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THE ROLE OF ALLEE EFFECT IN CANNIBALISTIC SPECIES: AN ACTION PLAN TO SUSTAIN THE DECLINING COD POPULATION

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Abstract. Atlantic cod collapsed in the late 20th century after being harvested heavily for 50 years. This paper aims to design conservation guidelines for the cod population, which is diminishing due to predation by grey seals and cannibalism. For this purpose, we first designed a continuous time ecological model (with and without the Allee effect) using a system of differential equations consisting of juvenile Atlantic cod, adult Atlantic cod, and grey seals. The developed model has set forth global existence, non-negativity, and long-term behavior. Subsequently, to handle the extinction problem cost-effectively, Pontryagin's principle is employed to construct the optimal control, which is then numerically solved using an iterative forward–backward method. We numerically explored the impact of the Allee effect on cod survival within the original model and its two extended versions (i) stochastic and (ii) reaction–diffusion, to thoroughly understand the possible consequences wherein a population has cannibalistic tendencies. The numerical comparison between the non-Allee and Allee models (Ordinary, Stochastic, Reaction–Diffusion) reveals that the Allee effect may significantly promote recovery and benefit the cannibalistic population. We adopted a partial rank correlation coefficient (PRCC) to conduct a global sensitivity analysis to estimate the most sensitive parameters responsible for cod prevalence.

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1. Introduction

The world's biodiversity is deteriorating at an unprecedented rate. The list of declining species is long (https://www.nationalgeographic.com/environment/article/ipbes-un-biodiversity-report-warns-one-million-species-at-risk). The Atlantic cod (*Gadus morhua*) is perceived as one of the greatest threat to global biodiversity in this list [1]. The Atlantic cod is an important species in many of the world's ocean systems from an economic, ecological, and cultural perspective [2–4]. The Atlantic cod is a medium to large-sized marine fish inhabiting cold water ($0^o - 15^o$ C). The habitat requirement of cod is not known clearly. They inhabit all water overlying the continental shelves of the Northwest and Northeast Atlantic oceans. Cod is also abundant in the Skagerrak and Kattegat and the southern parts of the Baltic Sea [5]. Cod was once considered the world's most fertile fish. Since the 1990s, this fishing has suffered depletion and has not recovered. The primary reasons for

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declination are its cannibalistic nature, predation by grey seals, and climate change [6]. The productivity of the stock is much lower than in the 1970s when a similar decline occurred [7].

In this article, we will focus on cod mortality by cause of over-fishing, predation, and juvenile mortalities owing to cannibalism [8, 9]. Cannibalism is defined as the act of killing and consuming its conspecifics. It is prevalent in natural predator-prey communities, ubiquitous in most animal groups, including fishes, and significantly regulates population dynamics [10]. Cannibalism is size-selective predation that affects the entire population's abundance and size structure. Cannibalism in Atlantic cod is a frequent phenomenon in aquaculture and wild fish stocks, and if it is not appropriately managed, the losses can be devastating [11, 12]. Due to this, cannibalism in Atlantic cod has received considerable attention in field and aquaculture studies. It is assumed to significantly affect recruitment of the cod fishery as well as contribute to high economic losses in the aquaculture industry [11, 13]. Past observations suggest that environmental heterogeneity benefits juveniles because it reduces the possibility of predation. Thus, to make juvenile survival and production as much as possible, it is crucial to estimate the potential influences of the size distribution of juveniles on cannibalism and growth [14]. A recent study suggested that gray seal predation also impeded the cod population's recovery. Gray seals are generalist predators with pronounced seasonal shifts in distribution and foraging behavior [15]. They exhibit a diverse diet, preying on various pelagic and demersal fishes, including Atlantic cod ([16, 17]). The population breeding on Sable Island is a notable example of the exponential growth of a long-lived marine mammal, experiencing an annual increase of 12.8% for over 25 years [18]. It is crucial to comprehend the factors, such as cannibalism and predation, that hinder the recovery of depleted populations to devise successful strategies for restoration.

Another vital factor known as the Allee effect, interpreted as the difficulty in finding mates at low-density [19], can influence cod population density. The theory underlying Allee effects is well established, and empirical evidence has been detected in various systems [20, 21]. Despite this knowledge, implications of Allee effects for recovery plans, practical conservation decisions, and harvesting strategies have yet to be implemented [22]. A primary reason for this is that it can be difficult to document an Allee effect in populations that have declined or have little data. Conservation biologists and managers are usually more interested in demographic Allee effects because they ultimately govern the extinction or recovery of species at low populations. In most articles, the Allee effect is considered to harm the population. In this article, we will validate the hypothesis of whether the Allee effect is beneficial for the cannibalistic population or not, as established by Veprauskas and Cushing (2017) [23] for a discrete-time model.

In this article, we developed a new mathematical model to understand the extinction dynamics of the cod population. We started with the ODE model without the Allee effect and then extended it into one with the Allee effect. To demonstrate the resilience of the hypothesis regarding the Allee effect in cannibalistic populations, we extended the model into a stochastic model incorporating environmental fluctuation and a reaction–diffusion model. The aim is to check whether the Allee effect has harmful or beneficial impacts on all three systems. Stochastic models consider environmental fluctuations to account for changes in external conditions that impact the system dynamics. These fluctuations could represent variations in temperature, resource availability, or other environmental factors. However, self-diffusion models often describe the movement or diffusion of entities within the system. Self-diffusion reflects the inherent movement or spreading of particles, individuals, or other entities without an external driving force. We also aimed to minimize the decrease in the Atlantic cod population while minimizing the cost associated with implementing the control strategy. Putting everything together, here we mathematically validate four key factors affecting the ongoing threatening processes of cod: (i) the impact of cannibalistic behavior of cod, (ii) How the Allee effect impacts the cannibalistic population?, (iii) Whether the absence of grey seal is crucial for maintaining adult cod populations? [24], and (iv) What is the effect of control measures on cod survival? To our knowledge, this is the first time anyone has attempted to validate these hypotheses so far mathematically. We numerically compared all the designed models to assess whether the Allee effect benefits the cannibalistic population.

This article is organized as follows: In Section 2, we formulated our model system (with and without the Allee effect). In Section 3, we studied our proposed model's (without Allee effect) existence, uniqueness, dissipativeness, and persistence. Section 4 presents a few analyses of the designed model with the Allee effect.

Section 5 discusses the possible two extensions of the model: (1) stochastic and (2) reaction–diffusion. Numerical simulation and comparison are made in Section 6. Finally, the conclusion is presented in Section 7.

