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On The Analytical Solution of The Schrödinger Equation in Ellipsoidal Coordinates with Constant Potential

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This paper solves the Schrödinger equation in confocal ellipsoidal coordinates under a constant potential. An auxiliary approach is introduced to separate the equation, which allows for the analytical derivation of energy levels using a series solution method. The results show that the four types of ellipsoidal harmonics lead to the same energy expressions. The energy levels are analyzed in terms of their dependence on the geometrical properties of the coordinates and their degeneracy patterns. This study highlights the role of ellipsoidal geometry in defining energy spectra and demonstrates its usefulness for describing quantum systems with non-spherical symmetry. The findings are relevant to diatomic molecules, quantum chemistry, and nanoscale physics, particularly nanorods. The method provides a foundation for future work on more complex potentials, which could extend the range of problems addressed using this approach.

Keywords: Schrödinger equation, ellipsoidal coordinates, ellipsoidal harmonics, diatomic molecules

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I. INTRODUCTION

Research has shown that ellipsoidal coordinates are very useful for studying two-center problems in quantum mechanics, especially for systems with non-spherical shapes. Budziński's work on integrals over Slater-type orbitals in these coordinates highlights how they help accurately describe electron interactions in molecules, which is essential for modeling correlation effects [1]. Bates et al. used ellipsoidal coordinates to study the hydrogen molecular ion, showing that these coordinates are important for calculating energy levels and potentials in diatomic molecules with non-spherical effects [2]. Mekelleche and Baba-Ahmed further emphasized that ellipsoidal coordinates match the symmetry of two-center problems well, making it easier to compute overlap and nuclear attraction integrals for molecular systems [3].

Altogether, these studies illustrate that ellipsoidal coordinates provide a useful framework for solving the Schrödinger equation both analytically and numerically in cases where traditional spherical coordinates may not work as effectively. Recently, a notable application of the ellipsoidal coordinates appeared in nano-science, where they emerge in the context of quantum-scale nanoparticles, where in some cases, these nano-particles have an ellipsoidal shape [4].

The numerical solutions are often characterized by the well-known Lamé wave functions, which arise when solving the Schrödinger equation in ellipsoidal coordinates. T. Levitina and E. Brändas [5–7] have investigated quantum scattering of a particle affected by a potential that vanishes sufficiently fast, such as the Lennard-Jones potential. The importance of ellipsoidal coordinates become immediately significant, as non-spherical effects cannot be neglected. Taking these factors into account, the scattering data are expanded using perturbed Lamé wave functions to obtain the solutions of Schrödinger equation.

T. Kereselidze et al. [8] studied the energy spectra of a particle confined in a finite, strongly prolate ellipsoidal-shaped potential well. That is to say, the distance between foci is very large. It is demonstrated that quasiangular wave functions inside and outside the potential well coincide on the entire surface of a strongly prolate ellipsoid when the separation parameters are chosen appropriately. By equating the quasiradial wave function and its derivative on the ellipsoid's surface, the transcendental equation for the energy levels can be obtained. The most significant finding in these investigations is the pronounced interdependence between the energy spectrum of the particle and the geometrical shape of the potential.

Another research domain in which ellipsoidal coordinates play a critical role is the pursuit of a specific coordinate system to effectively handle and accurately calculate the ground state energy of diatomic molecules, such as H_2^+ [9–11]. This approach also addresses numerous challenges arising from the three-body Coulomb potential where three charged particles interact with each other.

Tolstikhin et al. [12], have introduced the hyperspherical elliptic coordinates for the study of a class of three-body systems. These new coordinates, which are well adapted to the singularities of the three-body potential, can be regarded as a generalization of the familiar spheroidal coordinates for diatomic molecules. However, as they are defined on the hypersphere in six-dimensional space, the long-standing difficulties of the Born-Oppenheimer ansatz, such as the laborious evaluation of non-adiabatic coupling terms, a separate treatment of the mass-polarization effect, and the necessity for the electronic translational factor, have been formally resolved by these new coordinates.

