordered Abelian group. For convenience, the quotient algebra is also sometimes called the Conway algebra.

A Left option G^L is said to be *dominated* if there exists another Left option $G^{L'}$ with $G^L \leq G^{L'}$; dually, a Right option G^R is *dominated* if there exists $G^{R'}$ with $G^R \geq G^{R'}$. It is known that deleting a dominated option preserves equality, that is:

$$\begin{array}{ll} \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\right\} = \left\{G^{\mathcal{L}} \setminus \left\{G^L\right\} \mid G^{\mathcal{R}}\right\} & \text{if } G^L \text{ is dominated} \\ \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\right\} = \left\{G^{\mathcal{L}} \mid G^{\mathcal{R}} \setminus \left\{G^R\right\}\right\} & \text{if } G^R \text{ is dominated} \end{array}$$

Another standard reduction is *bypassing reversible options*, which also preserves equality. By repeatedly removing dominated options and bypassing reversible options, one obtains the *canonical form* of G. Canonical forms are unique up to equality and thus serve as canonical representative of each equality class.

The algebra $\mathbb{G}/=$ contains ordered Abelian subgroups isomorphic to the integers and the dyadic rationals with the usual addition and order. We also denote by $\mathbb{Z}$ (resp. $\mathbb{D}$) the set of the positions in the equivalence classes corresponding to the integers (resp. the dyadic rationals). For a dyadic rational x, we also write x for the canonical form of its correspondence equivalence class. The correspondence is as follows: $0 \cong \{\,|\,\}$, $m \cong \{m-1 \mid\}$, $-m \cong \{\,|\,1-m\}$, and $\frac{n}{2^d} \cong \left\{\frac{n-1}{2^d} \mid \frac{n+1}{2^d}\right\}$, where m and d are positive integers and n is an odd integer. In addition, there are many non-numeric positions, such as $* \cong \{0 \mid 0\}$, $\uparrow \cong \{0 \mid *\}$, and $\uparrow * \cong \{0, * \mid 0\}$.

We now introduce the notions of stops, which give rise to one of the representative features.

Definition 2.1: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. For a position G, we define $\mathrm{L}_{\mathcal{A}}(G)$ and $\mathrm{R}_{\mathcal{A}}(G)$ as follows.

$$\mathrm{L}_{\mathcal{A}}(G) = \begin{cases} G & \text{if } G \in \mathcal{A} \\ \max\{\mathrm{R}_{\mathcal{A}}(G^L) : G^L \in G^{\mathcal{L}}\} & \text{otherwise} \end{cases}$$

$$\mathrm{R}_{\mathcal{A}}(G) = \begin{cases} G & \text{if } G \in \mathcal{A} \\ \min\{\mathrm{L}_{\mathcal{A}}(G^R) : G^R \in G^{\mathcal{R}}\} & \text{otherwise} \end{cases}$$

We call $\mathrm{L}_{\mathbb{D}}(G)$ (resp. $\mathrm{R}_{\mathbb{D}}(G)$) the *left* (resp. *right*) *stops*, and $\mathrm{L}_{\mathbb{Z}}(G)$ (resp. $\mathrm{R}_{\mathbb{Z}}(G)$) the *integer left* (resp. *right*) *stops*.

The stops are known to be invariant: that is, if $G = H$ then $\mathrm{L}_{\mathcal{A}}(G) = \mathrm{L}_{\mathcal{A}}(H)$ and $\mathrm{R}_{\mathcal{A}}(G) = \mathrm{R}_{\mathcal{A}}(H)$ for $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. The stops also has the following proposition.

Proposition 2.2: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, G be a position, and $x \in \mathcal{A}$. Then $\mathrm{L}_{\mathcal{A}}(G) < x \Rightarrow G < x$ and $\mathrm{R}_{\mathcal{A}}(G) > x \Rightarrow G > x$.

Although the case of $\mathcal{A} = \mathbb{Z}$ in the above proposition is not directly shown in [6], it can be shown in a similar way to the case of $\mathcal{A} = \mathbb{D}$.

Proposition 2.3 (Simplicity Theorem): Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$ and G be a position. If there exists $x \in \mathcal{A}$ such that $G^{\mathcal{L}} \lhd\!\parallel x \lhd\!\parallel G^{\mathcal{R}}$, then $G \in \mathcal{A}$. More specifically, G is the simplest position among $\{\, y \in \mathcal{A} : G^{\mathcal{L}} \lhd\!\parallel y \lhd\!\parallel G^{\mathcal{R}} \,\}$.