2. Formulation of model

Atlantic cod has experienced more significant and profound population depletion than other species [24]. Despite decades of scientific-based management, most marine fisheries have not seen significant success in their management efforts. This lack of achievement suggests that the existing theories developed to comprehend the determinants impacting fish population sizes and their management are flawed ([25, 26]). Biological stock/recruitment models, rational choice theory in social sciences, and ecological adaptive management approaches have all been utilized to tackle issues in marine resource management. However, without integration with historical insights, these theories do not explain the current depletion observed in cod fisheries and the broader North Atlantic ground fishery. The decline of the population is expected to continue even in the absence of fishing unless productivity improves. When a fish population begins declining towards the threshold known in fisheries management as the "critical zone" − where its capacity to recover is deemed at risk − immediate action becomes imperative. Initiating recovery efforts before the stock plummets too severely significantly enhances the likelihood of restoring it to a robust state. However, if the stock descends deeply into the critical zone, the surviving fish become considerably less resilient to influences like predators and the impacts of climate change. In this article, we will discuss the case of Atlantic Cod to design some conservation measures that can be applied to any species with the same kind of threat. Atlantic cod has failed to recover since the 1990s mainly due to its cannibalistic nature and an emergent effect resulting from increasing grey seal abundance. We proposed a mathematical model consisting of cod and grey seals. We further divided the cod population into two classes: Juvenile (J-(0-7)y) and Adult cod ($A > 7$y). We assume that the cod population grows logistically, and only juvenile cod is predated by the grey seal [27]. We incorporated a Holling type II functional response [28] for grey seals preying on Atlantic cod's 0–7 age group [1]. Many researchers detected cannibalism to be the main mortality factor in cod after metamorphosis [12, 13, 29–33]. We have also included this fact by incorporating the cannibalism term $\omega_1 AJ$ into our model. Over the last century, humans emerged as a globally dominant predator of wild marine fish, primarily due to significant advances in fishing technology. Therefore, incorporating harvesting terms h_1 and h_2 in adult cod and grey seal populations is also reasonable in the model formation. The model, with the considered assumption, will be formulated as follows:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = r_1 A\left(1 - \frac{A}{K_1}\right) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta + J} - \rho J, \tag{2.1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \rho J + \omega_2 AJ - h_1 A, \tag{2.2}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = r_2 G\left(1 - \frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta + J} - h_2 G, \tag{2.3}$$

with $J(0) = J_0 \geq 0, A(0) = A_0 \geq 0$ and $G(0) = G_0 \geq 0$. All the parameters used in system (2.1)–(2.3) are positive. Figure 1 is the schematic representation of the model system. To see the description of parameter values of the formulated model, refer to Table 1.

Besides its relevance to marine conservation and fisheries, Atlantic cod also provides an excellent basis for investigating the influence of the Allee effect on population recovery. A demographic Allee effect has been observed in the cods through extensive analyses of several populations in past literature. In this paper, we will investigate whether the Allee effect has a positive or negative role in the survival of cod populations. The population growth rate is generally assumed to be relatively faster at low abundance because of low competition (*i.e.*, negative density dependence). However, population growth can be limited in small populations due to the Allee effect. Now, we investigate how an Allee effect can influence recovery dynamics in cannibalistic species. We modify the system (2.1)–(2.3) to illustrate the impact of the strong Allee effect in the cod population [34, 35].

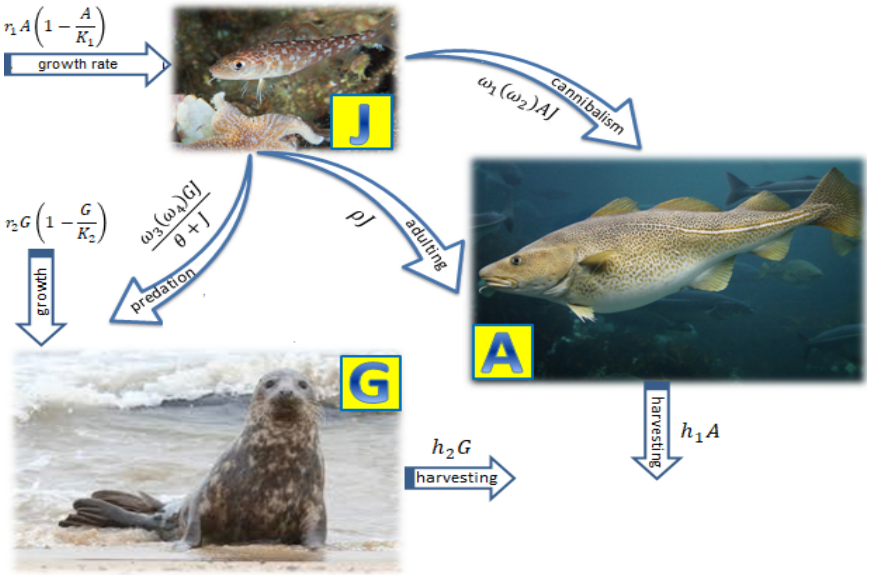


FIGURE 1. Schematic diagram for three species food model system (2.1)–(2.3).

TABLE 1. Description of parameter values used in model (2.1)-(2.3).

Variables/parameters	Unit	Description
J	Juvenile Atlantic cod per unit area	Density of juvenile Atlantic cod population
A	Adult Atlantic cod per unit area	Density of adult Atlantic cod population
G	Grey seal per unit area	Density of grey seal population
r_1	Per day	Intrinsic growth rate of adult Atlantic cod population
r_2	Per day	Intrinsic growth rate of grey seal population
K_1	Cod per unit area	Carrying capacity of Cod
K_2	Grey seal per unit area	Carrying capacity of grey seal
ω_1	Per day	Rate of cannibalism in cod
ω_2	Per day	Biomass conversion rates of juvenile cod to adult cod
ω_3	Per day	Predation rate of juvenile cod by grey seal
ω_4	Per day	Biomass conversion rates of juvenile cod to grey seal
ρ	Per day	Conversion rate of juvenile cod to adult cod
θ	Per unit area	Measures of the extent to which the environment provides refuges to juvenile cod, thus decreasing the maximal predation rates
h_1	Per day	Harvesting rate of adult Atlantic cod
h_2	Per day	Harvesting rate of grey seal

With the above assumptions, the model (2.1)–(2.3) can be formulated as follows:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = r_1 A\left(1-\frac{A}{K_1}\right)(A+\beta) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta+J} - \rho J, \tag{2.4}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \rho J + \omega_2 AJ - h_1 A, \tag{2.5}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = r_2 G\left(1-\frac{G}{K}\right) + \frac{\omega_4 GJ}{\theta + J} - h_2 G. \tag{2.6}$$

with $J(0) = J_0 \geq 0, A(0) = A_0 \geq 0$ and $G(0) = G_0 \geq 0$. We focus on the case where the Allee effect damps the growth rate of prey. Here, parameter β represents the Allee effect. When $0 < \beta < 1$, we call it a weak Allee effect, and for $\beta < 0$, the Allee effect becomes strong [36, 37].

3. Analysis of model (2.1)–(2.3)

In this section, we shall only discuss the positivity and boundedness of the system (2.1)–(2.3) to ensure that the model is well-posed.