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This paper is dedicated to exploring the energy levels of bound states for a particle in confocal ellipsoidal coordinates under the influence of a constant potential. The findings may prove to be useful in the fields of diatomic molecules, quantum chemistry, and quantum-scale nanorods [13–15]. It is and is organized as follows: In Section II, the confocal ellipsoidal coordinate system, which is utilized in the analysis, will be discussed in detail. In Section III, the separability of the Schrödinger equation will be analyzed and explored. In Section IV, the Schrödinger equation will be solved using the series solution method, leading to the derivation of four recursion relations. In Section V, the formula of energy levels will be derived. Finally, in section VI, the results of the study will be presented and discussed.

II. ELLIPSOIDAL COORDINATES

Confocal ellipsoidal coordinates, as described in [16], are defined by the intersection of three types of non-degenerate second-degree surfaces: ellipsoids, one-sheeted hyperboloids, and two-sheeted hyperboloids. Following the original Lamé notation, (λ, μ, ν) are parameterizing variables for each one of these surfaces. That is to say, the family of ellipsoids is represented by the equation

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - h_3^2} + \frac{z^2}{\lambda^2 - h_2^2} = 1, \quad \lambda^2 \in \left(h_2^2, +\infty\right), \tag{1}$$

the family of one-sheeted hyperboloids is given by

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - h_3^2} + \frac{z^2}{\mu^2 - h_2^2} = 1, \quad \mu^2 \in \left(h_3^2, h_2^2\right), \tag{2}$$

and finally, the family of two-sheeted hyperboloids is given by

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - h_3^2} + \frac{z^2}{\nu^2 - h_2^2} = 1, \quad \nu^2 \in \left(0, h_3^2\right). \tag{3}$$

Where the domains of the parameters in the first octant satisfy

$$0 \leqslant \nu^2 \leqslant h_3^2 \leqslant \mu^2 \leqslant h_2^2 \leqslant \lambda^2 < +\infty, \tag{4}$$

and h_2 and h_3 are two of the three semi-focal distances of the surfaces, which remain fixed for a specific coordinate.

One can define an orthogonal coordinate system with the constraint $xyz \neq 0$, known as the confocal ellipsoidal coordinate system by solving equations (1-3) for x^2, y^2, and z^2. Where any point within the first octant in the cartesian space, can be uniquely identified by the intersection of an ellipsoid, a single-sheet hyperboloid , and a two-sheet hyperboloid as mentioned above, see FIG. (1). Then the transformation is given by :

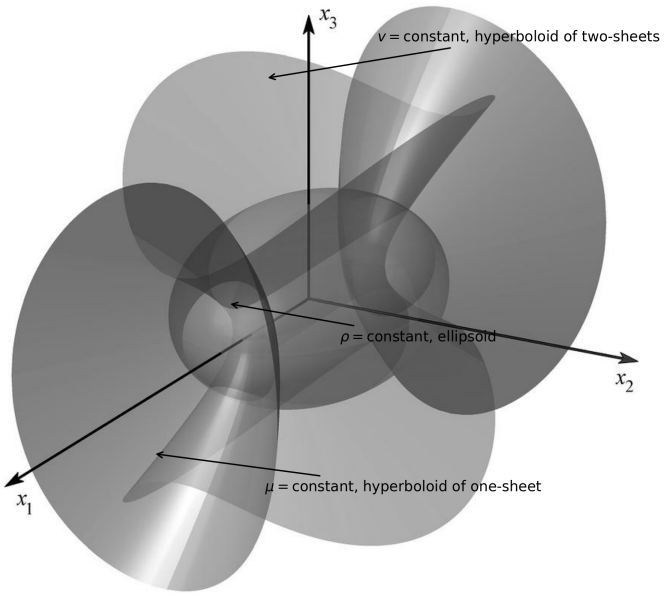


FIG. 1. The intersection of an ellipsoid, a single-sheet hyperboloid, and a two-sheet hyperboloid.