The notion "simplest" in the simplicity theorem is defined in terms of *birthday*. Intuitively, the birthday of G is the height of the game tree of G.

The following proposition follows from the simplicity theorem.

Proposition 2.4: If $G^{\mathcal{L}} = \emptyset \vee G^{\mathcal{R}} = \emptyset$ then $G \in \mathbb{Z}$.

We introduce two technical propositions.

Proposition 2.5: $G^L \lhd\!\parallel G \lhd\!\parallel G^R$ for any G, G^L, and G^R.

Proposition 2.6: For any positions G, H, J, if $G \lhd\!\parallel H \leq J$ or $G \leq H \lhd\!\parallel J$, then $G \lhd\!\parallel J$.

The proof of Proposition 2.6 is not explicitly written in

[1], [6], so we will provide it here.

Proof: Assume that $G \geq J$. Then we have $H \leq G$ or $J \leq H$ because $H \leq J \leq G$ or $J \leq G \leq H$, respectively. It is a contradiction with $G \lhd\!\!| H$ or $H \lhd\!\!| J$, respectively. □

Proposition 2.7 (Translation Theorem): Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, $x \in \mathcal{A}$, and $G \notin \mathcal{A}$. Then $G + x = \left\{G^{\mathcal{L}} + x \,|\, G^{\mathcal{R}} + x\right\}$.

The proof for the case $\mathcal{A} = \mathbb{Z}$ is not explicitly given in [1], [6], so we include the details here for completeness, adapting the argument in [5].

Proof: The case $\mathcal{A} = \mathbb{D}$ is treated in [1], [6]. It therefore remains to consider the case $\mathcal{A} = \mathbb{Z}$ and $G \in \mathbb{D} \setminus \mathbb{Z}$.

The case $x = 0$ is immediate. We prove the case $x > 0$; the case $x < 0$ is analogous. It suffices to consider the case where x is in canonical form, namely $x = \{x - 1 \,|\, \}$. Since $G + x = \left\{G^{\mathcal{L}} + x, G + (x-1) \,|\, G^{\mathcal{R}} + x\right\}$, it suffices to show that the option $G + (x - 1)$ is dominated, i.e., that there exists G^L with $G^L + x > G + (x - 1)$.

Let l be an integer with $l \leq G < l + 1$. From $G \in \mathbb{D} \setminus \mathbb{Z}$ and Proposition 2.5, we have $l < G \lhd\!\!| G^{\mathcal{R}}$. If every $G^L \lhd\!\!| l$, then $G \in \mathbb{Z}$ by Proposition 2.3, which contradicts $G \in \mathbb{D} \setminus \mathbb{Z}$. Hence there exists G^L such that $G^L \geq l$. Therefore $G^L - G \geq l - G > -1$, and consequently $G^L + x > G + (x - 1)$. □

Finally, we introduce the notions of *integer pairs* $\{\mathbb{Z} \,|\, \mathbb{Z}\}$ and *number pairs* $\{\mathbb{D} \,|\, \mathbb{D}\}$, which play an important role in this paper.

$$\{\mathbb{Z} \,|\, \mathbb{Z}\} = \left\{ \{m \,|\, n\} \;:\; m, n \in \mathbb{Z} \right\}$$
$$\{\mathbb{D} \,|\, \mathbb{D}\} = \left\{ \{x \,|\, y\} \;:\; x, y \in \mathbb{D} \right\}$$

By the simplicity theorem, it is easy to see that $n = \{n - 1 \,|\, n + 1\}$ for any $n \in \mathbb{Z}$. Since $\frac{n}{2^d} \cong \left\{\frac{n-1}{2^d} \,|\, \frac{n+1}{2^d}\right\}$, we obtain the following proposition.

Proposition 2.8: For $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, we have $\mathcal{A} \subset \{\mathcal{A} \,|\, \mathcal{A}\}$.

Proposition 2.9: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$ and $x_1, x_2, y \in \mathcal{A}$ with $\{x_1 \,|\, x_2\} \in \{\mathcal{A} \,|\, \mathcal{A}\} \setminus \mathcal{A}$. Then $\{x_1 \,|\, x_2\} \lhd\!\!| y \Leftrightarrow x_2 \leq y$ and $\{x_1 \,|\, x_2\} \rhd\!\!| y \Leftrightarrow x_1 \geq y$.