Theorem 3.1. *All solutions of system (2.1)–(2.3) starting from $\mathbb{R}^+_3$ remain positive for all time. Furthermore, $\limsup_{t\to\infty} P(t) \leq \left(\frac{r_1K_1}{4} + \frac{r_2K_2}{4} + \gamma K_1\right)$ where, $P(t) = J(t) + A(t) + G(t)$ and $\gamma = \min\{h_1, h_2\}$.*

Proof. As $P(t) = J(t) + A(t) + G(t)$, taking time derivative on both the sides we get

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}J}{\mathrm{d}t} + \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\mathrm{d}G}{\mathrm{d}t},$$

Now, from model (2.1)–(2.3), we get

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= r_1 A\left(1-\frac{A}{K_1}\right) + (\omega_2 - \omega_1) AJ + (\omega_4 - \omega_3)\frac{GJ}{\theta + J} + r_2 G\left(1-\frac{G}{K_2}\right) - h_1 A - h_2 G,\\
&\leq r_1 A\left(1-\frac{A}{K_1}\right) + r_2 G\left(1-\frac{G}{K_2}\right) - h_1 A - h_2 G,\\
\frac{\mathrm{d}P}{\mathrm{d}t} + \gamma P &\leq r_1 A\left(1-\frac{A}{K_1}\right) + r_2 G\left(1-\frac{G}{K_2}\right) + \gamma J + (\gamma - h_1) A + (\gamma - h_2) G,
\end{aligned}$$

since, $\gamma = \min\{h_1, h_2\}$

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} + \gamma P &\leq r_1 A\left(1-\frac{A}{K_1}\right) + r_2 G\left(1-\frac{G}{K_2}\right) + \gamma J,\\
&\leq r_1\left(A - \frac{A^2}{K_1}\right) - r_2\left(\frac{G^2}{K_2} - G\right) + \gamma K_1, && (\because J \leq K_1)\\
&\leq -r_1\left(\frac{A}{\sqrt{K_1}} - \frac{\sqrt{K_1}}{2}\right)^2 + \frac{r_1K_1}{4} - r_2\left(\frac{G}{\sqrt{K_2}} - \frac{\sqrt{K_2}}{2}\right)^2 + \frac{r_2K_2}{4} + \gamma K_1,\\
&\leq \frac{r_1K_1}{4} + \frac{r_2K_2}{4} + \gamma K_1, && (\because A \leq K_1 \text{ and } G \leq K_2)\\
P(t) &\leq \left(\frac{r_1K_1}{4} + \frac{r_2K_2}{4} + \gamma K_1\right) + \mathrm{e}^{-\gamma t}\left(P(0) - \frac{r_1K_1}{4} - \frac{r_2K_2}{4} - \gamma K_1\right).
\end{aligned}$$

As $t \to \infty$,

$$P \leq \left(\frac{r_1K_1}{4} + \frac{r_2K_2}{4} + \gamma K_1\right).$$

□

For a biologically functional system, all populations must be kept within a close bound with their environment.

3.1. Stability analysis

Formulated model (2.1)–(2.3) has following possible equilibrium points:

1. The trivial equilibrium $E_0 = (0, 0, 0)$ always exists. Its variational matrix is given by
$$\mathcal{V}(E_0) = \begin{pmatrix} -\rho & r_1 & 0 \\ \rho & -h_1 & 0 \\ 0 & 0 & -h_2 + r_2 \end{pmatrix}.$$
The eigenvalues are $r_2 - h_2$, $\frac{1}{2}(-h_1 - \rho \pm \sqrt{{h_1}^2 - 2h_1\rho + 4r_1\rho + \rho^2})$. Hence, the trivial equilibrium is locally asymptotically stable if $r_2 < h_2$ and $r_1 < h_1$. This means that when the growth rate is less than the harvesting rate, small deviations from the zero equilibrium will eventually dampen, leading all population to become extinct. The stability of this equilibrium indicates the long-term extinction of the entire system.
2. The cod (juvenile and adult) free equilibrium $E_1 = (0, 0, \frac{K_2(r_2-h_2)}{r_2})$, exists if $h_2 < r_2$.

   Its variational matrix is given by
$$\mathcal{V}(E_1) = \begin{pmatrix} -\rho - \frac{K_2(r_2-h_2)\omega_3}{r 2\theta} & r_1 & 0 \\ \rho & -h_1 & 0 \\ \frac{K_2(r_2-h_2)\omega_4}{r_2\theta} & 0 & r_2(1 - \frac{r_2-h_2}{r_2}) - r_2 \end{pmatrix}.$$
The eigenvalues are $h_2 - r_2$, $\frac{M \pm \sqrt{N}}{L}$,
where, $M = h_2K_2\omega_3 - K_2r_2\omega_3 - h_1r_2\theta - r_2\theta\rho$,
$N = (h_1r_2\theta + r_2\theta\rho - h_2K_2\omega_3 + K_2r_2\omega_3)^2 - 4r_2\theta(h_1r_2\theta\rho - r_1r_2\theta\rho - h_1h_2K_2\omega_3 + h_1K_2r_2\omega_3)$,
$L = 2r_2\theta$.
Hence, the equilibrium is asymptotically stable if $r_1 < h_1 + \frac{h_1K_2(r_2-h_2)\omega_3}{r_2\theta\rho}$. This means that when the growth rate of Atlantic cod is less than a critical value, the neighboring solution of this equilibrium will converge to the equilibrium E_1, leading the cod population to become extinct. The stability of this equilibria indicates the potential for long-term cod extinction.
3. The equilibrium $E_2 = \left(\frac{-R+P}{Q}, \frac{S+P}{T}, 0\right)$ always exists where,
$$\begin{aligned} R &= h_1K_1(\omega_1 - 2\omega_2) + r_1(\rho + K_1\omega_2), \\ P &= \sqrt{(r_1\rho + h_1K_1\omega_1)^2 - 2K_1r_1(2h_1\rho - r_1\rho + h_1K_1\omega_1)\omega_2 + {K_1}^2{r_1}^2{\omega_2}^2}, \\ Q &= 2K_1(\omega_1 - \omega_2)\omega_2, \\ S &= -r_1\rho - h_1K_1\omega_1 + K_1r_1\omega_2, \\ T &= 2r_1\omega_2, \end{aligned}$$
The variational matrix is given by
$$\mathcal{V}(E_2) = \begin{pmatrix} -\left(\rho + \frac{\omega_1(P+S)}{T}\right) & \frac{P(2\omega_2-\omega_1)-\omega_1S-\rho T}{Q} & -\frac{\omega_3(P-R)}{(P-R-\theta Q)} \\ \frac{U+P}{2r_1} & \frac{(U-P)\omega_2}{Q} & 0 \\ 0 & 0 & -h_1 + r_2 + \frac{\omega_4(P-R)}{(P-R-\theta Q)} \end{pmatrix}.$$
where,

$U = r_1\rho - h_1K_1\omega_1 + K_1r_1\omega_2$.

The eigenvalues are $r_2 - h_2 + \dfrac{(P-R)\omega_4}{P-R-Q\theta}, \dfrac{1}{2K_1QT^2}\Big(-K_1TW \pm V\Big)$,

where,

$$W = \big(QT(h_1+\rho) + Q(P+S)\omega_1 + (P-R)T\omega_2\big),$$

$$V = \sqrt{K_1T^2(K_1W)^2 - 4Q(T(Q(2r_1(P+S) + K_1(h_1-r_1)T)\rho + K_1(h_1Q(P+S)}$$

$$\overline{+(-P+R)T\rho)\omega_1) + (Qr_1(P+S)(2(P+S) - K_1T) + K_1(P-R)T^2\rho)\omega_2))}.$$

The equilibrium point is locally asymptotically stable if all the eigenvalues are negative. Negative real parts of eigenvalues associated with equilibrium points signify stability. This means that if the conditions are met, and the system experiences a small disturbance or deviation from the equilibrium point E_2, it eventually returns to the equilibrium, and grey seals will face extinction.