$$\begin{aligned} x^2 &= \frac{\lambda^2\mu^2\nu^2}{h_2^2 h_3^2} \\ y^2 &= \frac{\left(\lambda^2 - h_3^2\right)\left(\mu^2 - h_3^2\right)\left(h_3^2 - \nu^2\right)}{(h_2^2 - h_3^2)h_3^2} \\ z^2 &= \frac{\left(\lambda^2 - h_2^2\right)\left(h_2^2 - \mu^2\right)\left(h_2^2 - \nu^2\right)}{(h_2^2 - h_3^2)h_2^2}. \end{aligned} \tag{5}$$

Due to the symmetry of the confocal ellipsoidal coordinate system, the geometric structure of the system in each of the eight Cartesian octants is the same. By taking into account the positive and negative signs of the square roots in the expressions for (x, y, z) given in (5), the points in the remaining seven octants can be obtained. This symmetry provides a convenient way to describe the full three-dimensional space using the confocal ellipsoidal coordinates.

The Laplace operator, can be defined in various manners. One such approach involves utilizing the so-called thermometric parameters (ξ, η, ζ), which are historically attributed to Lamé's work on solving the heat equation in ellipsoidal coordinates [16]. The operator takes the form

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$$\begin{aligned}\Delta =&\frac{1}{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}\left[(\lambda^2-h_3^2)(\lambda^2-h_2^2)\frac{\partial^2}{\partial\lambda^2}+\lambda\left(2\lambda^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\lambda}\right]\\&+\frac{1}{(\mu^2-\lambda^2)(\mu^2-\nu^2)}\left[(\mu^2-h_3^2)(\mu^2-h_2^2)\frac{\partial^2}{\partial\mu^2}+\mu\left(2\mu^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\mu}\right]\\&+\frac{1}{(\nu^2-\lambda^2)(\nu^2-\mu^2)}\left[(\nu^2-h_3^2)(\nu^2-h_2^2)\frac{\partial^2}{\partial\nu^2}+\nu\left(2\nu^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\nu}\right],\end{aligned}\tag{6}$$

where the operators of the thermometric parameters (ξ,η,ζ) are related to (λ,μ,ν) by

$$\begin{aligned}\frac{\partial^2}{\partial\xi^2}&=(\lambda^2-h_3^2)(\lambda^2-h_2^2)\frac{\partial^2}{\partial\lambda^2}+\lambda\left(2\lambda^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\lambda}\\-\frac{\partial^2}{\partial\eta^2}&=(\mu^2-h_3^2)(\mu^2-h_2^2)\frac{\partial^2}{\partial\mu^2}+\mu\left(2\mu^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\mu}\\\frac{\partial^2}{\partial\zeta^2}&=(\nu^2-h_3^2)(\nu^2-h_2^2)\frac{\partial^2}{\partial\nu^2}+\nu\left(2\nu^2-h_3^2-h_2^2\right)\frac{\partial}{\partial\nu}.\end{aligned}\tag{7}$$

Ellipsoidal coordinates belong to one of the most general coordinates in three-dimensions where the Laplace equation is separable [17]. This fact makes this family of coordinates very important and applicable to numerous physical systems. However, we will solve the Schrödinger equation with constant potential; the Schrödinger equation then reads as follows:

$$\begin{aligned}&(\mu^2-\nu^2)\frac{\partial^2\Phi}{\partial\xi^2}+(\lambda^2-\nu^2)\frac{\partial^2\Phi}{\partial\eta^2}+(\lambda^2-\mu^2)\frac{\partial^2\Phi}{\partial\zeta^2}\\&+(\mu^2-\nu^2)(\lambda^2-\nu^2)(\lambda^2-\nu^2)\chi^2\Phi=0,\end{aligned}\tag{8}$$

where, for bound states, $\chi^2=\frac{2m}{\hbar^2}(V-E)$ and V is the constant potential where $E<V$.