Proof: From Proposition 2.7, we have $\{x_1 \,|\, x_2\} - y = \{x_1 - y \,|\, x_2 - y\}$. Hence $\{x_1 \,|\, x_2\} \lhd\!\!| y \Leftrightarrow \{x_1 \,|\, x_2\} - y \lhd\!\!| 0 \Leftrightarrow \{x_1 - y \,|\, x_2 - y\} \lhd\!\!| 0 \Leftrightarrow \{x_1 - y \,|\, x_2 - y\} \in \mathcal{R} \cup \mathcal{N} \Leftrightarrow x_2 - y \in \mathcal{R} \cup \mathcal{P} \Leftrightarrow x_2 - y \leq 0 \Leftrightarrow x_2 \leq y$. The second assertion follows by the same argument. □

2.2 Ruleset

We introduce the concept of a ruleset as a formalization of the combinatorial games studied in this paper (we restrict our attention to short partizan games under normal play).

Definition 2.10: A *ruleset* is a triple $\langle \mathcal{G}, \xrightarrow[L]{}, \xrightarrow[R]{} \rangle$, where $\mathcal{G}$ is the set of *game positions*, and both $\xrightarrow[L]{}$ and $\xrightarrow[R]{}$ are well-founded and finitely branching relations on $\mathcal{G}$.

Here, $G \xrightarrow[L]{} G'$ (resp. $G \xrightarrow[R]{} G'$) represents a move by the Left (resp. Right) player in the given combinatorial game. We adopt the normal play convention, under which the player who makes the last move is the winner.

For a game position G with $\{G^{L_1}, \ldots, G^{L_m}\} = \{G' \mid G \xrightarrow[L]{} G'\}$ and $\{G^{R_1}, \ldots, G^{R_n}\} = \{G' \mid G \xrightarrow[R]{} G'\}$, the game position G is assigned a position $H \cong \left\{H^{L_1}, \ldots, H^{L_m} \mid H^{R_1}, \ldots, H^{R_n}\right\}$ in the Conway algebra, where H^{L_i} and H^{R_j} are the assigned positions of G^{L_i} and G^{R_j}, respectively. According to convention, we identify a game position with the assigned position, and import various notions from the Conway algebra into game positions. We call the canonical form of the assigned position of G the *position value* (the *game value*, simply the *value*) of G.

3. Basic Diamond Properties

In this section, we introduce the concept of the $\diamond_{\mathbb{Z}}$-property and the $\diamond_{\mathbb{D}}$-property, and explain how it relates to the integers $\mathbb{Z}$, the numbers $\mathbb{D}$, the integer pairs $\{\mathbb{Z} \,|\, \mathbb{Z}\}$, and the number pairs $\{\mathbb{D} \,|\, \mathbb{D}\}$.

Lemma 3.1: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, $G_0 \in \{\mathcal{A} \,|\, \mathcal{A}\}$ and $G_1 \in \{\mathcal{A} \,|\, \mathcal{A}\} \setminus \mathcal{A}$. Then the following properties hold for any $x \in \mathcal{A}$:

- $\mathrm{R}_{\mathcal{A}}(G_1) \leq \mathrm{R}_{\mathcal{A}}(G_0)$ and $G_0 \lhd\!\!| x \Rightarrow G_1 \lhd\!\!| x$
- $\mathrm{L}_{\mathcal{A}}(G_1) \geq \mathrm{L}_{\mathcal{A}}(G_0)$ and $G_0 \rhd\!\!| x \Rightarrow G_1 \rhd\!\!| x$

Proof: We will prove only the first claim. The second can be shown in a similar way. Let $G_1 = \{z_1 \,|\, z_2\}$.

In the case of $G_0 = y \in \mathcal{A}$: We have $z_2 \leq y$ and $y < x$ from $\mathrm{R}_{\mathcal{A}}(G_1) \leq \mathrm{R}_{\mathcal{A}}(G_0)$ and $G_0 \lhd\!\!| x$, respectively. It follows from Proposition 2.9 that $G_1 = \{z_1 \,|\, z_2\} \lhd\!\!| y < x$. Hence, by Proposition 2.6, we obtain $G_1 \lhd\!\!| x$.