4. For establishing the existence of non-trivial equilibrium, we reduce system (2.1)–(2.3) into an equation in terms of J (by finding A in terms of J from (2.2) and finding G in terms of J from (2.3) and then putting both in (2.1)), becomes

$$k(J) = aJ^4 + bJ^3 + cJ^2 + dJ + e, \tag{3.1}$$

where,

$$\begin{aligned}
a &= K_1r_2\rho\omega_1\omega_2 - K_1r_2\rho{\omega_2}^2,\\
b &= -r_1r_2\rho(\rho + K_1\omega_2) - K_1(h_1r_2\rho(\omega_1 - 2\omega_2) + \omega_2(r_2(2\theta\rho(\omega_2-\omega_1) + K_2\omega_2\omega_3)\\
&\quad + K_2\omega_2\omega_3(\omega_4 - h_2))),\\
c &= -{h_1}^2K_1r_2\rho - 2r_1r_2\theta\rho(\rho + K_1\omega_2) + K_1\theta\omega_2(r_2\theta\rho(\omega_1-\omega_2) + K_2(h_2-r_2)\omega_2\omega_3)\\
&\quad + h_1K_1(r_1r_2\rho + r_2(4\theta\rho\omega_2 - 2\theta\rho\omega_1 + 2K_2\omega_2\omega_3) + 2K_2\omega_2\omega_3(\omega_4 - h_2)),\\
d &= -r_1r_2\theta^2\rho(\rho + K_1\omega_2) + h_1K_1\theta(r_2\rho(2r_1 - \theta\omega_1 + 2\theta\omega_2) + 2K_2(r_2-h_2)\omega_2\omega_3)\\
&\quad - {h_1}^2K_1(r_2(2\theta\rho + K_2\omega_3) + K_2\omega_3(\omega_4 - h_2)),\\
e &= h_1K_1\theta((-h_1+r_1)r_2\theta\rho + h_1K_2(h_2-r_2)\omega_3),
\end{aligned}$$

then the discriminant of the quartic function becomes
$\Delta = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2$.
Also, $\mathcal{T} = 8ac - 3b^2$, $\mathcal{U} = b^3 + 8da^2 - 4abc$, $\mathcal{V} = c^2 - 3bd + 12ae$, $\mathcal{W} = 64a^3e - 16a^2c^2 + 16ab^2c - 16a^2bd - 3b^4$.
Now,

- Case 1: If $\Delta < 0$, then there will be two real distinct and two complex conjugates roots.
- Case 2: If $\Delta > 0$ then
  - (a) If $\mathcal{T} < 0$ and $\mathcal{W} < 0$ then there will be 4 real and distinct roots.

– (b) If $\mathcal{T} > 0$ and $\mathcal{W} > 0$, then there will be two pairs of complex conjugate roots.

- Case 3: If $\Delta = 0$, then
  – (a) If $\mathcal{T} < 0$ and $\mathcal{W} < 0$ and $\mathcal{V} \neq 0$, then there will be two real simple and one real double roots.
  – (b) If $\mathcal{W} > 0$ or $\mathcal{T} > 0(\mathcal{W} \neq 0$ or $\mathcal{U} \neq 0)$ then there will be one real double and a pair of complex conjugate roots.
  – (c) If $\mathcal{V} = 0$ and $\mathcal{W} \neq 0$, then there will be one real triple and one real simple root.
  – (d) If $\mathcal{W} = 0$ then
    - (i) If $\mathcal{T} < 0$ then there will be 2 real double roots.
    - (ii) If $\mathcal{T} > 0$ and $\mathcal{U} = 0$ then there will be 2 double complex conjugate roots.
    - (iii) If $\mathcal{V} = 0$ then there will be 4 equal roots.

Example 3.2. Let us set $r_1 = 0.49,\ h_1 = 0.12,\ r_2 = 1.2,\ h_2 = 0.7,\ \omega_1 = 0.9,\ \omega_2 = 0.25,\ \omega_3 = 1.1, \omega_4 = 0.95, \theta = 25, \rho = 1/(7 * 365), K_1 = 1000$ and $K_2 = 100$ the eqn (3.1) becomes

$$k(J) = 0.0763209J^4 - 9965.01J^3 - 76323.8J^2 + 80154.2J - 19787. \tag{3.2}$$

For these parameters: $\Delta = 7.65037X10^{18} > 0$, $\mathcal{T} = -2.97951X10^8 < 0$ and $\mathcal{W} = -2.95916X10^{16} < 0$. Hence, from Case 2 (a), there will be four real and distinct roots.
Since we are interested in positive values of all three species (J, A, G), calculating mathematically, we get two positive equilibriums, (i) $E_1^* = (0.479992, 100.154, 43.158)$ and (ii) $E_2^* = (0.479959, 18.2116, 43.1579)$.

For the stability of E^*, we calculate the jacobian matrix as follows,

$$V(E^*) = \begin{pmatrix} -\rho - A^*\omega_1 + \frac{G^*J^*\omega_3}{(J^*+\theta)^2} - \frac{G\omega_3}{J^*+\theta} & (1 - \frac{A^*}{K_1})r_1 - \frac{A^*r_1}{K_1} - J^*\omega_1 & -\frac{J^*\omega_3}{J^*+\theta} \\ \rho + A^*\omega_2 & -h_1 + J^*\omega_2 & 0 \\ -\frac{G^*J^*\omega_4}{(J^*+\theta)^2} + \frac{G^*\omega_4}{J^*+\theta} & 0 & -h_2 + (1 - \frac{G^*}{K_2})r_2 - \frac{G^*r_2}{K_2} + \frac{J^*\omega_4}{J^*+\theta} \end{pmatrix}.$$

The characteristic equation becomes-:

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0,$$

where,

$$B_1 = h_1 + h_2 - r_2 + \rho + A^*\omega_1 - J^*\omega_2 + \frac{2G^*r_2}{K_2} - \frac{G^*J^*\omega_3}{(J^*+\theta)^2} - \frac{-G^*\omega_3 + J^*\omega_4}{J^*+\theta},$$

$$B_2 = h_1(h_2 - r_2) + \frac{2G^*h_1r_2}{K_2} + \rho(h_1 - r_1 - r_2) + h_2\rho + A^*h_1\omega_1 + A^*h_2\omega_1 - A^*r_2\omega_1 + J^*\rho\omega_1$$
$$- (A^*r_1 + J^*(h_2 - r_2 + \rho))\omega_2 + \frac{2(A^*(K_2r_1\rho + G^*K_1r_2\omega_1) + A^{*2}K_2r_1\omega_2 + G^*K_1r_2(\rho - J^*\omega_2))}{K_1K_2}$$
$$- \frac{G^*\theta(-2G^*r_2 + K_2(-h_1 - h_2 + r_2 + J^*\omega_2))\omega_3 + J^*K_2(J^*+\theta)(h_1 + \rho + A^*\omega_1 - J^*\omega_2)\omega_4}{K_2(J^*+\theta)^2},$$

$$
\begin{aligned}
B_3 = -\frac{1}{K_1K_2(J^*+\theta)^2}\big(&2A^{*2}r_1(J^*+\theta)\omega_2(-(h_2k_2+2G^*r_2-K_2r_2)(J^*+\theta)+J^*K_2\omega_4)\\
&+A^*r_1(J^*+\theta)(2\rho-K_1\omega_2)(-(h_2K_2+2G^*r_2-K_2r_2)(J^*+\theta)+J^*K_2\omega_4)\\
&+h_1K_1(h_2K_2(-(J^*+\theta)^2(\rho+A^*\omega_1)-G^*\theta\omega_3)+(2G^*-K_2)r_2(-(J^*+\theta)^2(\rho+A^*\omega_1)-G^*\theta\omega_3)\\
&+J^*K_2(J^*+\theta)(\rho+A^*\omega_1)\omega_4)+K_1(2G^{*2}J^*r_2\theta\omega_2\omega_3+h_2K_2(-(J^*+\theta)^2\rho](-r_1+J^*(\omega_1-\omega_2))\\
&+G^*J^*\theta\omega_2\omega_3)+G^*r_2(-2(J^*+\theta)^2\rho(-r_1+J^*\omega_1-J^*\omega_2)-J^*K_2\theta\omega_2\omega_3)\\
&+K_2(J^*+\theta)\rho(-r_1+J^*\omega_1-J^*\omega_2)(r_2(J^*+\theta)+J\omega_4))\big).
\end{aligned}
$$

By Routh-hurwitz criterion, we can say that E^* is locally asymptotically stable if $B_1 > 0, B_3 > 0$ and $B_1B_2 - B_3 > 0$.

Remark 3.3. Setting the parameters as in example 3.2, we find that

- (i) For $E_1^* = (0.479959, 100.154, 43.1579)$, $B_1 = 92.485 > 0, B_3 = 0.520667 > 0$ and $B_1B_2 - B_3 = 4500.48 > 0$, and
- (ii) For $E_2^* = (0.479959, 18.2116, 43.1579)$, $B_1 = 18.7368 > 0, B_3 = -0.0946749 \not> 0$ and $B_1B_2 - B_3 = 174.073 > 0$.

Hence, for this set of parameters, positive equilibrium E_1^* is locally asymptotically stable and E_2^* is unstable.

3.2. Dissipative and Persistence

Biologically, the persistence of a system means the long-term survival of each component, no matter what the initial populations are. Mathematically, it means strictly positive solutions do not have omega limit points on the boundary of the non-negative cone [38]. We first state the well-known conclusion as Lemma 3.4.

Lemma 3.4. *If $m > 0$, $n > 0$ and $\dot{y}(t) \leq (\geq) y(t)\big(n - my^{\beta}(t)\big)$, where $\beta > 0$(constant), $t \geq 0$ and $y(0) > 0$, then we have $y(t) \leq (\geq)\frac{n}{m}^{\frac{1}{\beta}}\left[1 + \left(\frac{ny^{-\beta}(0)}{m} - 1\right)e^{-nmt}\right]^{\frac{-1}{\beta}}$.*

Theorem 3.5. *The non-negative solution of system (2.1)–(2.3) satisfies*

$$
\begin{aligned}
\lim_{t\to\infty} J(t) &\leq K_1,\\
\lim_{t\to\infty} A(t) &\leq \frac{\rho K_1}{h_1 - \omega_2 K_1},\\
\lim_{t\to\infty} G(t) &\leq \frac{K_2(r_2 + \omega_2 K_1 - h_2)}{r_2},
\end{aligned}
$$

if $\omega_2K_1 - h_1 < 0$ and $r_2 + \omega_2K_1 - h_2 > 0$.

Proof. From the system 2.1 we can deduce that,

$$
\begin{aligned}
J(t) &\leq K_1, &(3.3)\\
\frac{\mathrm{d}A}{\mathrm{d}t} &\leq \rho K_1 + (\omega_2K_1 - h_1)A, &(3.4)\\
\frac{\mathrm{d}G}{\mathrm{d}t} &\leq r_2G\left(1 - \frac{G}{K_2}\right) + \omega_2GK_1 - h_2G,
\end{aligned}
$$

$$= G\big(r_2 + \omega_2 K_1 - h_2 - \frac{r_2 G}{K_2}\big). \tag{3.5}$$

By integrating (3.4) and taking limit $(t \to \infty)$ we see that,

$$\lim_{t\to\infty} A(t) \leq \frac{\rho K_1}{h_1 - \omega_2 K_1}.$$

By integrating (3.5), taking limit $(t \to \infty)$ and using lemma 3.4 we see that,

$$\lim_{t\to\infty} G(t) \leq \frac{K_2(r_2 + \omega_2 K_1 - h_2)}{r_2}.$$

Hence, the proof. □

Biologically, this theorem indicates the sufficient conditions for which all these species have their bounds *i.e.* their number can't surpass some definite value if some conditions hold. Hence, a dissipative system [39].

Theorem 3.6. *Assume that upper bound conditions hold and in addition* $\omega_2 C_1 - h_1 < 0$ *and*

$$\frac{\omega_1 \rho K_1}{h_1 - \omega_2 K_1} < \frac{\omega_3 K_2(r_2 + \omega_2 K_1 - h_2)}{r_2} + \rho \text{ then the system persists.}$$

Proof. From system (2.1)–(2.3), we get

$$\frac{\mathrm{d}J}{\mathrm{d}t} \geq J\Big[\frac{\omega_1 \rho K_1}{h_1 - \omega_2 K_1} - \frac{\omega_3 K_2(r_2 + \omega_2 K_1 - h_2)}{r_2} - \rho\Big].$$

By integrating and taking limit $(t \to \infty)$,

$$\lim_{t\to\infty} J > e^c (= C_1). \quad \{\text{where } c, C_1 \text{ are constants}\} \tag{3.6}$$

Now,

$$\frac{\mathrm{d}A}{\mathrm{d}t} > A(\omega_2 C_1 - h_1) \qquad \{\text{Using eqn (3.6)}\}$$

By integrating and taking limit $(t \to \infty)$,

$$\lim_{t\to\infty} A > e^d (= D_1). \qquad \{\text{where d,} D_1 \text{ are constants}\} \tag{3.7}$$

Lastly,

$$\frac{\mathrm{d}G}{\mathrm{d}t} \geq G\Big[r_2 - h_2 - \frac{r_2 G}{K_2}\Big].$$

Using Lemma 1 and taking limit $(t \to \infty)$,

$$\lim_{t\to\infty} G \geq \frac{K_2(r_2 - h_2)}{r_2}. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we say system persists. □

Biologically, if the given conditions hold then all the three species survive, no matter what the initial population is.