III. SEPARATION OF VARIABLES

The Schrödinger equation (8) is not separable by itself, to overcome this difficulty, we define an auxiliary partial differential equation introduced as:

$$\begin{aligned}&(\mu^2-\nu^2)\frac{\partial^2\Psi}{\partial\xi^2}+(\lambda^2-\nu^2)\frac{\partial^2\Psi}{\partial\eta^2}+(\lambda^2-\mu^2)\frac{\partial^2\Psi}{\partial\zeta^2}\\&+(\mu^2-\nu^2)(\lambda^2-\nu^2)(\lambda^2-\nu^2)\chi'^2\Psi=0.\end{aligned}\tag{9}$$

Where the new χ'^2 is measured with respect to the squares of Lamé coordinates, or:

$$\chi'^2=\frac{\chi^2}{\lambda^2\mu^2\nu^2}.\tag{10}$$

The new energy E' and potential V', which are measured with respect to the squared values of the Lamé parameters (λ,μ,ν), are defined in the same manner as in equation (10). Solving the auxiliary equation (9) allows us to determine the arbitrary E. As a consequence, the values of E' can be found from (10), which is the goal of this analysis. It is important to mention here that the operators of wave functions Φ and Ψ do not commute; they do not share the same eigenstates.

After some straightforward algebra, we arrive at the following form:

$$\begin{aligned}&\left[(\mu^2-\nu^2)\frac{\partial^2}{\partial\xi^2}+(\mu^4\nu^2-\nu^4\mu^2)\frac{\chi^2}{\lambda^2\mu^2\nu^2}\right]\Psi\\&+\left[(\lambda^2-\nu^2)\frac{\partial^2}{\partial\eta^2}-(\lambda^4\nu^2-\nu^4\lambda^2)\frac{\chi^2}{\lambda^2\mu^2\nu^2}\right]\Psi\\&+\left[(\lambda^2-\mu^2)\frac{\partial^2}{\partial\zeta^2}+(\lambda^4\mu^2-\mu^4\lambda^2)\frac{\chi^2}{\lambda^2\mu^2\nu^2}\right]\Psi=0.\end{aligned}\tag{11}$$

At this stage, the partial differential equation (11) is separable, meaning it is linear in both the Lamé coordinates (λ,μ,ν) and their corresponding thermometric parameters (ξ,η,ζ). This equation can be solved using the method of separation of variables technique if we introduce the following identities

$$(\mu^2-\nu^2)+(\nu^2-\lambda^2)+(\lambda^2-\mu^2)=0\tag{12}$$

$$\lambda^2(\mu^2-\nu^2)+\mu^2(\nu^2-\lambda^2)+\nu^2(\lambda^2-\mu^2)=0,\tag{13}$$

multiplying equation (12) by $B\Psi$ and equation (13) by $A\Psi$, and then substituting the results into equation (11). Next, we can express $\Psi(\lambda,\mu,\nu)$ as a product of three independent functions:

$$\Psi(\lambda,\mu,\nu)=L(\lambda)M(\mu)N(\nu).$$

As a result, the separable equation is:

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$$\begin{aligned}
&\frac{(\mu^2-\nu^2)}{L(\lambda)}\left[(\lambda^2-h_3^2)(\lambda^2-h_2^2)\tilde{D}^2L+\lambda(2\lambda^2-h_3^2-h_2^2)\tilde{D}L+(A\lambda^2+B+\frac{\chi^2}{\lambda^2})L\right]\\
&-\frac{(\lambda^2-\nu^2)}{M(\mu)}\left[(\mu^2-h_3^2)(\mu^2-h_2^2)\tilde{D}^2M+\mu(2\mu^2-h_3^2-h_2^2)\tilde{D}M+(A\mu^2+B+\frac{\chi^2}{\mu^2})M\right]\\
&+\frac{(\lambda^2-\mu^2)}{N(\nu)}\left[(\nu^2-h_3^2)(\nu^2-h_2^2)\tilde{D}^2N+\nu(2\nu^2-h_3^2-h_2^2)\tilde{D}N+(A\nu^2+B+\frac{\chi^2}{\nu^2})N\right]=0.
\end{aligned} \tag{14}$$