In the case of $G_0 = \{y_1 \,|\, y_2\} \in \{\mathcal{A} \,|\, \mathcal{A}\} \setminus \mathcal{A}$: We have $z_2 \leq y_2$ and $y_2 \leq x$ from $\mathrm{R}_{\mathcal{A}}(G_1) \leq \mathrm{R}_{\mathcal{A}}(G_0)$ and $G_0 \lhd\!\!| x$, respectively. It follows from Proposition 2.9 that $G_1 = \{z_1 \,|\, z_2\} \lhd\!\!| y_2 \leq x$. Hence, by Proposition 2.6, we obtain $G_1 \lhd\!\!| x$. □

The property given in this lemma does not hold in general. For instance, it fails if we allow $G_1 \in \mathcal{A}$. A concrete counter example is $G_0 = *$ and $G_1 = 0$ because $\mathrm{R}_{\mathcal{A}}(G_1) = 0 = \mathrm{R}_{\mathcal{A}}(G_0)$ and $G_0 \lhd\!\!| 0$ but $G_1 = 0 \not\lhd\!\!| 0$.

Within the framework of this paper, positions that are fuzzy to each other — such as 0 and $*$ — do not appear simultaneously among the Left (or Right) options, as in this example. This restriction prevents canonical forms from becoming quite complicated.

Definition 3.2: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. For a position G, the sets of *guide left options* and *guide right options*, denoted by $\mathrm{gd}^L_{\mathcal{A}}(G)$ and $\mathrm{gd}^R_{\mathcal{A}}(G)$, are defined as follows:

$$\mathrm{gd}^L_{\mathcal{A}}(G) = \begin{cases} \emptyset & \text{if } G \in \mathcal{A} \\ \{G^L : G^L = \mathrm{L}_{\mathcal{A}}(G)\} & \text{if } G \notin \mathcal{A} \text{ and } \mathrm{L}_{\mathcal{A}}(G) \in G^{\mathcal{L}} \\ \{G^L : \mathrm{R}_{\mathcal{A}}(G^L) = \mathrm{L}_{\mathcal{A}}(G)\} & \text{otherwise} \end{cases}$$

$$\mathrm{gd}^R_{\mathcal{A}}(G) = \begin{cases} \emptyset & \text{if } G \in \mathcal{A} \\ \{G^R : G^R = \mathrm{R}_{\mathcal{A}}(G)\} & \text{if } G \notin \mathcal{A} \text{ and } \mathrm{R}_{\mathcal{A}}(G) \in G^{\mathcal{R}} \\ \{G^R : \mathrm{L}_{\mathcal{A}}(G^R) = \mathrm{R}_{\mathcal{A}}(G)\} & \text{otherwise} \end{cases}$$

Definition 3.3: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. We say that a position G has the *diamond property for $\mathcal{A}$ or simply $\diamond_{\mathcal{A}}$-property* if there exist $G^L \in \mathrm{gd}^L_{\mathcal{A}}(G)$, $G^R \in \mathrm{gd}^R_{\mathcal{A}}(G)$, and $x \in \mathcal{A}$ such that $G^L \lhd\!\lhd x \lhd\!\lhd G^R$.

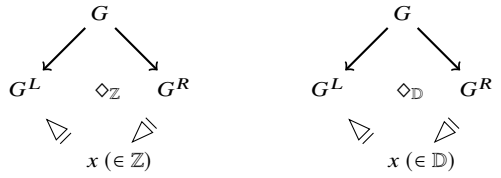


Fig. 3 $\diamond_{\mathbb{Z}}$-property and $\diamond_{\mathbb{D}}$-property

Lemma 3.4: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. If a position G has the $\diamond_{\mathcal{A}}$-property and $G^{\mathcal{L}}, G^{\mathcal{R}} \subset \{\mathcal{A} \mid \mathcal{A}\}$, then $G \in \mathcal{A}$.

Proof: Assume that $G \notin \mathcal{A}$. From the $\diamond_{\mathcal{A}}$-property, there exist a guide left option G^L, a guide right option G^R, and $x \in \mathcal{A}$ such that $G^L \lhd\!\lhd x \lhd\!\lhd G^R$.

We first show $G^{L'} \lhd\!\lhd x$ for any $G^{L'} \in G^{\mathcal{L}} \setminus \{G^L\}$. Since G^L is a guide left option, we have $\mathrm{R}_{\mathcal{A}}(G^{L'}) \leq \mathrm{R}_{\mathcal{A}}(G^L)$.