4. Analysis of model (2.4)–(2.6)

The positivity, boundedness, and stability of the model system (2.4)–(2.6) can be done similarly as done for model (2.1)–(2.3) and hence omitted to avoid the unnecessary length of the paper.

4.1. A Threshold value for model (2.4)–(2.6)

Now, since we are interested in exploring the Allee effect in cod survival and extinction, we will find a threshold number (say R_0^{J+A}) similar to the basic reproduction number in an epidemic model [40]. For this food chain model, R_0^{J+A} can serve as a threshold parameter that predicts whether the cod population will persist uniformly or tend to extinction. In the situation of a surplus predator, we can determine which control measures, and at what magnitude and combinations, would be most effective in increasing R_0^{J+A} above 1, providing essential guidance for designing policy measures to save cod extinction. Now, we state the following theorem:

Theorem 4.1. *Let* $\mathcal{R}_0^{J+A} = \left|\frac{h_1K_2(h_2-r_2)\omega_3}{r_2\theta\rho(\beta r_1+h_1)}\right|$ *be a threshold. Then, if*

– $\mathcal{R}_0^{J+A} > 1$ *and* $r_2 < h_2$ *cod population (juvenile and adult) persists as* $t \to \infty$.

– $\mathcal{R}_0^{J+A} < 1$, *cod population may extinct or persists uniformly.*

Proof. As by Garrione and Rebelo (2016) [41], we found $\mathcal{F}$ and V

$\mathcal{F} = \begin{pmatrix} \frac{(h_2K_2-K_2r_2)\omega_3}{r_2\theta} & 0 \\ 0 & 0 \end{pmatrix}$ and $V = \begin{pmatrix} \rho & r_1\beta \\ -\rho & h_1 \end{pmatrix}$.

Then the threshold value for cod (juvenile and adult), $\mathcal{R}_0^{J+A}$ is the spectral radius of matrix $\mathcal{F}V^{-1}$, *i.e.*,

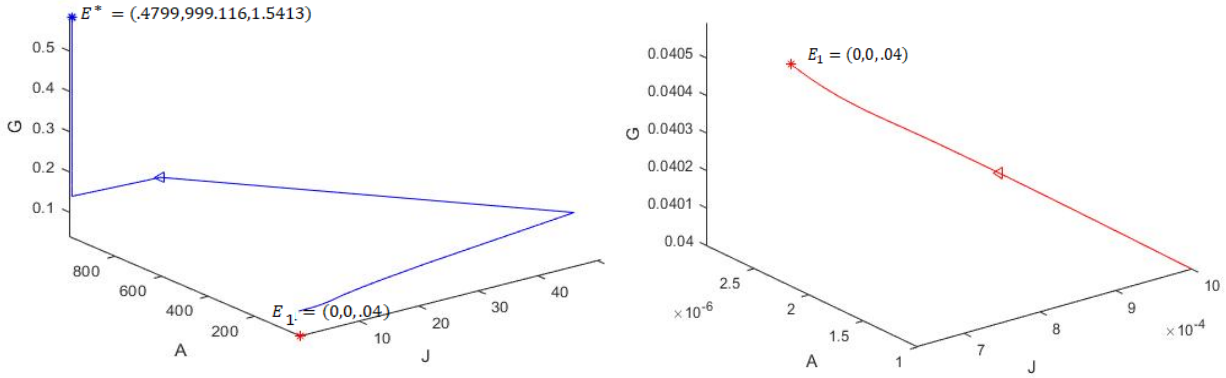
$$\mathcal{R}_0^{J+A} = \left|\frac{h_1K_2(h_2-r_2)\omega_3}{r_2\theta\rho(\beta r_1+h_1)}\right|.$$

□

Example 4.2. For $r_1 = 0.49$, $r_2 = 1.2$, $h_1 = 0.12$, $h_2 = 1.1994$, $\omega_1 = 0.9$, $\omega_2 = 0.25, \omega_3 = 1.1$, $\omega_4 = 0.95$, $\theta = 25$, $\rho = 0.00039$, $K_1 = 1000$, $K_2 = 100$, $\beta = 1.5$, we have $\mathcal{R}_0^{J+A} = 0.791723 < 1$. Here, we observe the existence and stability of two equilibrium points, (i) $E_1 = (0, 0, 0.05)$, extinction equilibrium, which is attained as we set the initial condition as [0.001, 0.000001, 0.04] (*cf.* Fig. 2a), and (ii) $E^* = (0.4799, 999.116, 1.5413)$, positive equilibrium, which is attained as we set the initial condition as [0.1, 10, 0.1] (*cf.* Fig. 2b). Hence the two stable equilibrium can exist at $R_0^{A+J} < 1$. This means survival is possible for a cannibalistic population even if $R_0^{J+A} < 1$. This analysis is in agreement with [23]. They showed that a cannibalistic population can survive under a strong Allee effect when $R_0^{J+A} < 1$.
Now, if we take $h_2 = 1.21 > r_2$ and all the parameters as the same as above, then we observe that $\mathcal{R}_0^{J+A} = 13.1954 > 1$. In this case, the equilibrium E_1 does not exist and E^* is locally asymptotically stable. Therefore, any initial condition in the basin of attraction of E^* will be attracted to E^*, and the dynamics go to non extinction of cod (*cf.* Fig. 3). This situation is favorable as it guarantees the persistence of all species.

Allee effect has tremendous potential on population survival and extinction that remains unexplored in most marine species. This limits our ecological understanding and probable success in conserving marine species. This analysis shows that adult and juvenile survival can benefit from the Allee effect and provide survival even when $R_0^{J+A} < 1$. This is because as adult cod decline due to the Allee effect, in turn, juvenile gets a better place to

survive due to less cannibalism and eventually become fully grown adults. This interesting phenomenon suggests that if an initial threshold is maintained in a restrictive domain, the population can survive indefinitely even if $R_0^{J+A} < 1$.



(a) (b)

FIGURE 2. (a) Stability of equilibrium E^* for $R_0^{J+A} < 1$ with the initial condition [0.1, 10, 0.1]. (b) Stability of equilibrium E_1 for $R_0^{J+A} < 1$ with the initial condition [0.001, 0.000001, 0.04].

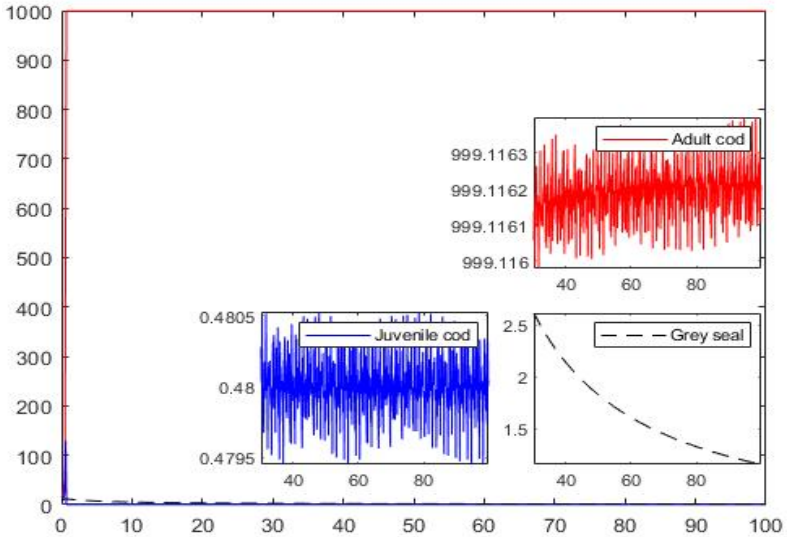


FIGURE 3. Existence and Stability of equilibrium E^* equilibrium for $R_0^{J+A} > 1$ with any initial condition.