The operators $\tilde{D}$ and $\tilde{D^2}$ in equation (14) represent the first and second ordinary derivatives with respect to the independent variables λ, μ, ν, respectively. In order for equation (14) to be satisfied, each of the three brackets must separately equal zero. It should be noted that these brackets are identical and only differ in their domains, which are specified in (4). The resulting ordinary differential equations take the form:

$$\begin{aligned}
&(\rho^2-h_2^2)(\rho^2-h_3^2)H''(\rho)+\rho(2\rho^2-h_2^2-h_3^2)H'(\rho)\\
&+\left(A\rho^2+B+\frac{\chi^2}{\rho^2}\right)H(\rho)=0.
\end{aligned} \tag{15}$$

Where ρ stands for one of the variables λ, μ, and ν, and $H(\rho)$ represents their functions. The constants $A = -n(n+1)$ and $B = (h_2^2+h_3^2)p$ for arbitrary p, are the separation constants.

IV. SERIES SOLUTION

The confocal ellipsoidal coordinate system, unlike the spherical coordinate system, has four classes of harmonics [18]. These harmonics are the functions that satisfy the differential equation (15) and take the following forms:

$$\begin{aligned}
K &= H(\rho)\\
L &= \sqrt{|\,\rho^2-h_3^2\,|}H(\rho)\\
M &= \sqrt{|\,\rho^2-h_2^2\,|}H(\rho)\\
N &= \sqrt{|\,\rho^2-h_3^2\,|}\sqrt{|\,\rho^2-h_2^2\,|}H(\rho).
\end{aligned} \tag{16}$$

In the above expressions, h_3^2 and h_2^2 are interchangeable, and solving for one of them suffices without loss of generality. Additionally, it is important to take the absolute values under the square roots when $\rho = \mu$ or $\rho = \nu$ to avoid negative sign under the square roots. See their domains as specified in (4).

The process involves substituting these forms into equation (15), resulting in the generation of four separate ordinary differential equations. Each of these equations represents a unique class, as previously mentioned. In other words, the separation of variables yields three ODEs, each containing four classes, resulting in a total of 12 ODEs. Subsequently, these are solved using series solutions.

A. Class K

The first class is straightforward, expanding $H(\rho)$ in power series

$$H(\rho)=\sum_{j=0}^{\infty}a_j\rho^{n-2j}, \tag{17}$$

where n is the highest power, after substituting the power series expansion (17) in equation (15) we get the first recursion relation

$$\begin{aligned}
&a_{j+2}[(n-2j-4)(n-2j-3)-n(n+1)]\\
&-a_{j+1}(h_2^2+h_3^2)[(n-2j-2)^2-p]\\
&+a_j[(h_2h_3)^2(n-2j)(n-2j-1)+\chi^2]=0.
\end{aligned} \tag{18}$$

B. Classes L and M

The next two classes are treated differently, with the understanding that L and M have similar expressions; h_2^2 in place of h_3^2. For example, let's consider class L. The power series expansion for this class takes the following form:

$$\begin{aligned}
L_n(\rho) &= \sqrt{|\,\rho^2-h_3^2\,|}R_{n-1}(\rho)\\
R_{n-1}(\rho) &= \sum_{j=0}^{\infty}b_j\rho^{n-2j-1}.
\end{aligned} \tag{19}$$

In the above expression, $R_{n-1}(\rho)$ is one-degree lower than $L_n(\rho)$. By substituting the expression for $R_{n-1}(\rho)$ given in (19) into equation (15), and performing some algebraic manipulations, the second and third ordinary differential equations for the classes L and M take the following form:

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$$(\rho^2-h_2^2)(\rho^2-h_3^2)\left[R''(\rho)+\frac{2\rho R'(\rho)}{\rho^2-h_3^2}-\frac{h_3^2R(\rho)}{(\rho^2-h_3^2)^2}\right]$$
$$+\rho(2\rho^2-h_2^2-h_3^2)\left[R'(\rho)+\frac{\rho R(\rho)}{\rho^2-h_3^2}\right]+\left[(h_2^2+h_3^2)p-n(n+1)\rho^2+\frac{\chi^2}{\rho^2}\right]R(\rho)=0. \quad (20)$$

Further simplifications and rearrangement give the following form which is solved by series solution.

$$(\rho^2-h_2^2)(\rho^2-h_3^2)R''(\rho)+\rho(4\rho^2-3h_2^2-h_3^2)R'(\rho)$$
$$+\left((h_2^2+h_3^2)p-h_2^2-(n-1)(n+2)\rho^2+\frac{\chi^2}{\rho^2}\right)R(\rho)=0. \quad (21)$$

Note that $R_{n-1}(\rho)$ in the previous equation is written as $R(\rho)$ for simplicity. The latter ODE has the recursion relation:

$$b_{j+2}\left[(n-2j-5)(n-2j-2)-(n-1)(n+2)\right]$$
$$+b_{j+1}\left[-(n-2j-3)\{(h_2^2+h_3^2)(n-2j-4)+3h_2^2+h_3^2\}+(h_2^2+h_3^2)p-h_2^2\right] \quad (22)$$
$$+b_j\left[(h_2h_3)^2(n-2j-2)(n-2j-1)+\chi^2\right]=0.$$

C. Class N

On the other hand, the last class N contains two roots, and takes the form

$$N_n(\rho)=\sqrt{|\,\rho^2-h_3^2\,|}\sqrt{|\,\rho^2-h_2^2\,|}S_{n-2}(\rho)$$
$$S_{n-2}(\rho)=\sum_{j=0}^{\infty}c_j\rho^{n-2j-2}. \quad (23)$$

Direct substitution of $N_n(\rho)$ into equation (15), we get an equation for $S_{n-2}(\rho)$ as:

$$(\rho^2-h_2^2)(\rho^2-h_3^2)\left[S''(\rho)+\frac{2\rho S'(\rho)}{\rho^2-h_3^2}+\frac{2\rho S'(\rho)}{\rho^2-h_2^2}-\frac{h_3^2S(\rho)}{(\rho^2-h_3^2)^2}-\frac{h_2^2S(\rho)}{(\rho^2-h_2^2)^2}\right]$$
$$+\rho(2\rho^2-h_2^2-h_3^2)\left[S'(\rho)+\frac{\rho S(\rho)}{\rho^2-h_3^2}+\frac{\rho S(\rho)}{\rho^2-h_2^2}\right] \quad (24)$$
$$+2\rho^2S(\rho)+\left[(h_2^2+h_3^2)p-n(n+1)\rho^2+\frac{\chi^2}{\rho^2}\right]S(\rho)=0.$$

By performing the same algebraic manipulations as in the second and third classes, we arrive at the following ordinary differential equation:

$$(\rho^2-h_2^2)(\rho^2-h_3^2)S''(\rho)+3\rho(2\rho^2-h_2^2-h_3^2)S'(\rho)$$
$$+\left((h_2^2+h_3^2)(p-1)-(n-2)(n+3)\rho^2+\frac{\chi^2}{\rho^2}\right)S(\rho)=0. \quad (25)$$

Where $S_{n-2}(\rho)$ is two-degrees lower than $N_n(\rho)$. Its corresponding recursion relation is as follows:

$$c_{j+2}[(n-2j-6)(n-2j-1)-(n-2)(n+3)]$$
$$-c_{j+1}(h_2^2+h_3^2)[(n-2j-4)(n-2j-2)+1-p] \quad (26)$$
$$+c_j[(h_2h_3)^2(n-2j-3)(n-2j-2)+\chi^2]=0.$$