- In the case of $G^{L'} \notin \mathcal{A}$: Then $G^{L'} \lhd\!\lhd x$ follows from Lemma 3.1.
- In the case of $G^{L'} \in \mathcal{A}$ and $\mathrm{R}_{\mathcal{A}}(G^{L'}) < \mathrm{R}_{\mathcal{A}}(G^L)$: Then $G^{L'} < G^L$ follows from Proposition 2.2, and hence $G^{L'} \lhd\!\lhd x$.
- In the case of $G^{L'} \in \mathcal{A}$ and $\mathrm{R}_{\mathcal{A}}(G^{L'}) = \mathrm{R}_{\mathcal{A}}(G^L)$: Since G^L is a guide left option, we have $G^L \in \mathcal{A}$. Hence $G^{L'} = G^L$, and $G^{L'} \lhd\!\lhd x$.

Thus, we have $G^{\mathcal{L}} \lhd\!\lhd x$. Similarly, we can also show $x \lhd\!\lhd G^{\mathcal{R}}$. Therefore, Proposition 2.3 derives $G \in \mathcal{A}$, which is a contradiction. □

We are now ready to state the theorem that forms the foundation of all the results in this paper.

Theorem 3.5: Suppose that a set $\mathcal{G}$ of positions is closed under options. For each $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, if $\mathcal{G}$ is partitioned into $\mathcal{D}$ and $\mathcal{S}$ such that:

(a) For any $G \in \mathcal{D}$, either $G \in \mathcal{A}$ is known or G has the $\diamond_{\mathcal{A}}$-property, and
(b) $G \in \mathcal{S} \Rightarrow G^{\mathcal{L}}, G^{\mathcal{R}} \subset \mathcal{D}$,

then the following properties hold:

(1) $G \in \mathcal{G} \Rightarrow G \in \{\mathcal{A} \mid \mathcal{A}\}$
(2) $G \in \mathcal{D} \Rightarrow G \in \mathcal{A}$

Proof: We proceed by induction on G ordered by the moving relation $\rightarrow$.

- In the case of $G \in \mathcal{A}$: It is trivial because of $\mathcal{A} \subset \{\mathcal{A} \mid \mathcal{A}\}$.
- In the case of $G \notin \mathcal{A}$ and $G \in \mathcal{S}$: From the assumption (b) and the induction hypothesis (2), we have $G^{\mathcal{L}}, G^{\mathcal{R}} \subset \mathcal{A}$. In either case $\mathcal{A} = \mathbb{Z}$ or $\mathcal{A} = \mathbb{D}$, any two positions in $\mathcal{A}$ can be compared. Hence, in $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$, we can remove all dominated options and keep at most one on each side. Therefore, $G \in \{\mathcal{A} \mid \mathcal{A}\}$.
- In the case of $G \notin \mathcal{A}$ and $G \in \mathcal{D}$: From the induction hypothesis (1), we have $G^{\mathcal{L}}, G^{\mathcal{R}} \subset \{\mathcal{A} \mid \mathcal{A}\}$. Since G has the $\diamond_{\mathcal{A}}$-property, Lemma 3.4 implies that $G \in \mathcal{A}$, which is a contradiction. Therefore, this case does not occur. □

The phrase "$G \in \mathcal{A}$ is known" in condition (a) of this theorem allows the use of arbitrary methods for determining that a position is in $\mathcal{A}$ (e.g., Proposition 2.4).

By considering the special case $\mathcal{D} = \mathcal{G}$ and $\mathcal{S} = \emptyset$, we obtain the following corollary.

Corollary 3.6: Suppose that a set $\mathcal{G}$ of positions is closed under options and $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. If for any $G \in \mathcal{G}$, either $G \in \mathcal{A}$ is known or G has the $\diamond_{\mathcal{A}}$-property, then $G \in \mathcal{A}$.

4. Various Diamond Properties

In this section, we introduce various diamond properties. We begin with those that are instances of both the $\diamond_{\mathbb{Z}}$-property and the $\diamond_{\mathbb{D}}$-property.