4.2. Optimal control analysis

As predicted by a few scientists, Atlantic cod will essentially go extinct within 20 years, despite the best attempts to manage it (https://www.sciencemag.org/news/2008/11/no-recovery-atlantic-cod-population). For this reason, more possible efforts are needed to reduce the risk of population extinction. We aim to minimize Atlantic cod's reduction while keeping the control strategy implementation cost low. For this purpose, we

include the control function $u_j's$ in our model. Control functions for such models are primarily functions of time appearing as coefficients in the model. While formulating an optimal control problem, an essential decision is to decide how and where to introduce the control in the system [42]. A planned strategy is required to reduce the declination of Atlantic cod. Therefore, we include controls (i) u_1, which represents efforts to increase artificial breeding, (ii) u_2, which represents the effort to isolate juvenile and adult cod populations, and (iii) u_3, which represents the effort to increase grey seal harvesting. The dynamics of the system (2.4)–(2.6) with control components is governed by the system of differential equations as follows:

$$\frac{\mathrm{d}J}{\mathrm{d}t} = (r_1 + u_1)A\left(1 - \frac{A}{K_1}\right)(A + \beta) - (\omega_1 - u_2)AJ - \frac{\omega_3 GJ}{\theta + J} - \rho J, \tag{4.1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \rho J + \omega_2 AJ - h_1 A, \tag{4.2}$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = r_2 G\left(1 - \frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta + J} - (h_2 + u_3)G, \tag{4.3}$$

subjected to the initial conditions $J(0) = J_0, A(0) = A_0, G(0) = G_0$. Next, by using bounded measurable control, we define our objective function as

$$\mathcal{J}(u_1, u_2, u_3) = \int_0^T (C_1 G(t) + C_2 u_1^2 + C_3 u_2^2 + C_4 u_3^2)\mathrm{d}t,$$

subject to the state system (4.1)–(4.3). The Lagrangian and Hamiltonian are calculated to find an optimal solution for the optimal control problem. The Lagrangian of the control problem is given by,

$$L = (C_1 G(t) + C_2 u_1^2 + C_3 u_2^2 + C_4 u_3^2)\mathrm{d}t.$$

The coefficients C_1, C_2, C_3, C_4 represent the balancing cost factors for grey seals and control strategies.

The optimal control problem involves finding u_1^*, u_2^*, u_3^* such that the associated state trajectories (J^*, A^*, G^*) is the solution of the controlled system of equation in the interval $[0, T]$ with the initial conditions and minimizing the cost functional $\mathcal{J}$. $\mathcal{J}(u_1, u_2, u_3) = min_{u_1,u_2,u_3\in\Delta}\{\mathcal{J}(u_1, u_2, u_3)|u_1, u_2, u_3 \in U\}$, where Δ is the set of admissible controls given by $\Delta = \{(u_1, u_2, u_3) \in L^1[0, T]|0 \le u_1 \le 1, 0 \le u_2 \le 1, 0 \le u_3 \le 1\}, t \in [0, T]$.

Pontryagin's Maximum Principle converts the system into a problem of minimizing a Hamiltonian function (H), defined as

$$\begin{aligned} H = {} & C_1 G(t) + C_2 u_1^2 + C_3 u_2^2 + C_4 u_3^2 + \lambda_J \left((r_1 + u_1)\, A\left(1 - \frac{A}{K_1}\right)(A + \beta) - (\omega_1 - u_2)AJ - \frac{\omega_3 GJ}{\theta + J} - \rho J\right) \\ & + \lambda_A\left(\rho J + \omega_2 AJ - h_1 A\right) + \lambda_G\left(r_2 G\left(1 - \frac{G}{K}\right) + \frac{\omega_4 GJ}{\theta + J} - (h_2 + u_3)G\right), \end{aligned}$$

where $\lambda_J, \lambda_A, \lambda_G$ are adjoint variables. Hence, the minimization function becomes:

$$H\left(J^*, A^*, G^*, \lambda_J(t), \lambda_A(t), \lambda_G(t), u_1, u_2, u_3\right),$$

where J^*, A^*, G^* be optimal state solution.

By corollary 4.1 of [43], we say that optimal control exists, and by applying Pontryagin's Maximum Principle [44], we obtain the following result:

Theorem 4.3. *For optimal control* $(u_1^*,\ u_2^*,\ u_3^*)$ *and solution* $J^*,\ A^*$ *and* G^* *of state system (4.1)–(4.3), adjoint variable* $\prod = (\lambda_J, \lambda_A, \lambda_G)$ *exists and satisfies,*

$$\begin{aligned}
\frac{d\lambda_J}{\mathrm{d}t} &= A\left(u_2+\omega_1\right)\lambda_J - \omega_2 A\lambda_A + \frac{G\theta}{\left(\theta+J\right)^2}\left(\omega_3\lambda_J - \omega_4\lambda_G\right) + \rho\left(\lambda_J - \lambda_A\right),\\
\frac{d\lambda_A}{\mathrm{d}t} &= \frac{\left(3A^2 - \beta K_1 - 2A\left(K_1-\beta\right)\right)\left(r_1+u_1\right)}{K_1}\lambda_J + h_1\lambda_A + J\left(\left(\omega_1-u_2\right)\lambda_J - \omega_2\lambda_A\right),\\
\frac{d\lambda_G}{\mathrm{d}t} &= -C_1 + \left(h_2 - r_2 + u_3\right)\lambda_G + \frac{2Gr_2}{K_2}\lambda_G + \frac{J}{\theta+J}\left(\omega_3\lambda_J - \omega_4\lambda_G\right),
\end{aligned} \tag{4.4}$$

with transversality conditions: $\lambda_J(T)=0, \lambda_A(T)=0, \lambda_G(T)=0$, *and,* $\lambda_J^*, \lambda_A^*, \lambda_G^*$ *be the solution of system (4.4).*

Further,

$$\begin{aligned}
u_1^*(t) &= \max\left\{0, \min\left\{1, \frac{A^*\left(\frac{A^*}{K_1}-1\right)\left(A^*+\beta\right)\lambda_J^*}{2C_2}\right\}\right\},\\
u_2^*(t) &= \max\left\{0, \min\left\{1, \frac{A^*J^*\lambda_J^*}{2C_3}\right\}\right\},\\
u_3^*(t) &= \max\left\{0, \min\left\{1, \frac{G^*\lambda_G^*}{2C_4}\right\}\right\}.
\end{aligned} \tag{4.5}$$