V. ENERGY LEVELS

For all four classes, the power series in the ordinary differential equations and their corresponding recursion

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relations (18, 22, 26) must terminate, meaning that the coefficient of the highest power must be set to zero to cutoff the series, that is to say, the coefficients a_j, b_j, and c_j of the highest power equal zero. Performing this algebra and recalling equation (10), all four classes result in the same energy expression.

$$E'_{density} = V' - \frac{\hbar^2}{2m}\frac{h_2^2 h_3^2}{(\lambda\mu\nu)^2}(n+1)(n+2), \qquad (27)$$

where n=0, 1, 2, ...

The expression for the energy density given above is in one dimension. To generalize it to three dimensions, we introduce three quantum numbers N_λ, N_μ, and N_ν, where $N_\lambda = (n_\lambda + 1)(n_\lambda + 2)$ and the remaining two are defined similarly. Thus, the energy density expression in three dimensions becomes:

$$E'_{density} = V' - \frac{\hbar^2}{2m}\frac{h_2^2 h_3^2}{(\lambda\mu\nu)^2}(N_\lambda + N_\mu + N_\nu), \qquad (28)$$

where the three quantum numbers take on the values $N_i = 2, 6, 12, 20, 30, ...$

Upon initial examination of equation (28), it is evident that the energy density depends on the semi-focal distances of the coordinate system, which remain constant for a specific confocal coordinate. Consequently, when employing this coordinate system, one must carefully select the appropriate, specified coordinate suitable for his work.

VI. DISCUSSIONS & CONCLUSION

In this paper, we successfully determined the energy levels of a particle subjected to a constant potential in confocal ellipsoidal coordinates by defining an auxiliary partial differential equation. The derived energy density formula exhibits two types of degeneracy, wherein distinct eigenstates share the same energy level. Upon initial inspection of formula (28), combinatorial analysis reveals that when the three quantum numbers N_i are equal, the degeneracy is 1. When two quantum numbers are identical but the third is different, the degeneracy is 3. In cases where all quantum numbers are distinct, the degeneracy amounts to 6. On the other hand, considering that confocal ellipsoidal coordinates possess four classes of harmonics, this implies that each of the aforementioned degeneracies must be multiplied by 4 to exhaust all possibilities. Consequently, the degeneracy values are $d = 4$, $d = 12$, or $d = 24$, respectively.

The energy levels are not equidistant and are directly depend on the geometric structure of the chosen system, in other words, they depend on h_2^2 and h_3^2. Moreover, the singularities appear at the foci of the system. This property may be useful in estimating the energy levels in diatomic molecules by situating nuclei at these foci, but the analysis as provided must undergo improvements to handle more intricate and realistic potentials, a task currently impossible due to the complexity of the 3-body Coulomb potential.

In particular cases, such as solving for infinite spherical or square wells, intriguing results emerge. Initially, by employing (5) and subsequently integrating the energy density formula given by Equation (27) to determine the total energy, each energy level is multiplied by a constant. This constant is expected to appear when solving a quantum system in two distinct non-degenerate coordinates, see [19] for instance. For the previous cases, this multiplying constant converges to $1/\pi^2$ for large n.

In an on-going study, we are developing the above technique to investigate three-body Coulomb potential of diatomic molecule ions such as H_2^+, N_2^+, and O_2^+. Additionally, we are interested in using other types of pseudo potentials to study various cases in ellipsoidal coordinates.

In conclusion, we have addressed the Schrödinger equation using confocal ellipsoidal coordinates with a constant potential, a coordinate system that is seldom employed due to its complex and ambiguous nature. However, in several areas of applied physics, like quantum chemistry [20] and electrostatics [21], it has demonstrated advantages over spherical coordinates. The goal is to enhance its analytical applications for wider areas of physics, particularly in quantum research.

STATEMENTS

A. Competing Interests and Funding

The authors declare there are no competing interests.

B. Data Availability

This manuscript does not report data.

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