Definition 4.1: We say that a position G has:

- the *$\diamond$-property* for $\mathcal{A}$ if there exist $G^L \in \mathrm{gd}^L_{\mathcal{A}}(G)$, $G^R \in \mathrm{gd}^R_{\mathcal{A}}(G)$, and H such that $G^L \xrightarrow[R]{} H \xleftarrow[L]{} G^R$,
- the *$\diamond^{\leq}$-property* for $\mathcal{A}$ if there exist $G^L \in \mathrm{gd}^L_{\mathcal{A}}(G)$, $G^R \in \mathrm{gd}^R_{\mathcal{A}}(G)$, G^{LR}, and G^{RL} such that $G^{LR} \leq G^{RL}$,
- the *$\diamond^{\lhd\!\lhd}_L$-property* for $\mathcal{A}$ if there exist $G^L \in \mathrm{gd}^L_{\mathcal{A}}(G)$, $G^R \in \mathrm{gd}^R_{\mathcal{A}}(G)$, and G^{LR} such that $G^{LR} \lhd\!\lhd G^R$, and
- the *$\diamond^{\lhd\!\lhd}_R$-property* for $\mathcal{A}$ if there exist $G^L \in \mathrm{gd}^L_{\mathcal{A}}(G)$, $G^R \in \mathrm{gd}^R_{\mathcal{A}}(G)$, and G^{RL} such that $G^L \lhd\!\lhd G^{RL}$.

We display these diamond properties in Fig.4.
In both cases $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$, by adding the assumption

(c) $G \in \mathcal{D} \Rightarrow G^{\mathcal{LR}}, G^{\mathcal{RL}} \subset \mathcal{D}$

to Theorem 3.5, we can assume that $G^{\mathcal{LR}}, G^{\mathcal{RL}} \subset \mathcal{A}$ as an induction hypothesis in the proof. Under the hypothesis, every diamond property in Definition 4.1 guarantees the $\diamond_{\mathcal{A}}$-property for each $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$.

$\diamond$-property: We have $H \in \mathcal{A}$ and $G^L \lhd\!\lhd H \lhd\!\lhd G^R$ because of $G^L \xrightarrow[R]{} H \xleftarrow[L]{} G^R$.

$\diamond^{\leq}$-property: We have $G^{LR}, G^{RL} \in \mathcal{A}$ and $G^L \lhd\!\lhd G^{LR}, G^{RL} \lhd\!\lhd G^R$ because of $G^L \xrightarrow[R]{} G^{LR} \leq G^{RL} \xleftarrow[L]{} G^R$.

$\diamond^{\lhd\!\lhd}_L$-property: We have $G^{LR} \in \mathcal{A}$ and $G^L \lhd\!\lhd G^{LR} \lhd\!\lhd G^R$

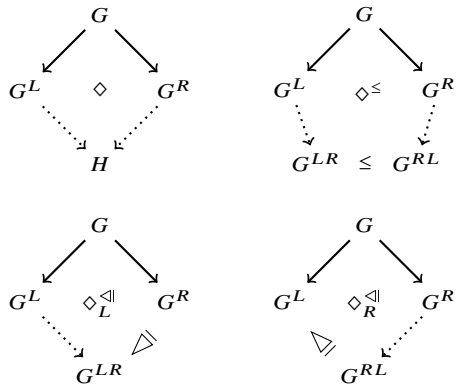


Fig. 4 Diamond properties in Definition 4.1

because of $G^L \xrightarrow[R]{} G^{LR} \lhd\!\!| \, G^R$.

$\diamond_R^{\lhd\!\!|}$**-property:** We have $G^{RL} \in \mathcal{A}$ and $G^L \lhd\!\!| \, G^{RL} \lhd\!\!| \, G^R$ because of $G^L \lhd\!\!| \, G^{RL} \xleftarrow[L]{} G^R$.

Hence we obtain the following corollary of Theorem 3.5.

Corollary 4.2: Let $\mathcal{A} \in \{\mathbb{Z}, \mathbb{D}\}$. Suppose that a set $\mathcal{G}$ of positions is closed under options. If $\mathcal{G}$ is partitioned into $\mathcal{D}$ and $\mathcal{S}$ such that:

(a) For any $G \in \mathcal{D}$, either $G \in \mathcal{A}$ is known or G has one of diamond properties for $\mathcal{A}$ in Definition 4.1,
(b) $G \in \mathcal{S} \Rightarrow G^{\mathcal{L}}, G^{\mathcal{R}} \subset \mathcal{D}$, and
(c) $G \in \mathcal{D} \Rightarrow G^{\mathcal{L}\mathcal{R}}, G^{\mathcal{R}\mathcal{L}} \subset \mathcal{D}$,

then the following properties hold:

(1) $G \in \mathcal{G} \Rightarrow G \in \{\mathcal{A} \,|\, \mathcal{A}\}$
(2) $G \in \mathcal{D} \Rightarrow G \in \mathcal{A}$

We also obtain the following corollary as the special case $\mathcal{D} = \mathcal{G}$ and $\mathcal{S} = \emptyset$. We will treat the case $\mathcal{A} = \mathbb{D}$ separately and present a stronger result later (Corollary 4.5).

Corollary 4.3: Suppose that a set $\mathcal{G}$ of positions is closed under options. If for any $G \in \mathcal{G}$, either $G \in \mathbb{Z}$ is known or G has one of diamond properties for $\mathbb{Z}$ in Definition 4.1, then $\mathcal{G} \subset \mathbb{Z}$.

Next, we introduce diamond properties that are instances of the $\diamond_{\mathbb{D}}$-property, although they are not instances of the $\diamond_{\mathbb{Z}}$-property.

Definition 4.4: We say that a position G has:

- the $\diamond_L^{\leq}$*-property* if there exist $G^L \in \mathrm{gd}_{\mathbb{D}}^L(G)$, $G^R \in \mathrm{gd}_{\mathbb{D}}^R(G)$, and G^{LR} such that $G^{LR} \leq G^R$,
- the $\diamond_R^{\leq}$*-property* if there exist $G^L \in \mathrm{gd}_{\mathbb{D}}^L(G)$, $G^R \in \mathrm{gd}_{\mathbb{D}}^R(G)$, and G^{RL} such that $G^L \leq G^{RL}$, and
- the $\triangle$*-property* if there exist $G^L \in \mathrm{gd}_{\mathbb{D}}^L(G)$, $G^R \in \mathrm{gd}_{\mathbb{D}}^R(G)$, such that $G^L \lhd\!\!| \, G^R$.

We display these diamond properties in Fig.5.

These three properties rely on the density of numbers. They imply that $G^L \lhd\!\!| \, G^R$. Under the assumption that $G^L, G^R \in \mathbb{D}$, we have $G^L < G^R$, and hence, by the density of numbers, there exists $x \in \mathbb{D}$ such that $G^L < x < G^R$. On the other hand, these properties do not hold for integers, because the density property failed for integers. Moreover, the requirement that $G^L, G^R \in \mathbb{D}$ even when $G \in \mathcal{D}$ implies that all positions in $\mathcal{G}$ must lie in $\mathcal{D}$. Hence, although we do not obtain the corollary (similar to Corollary 4.2) of Theorem 3.5 with respect to the diamond properties in Definition 4.4, we do obtain the corollary stated in Corollary 3.6.

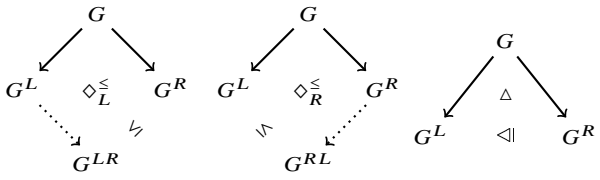


Fig. 5 Diamond properties in Definition 4.4

Corollary 4.5: Suppose that a set $\mathcal{G}$ of positions is closed under options. If for any $G \in \mathcal{G}$, either $G \in \mathbb{D}$ is known or G has one of diamond properties for $\mathbb{D}$ in Definition 4.1 and Definition 4.4, then $\mathcal{G} \subset \mathbb{D}$.

In the definitions of the various diamond properties, we require the existence of guide options $G^L \in \mathrm{gd}_{\mathcal{A}}^L(G)$ and $G^R \in \mathrm{gd}_{\mathcal{A}}^R(G)$. It follows from Proposition 2.4 that $G \in \mathbb{Z} \subset \mathbb{D}$ whenever $G^{\mathcal{L}} = \emptyset \vee G^{\mathcal{R}} = \emptyset$. Hence, even if we modify the requirement so that "all options" satisfy the remaining conditions, all the theorems and corollaries concerning the diamond properties will still hold. Under this modification, Corollary 4.3, restricted to the $\diamond^{\leq}$-property, and Corollary 4.5, restricted to the $\diamond^{\leq}$, $\diamond_L^{\leq}$, and $\diamond_R^{\leq}$-properties, recover the results of [4]. We recall that the F_2-property in [4] corresponds to the $\diamond^{\leq}$-property, and the F_1-property in [4] corresponds to the disjunction of the $\diamond_L^{\leq}$ and $\diamond_R^{\leq}$-properties.

In any ruleset, the set $\mathcal{G}$ of game positions is closed under options, and hence our results apply. As noted above, if either the set of Left or Right options is empty, then the position value is an integer. For convenience, we adopt the convention that a ruleset is regarded as satisfying the various diamond properties even in this case.

5. Case Study: Yashima Game on Bipartite Graphs

Yashima game, proposed by Arisawa [2], is played on a given undirected graph. In this section, we study Yashima game on bipartite graphs as a combinatorial game that satisfies the diamond property ($\diamond$-property). Using our results, we show that every game position in Yashima game on bipartite graphs takes a value of the very simple form $\{m \,|\, n\} \in \{\mathbb{Z} \,|\, \mathbb{Z}\}$.

To begin, recall that a graph is bipartite if and only if it is 2-colorable. We classify game positions of Yashima game on bipartite graphs as follows.

Definition 5.1: Let G be a game position of Yashima game on bipartite graphs.

- We call G a *different-color position* if there exists a

2-coloring in which the vertices of the Left and Right tokens receive different colors.
- Otherwise, G is called a *same-color position*.

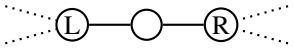
Let $\mathcal{D}$ (resp. $\mathcal{S}$) denote the set of all different-color (resp. same-color) positions.

With this partition, we obtain the following theorem.

Theorem 5.2: YASHIMA game on bipartite graphs satisfies the $\diamond$-property.

Proof: Let $G \in \mathcal{D}$ and suppose $G^L \underset{L}{\longleftarrow} G \underset{R}{\longrightarrow} G^R$. We claim that the Left move used in $G \underset{L}{\longrightarrow} G^L$ can also be played from G^R, yielding $G^R \underset{L}{\longrightarrow} G^{RL}$; symmetrically, the Right move used in $G \underset{R}{\longrightarrow} G^R$ can also be played from G^L, yielding $G^L \underset{R}{\longrightarrow} G^{LR}$. Once this is verified, it follows that $G^{RL} \cong G^{LR}$, and this common game position belongs to $\mathcal{D}$. Hence G has the $\diamond$-property.

In YASHIMA game, a planned move can be blocked by the opponent only in the following situation:



That is, the opponent moves onto exactly the vertex (the "central" vertex) to which the player intended to move. Such blocking is possible only in a same-color position. Since $G \in \mathcal{D}$, no such blocking move is possible for either player. □

For the different-color positions $\mathcal{D}$ and the same-color positions $\mathcal{S}$, conditions (b) and (c) of Corollary 4.2 follow directly from the rules of YASHIMA game. Therefore, we obtain the following corollary.

Corollary 5.3: Let G be a game position of YASHIMA game on bipartite graphs. Then the following properties hold:

(1) $G \in \{\mathbb{Z} \mid \mathbb{Z}\}$.
(2) If G is a different-color position, then $G \in \mathbb{Z}$.

A game with rules similar to YASHIMA game is known as TRON [3]. In TRON, after moving a token, one deletes the vertex just left, rather than deleting an edge as in YASHIMA game. Theorem 5.2 and Corollary 5.3 likewise hold for TRON on bipartite graphs.

Finally, Fig.6 illustrates a concrete example of a game position of YASHIMA game on a bipartite graph. The size of the fully expanded game tree for this game position reaches 104,241, so the number of follower game positions is enormous. For the analysis, one must also compute the canonical form of this game position. By using memoization, we can reduce the search space to 1,307 (the exact number depends on the implementation). Nevertheless, the computation typically require computer assistance. In contrast, the value $\{0 \mid -3\}$ of this game position is remarkably simple. Note that the value of the same game position, when regarded as a TRON game, is $\{1 \mid -1\}$, which is also simple.

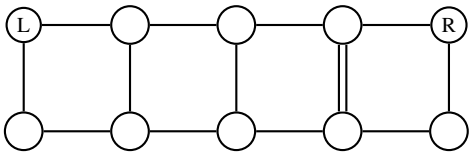


Fig. 6 YASHIMA game (position value: $\{0 \mid -3\}$)

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