Proof. The adjoint system can be obtained by

$$\frac{d\lambda_J}{\mathrm{d}t} = -\frac{\partial H}{\partial J}, \frac{d\lambda_A}{\mathrm{d}t} = -\frac{\partial H}{\partial A}, \frac{d\lambda_G}{\mathrm{d}t} = -\frac{\partial H}{\partial G}.$$

For optimality conditions, we differentiate H with respect to optimal controls u_1, u_2, u_3 and equate it to 0. On solving, we get (4.5) in $\Delta = \{(u_1,u_2,u_3) \in L^1[0,T] |\ 0 \le u_1 \le 1, 0 \le u_2 \le 1, 0 \le u_3 \le 1\},\ t \in [0,T]$. Hence, the proof. □

5. Extension of model: stochastic and reaction-diffusion

To verify the robustness of the Allee effect among the cannibalistic populations, we will expand our baseline model (2.1)–(2.3) and (2.4)–(2.6) in two different ways. Firstly, we incorporate fluctuations, representing potential variations in temperature, resource availability, or other environmental factors. Secondly, we consider a spatially explicit version of the model (2.1)–(2.3) and (2.4)–(2.6) that considers animal movement (regarded here as diffusion). At this point, we again emphasize that our primary objective is to gain insights into how Allee affects the cannibalistic population. The inclusion of these extensions aims to bolster our hypothesis in conjunction with the previously described model (2.4)–(2.6).

Stochastic model without Allee effect: Studying stochastic differential equations (SDE) is useful in understanding the responses of cod species to temperature variability and is essential to predict future climate change impacts in the oceans [45]. Therefore, it is relevant and crucial to study the stochastic model. We incorporate white noise to evaluate the effect of a randomly fluctuating environment. We assume fluctuations in seals' growth rate and contact rate of adult and juvenile cod as follows,

$$\omega_1 \to \omega_1 + \sigma_1 \mathrm{d}W_1(t),\ \omega_2 \to \omega_2 + \sigma_1 \mathrm{d}W_1(t),\ r_2 \to r_2 + \sigma_2 \mathrm{d}W_2(t),$$

where W_1, W_2 are mutually independent Brownian motions. σ_1, σ_2 represent the white noise intensities. Thus, the deterministic model (2.1)–(2.3) take the following stochastic form,

$$\mathrm{d}J = \left(r_1 A\left(1-\frac{A}{K_1}\right) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta+J} - \rho J\right)\mathrm{d}t + \sigma_1 JA\mathrm{d}W_1(t), \tag{5.1}$$

$$\mathrm{d}A = (\rho J + \omega_2 AJ - h_1 A)\mathrm{d}t + \sigma_1 JA\mathrm{d}W_1(t), \tag{5.2}$$

$$\mathrm{d}G = \left(r_2 G\left(1-\frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta+J} - h_2 G\right)\mathrm{d}t + \sigma_2 G\mathrm{d}W_2(t), \tag{5.3}$$

Stochastic model with Allee effect:

$$\mathrm{d}J = \left(r_1 A\left(1-\frac{A}{K_1}\right)(A+\beta) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta+J} - \rho J\right)\mathrm{d}t + \sigma_1 JA\mathrm{d}W_1(t), \tag{5.4}$$

$$\mathrm{d}A = (\rho J + \omega_2 AJ - h_1 A)\mathrm{d}t + \sigma_1 JA\mathrm{d}W_1(t), \tag{5.5}$$

$$\mathrm{d}G = \left(r_2 G\left(1-\frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta+J} - h_2 G\right)\mathrm{d}t + \sigma_2 G\mathrm{d}W_2(t), \tag{5.6}$$

For both the above described model we have, $(J(0), A(0), G(0)) \in \mathbb{R}^3_+$. Also, $w = \{W_1(t), W_2(t) : t \geq 0\}$ represents standard Brownian motion defined on a complete probability space $\{\Omega, F, P\}$ with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions.

Reaction-diffusion model without Allee effect: The spatial component of ecological interactions has been identified as an important factor in how the ecological communities are shaped. We introduce spatial variations to the model system (2.1)–(2.3). We assume all populations in the food chain perform active movement in 2-dimensional space (x and y direction). Animals move randomly because of various requirements like finding food, better opportunities for social interactions, *etc.* If all are confined to a fixed bounded domain Ω in $\mathbb{R}^3_+$, we led to consider the following reaction–diffusion system:

$$\frac{\partial J}{\partial t} - D_J\nabla^2 J = r_1 A\left(1-\frac{A}{K_1}\right) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta+J} - \rho J, \quad z\in\Omega, t>0 \tag{5.7}$$

$$\frac{\partial A}{\partial t} - D_A\nabla^2 A = \rho J + \omega_2 AJ - h_1 A, \quad z\in\Omega, t>0 \tag{5.8}$$

$$\frac{\partial G}{\partial t} - D_G\nabla^2 G = r_2 G\left(1-\frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta+J} - h_2 G. \quad z\in\Omega, t>0 \tag{5.9}$$

Reaction-diffusion model with Allee effect:

$$\frac{\partial J}{\partial t} - D_J\nabla^2 J = r_1 A\left(1-\frac{A}{K_1}\right)(A+\beta) - \omega_1 AJ - \frac{\omega_3 GJ}{\theta+J} - \rho J, \quad z\in\Omega, t>0 \tag{5.10}$$

$$\frac{\partial A}{\partial t} - D_A\nabla^2 A = \rho J + \omega_2 AJ - h_1 A, \quad z\in\Omega, t>0 \tag{5.11}$$

$$\frac{\partial G}{\partial t} - D_G\nabla^2 G = r_2 G\left(1-\frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta+J} - h_2 G. \quad z\in\Omega, t>0 \tag{5.12}$$

for both models, $z = (x, y) \in \Omega = [0, L] \times [0, L]$, $t > 0$, and with boundary conditions:

$$(\mathbf{n}.\nabla)J = (\mathbf{n}.\nabla)A = (\mathbf{n}.\nabla)G = 0, \quad z\in\partial\Omega, t>0$$

and initial conditions:

$$\begin{aligned} J(z,0) &= J(0) > 0, \\ A(z,0) &= A(0) > 0, \\ G(z,0) &= G(0) > 0. \end{aligned} \tag{5.13}$$

In system (5.7)–(5.9) and (5.10)–(5.12), ∇^2 represents Laplacian operator. In the above, the vector $\mathbf{n}$ is an outward unit normal vector to the boundary ($\partial\Omega$) of the habitat (Ω), and the homogeneous Neumann boundary conditions are considered. The homogeneous Neumann boundary conditions signify that the system is self-contained and there is no population flux across the boundary ($\partial\Omega$).

5.1. Boundedness and persistence

We will now show both the extended models' existence and the boundedness of solutions.

5.1.1. SDE model (5.1)–(5.3): Existence of unique nonnegative solution

To explore the dynamic behavior of our stochastic model, the primary consideration revolves around verifying nonnegativity and the solution's global existence. Therefore, in this subsection, we initially establish the global and nonnegative solution for the system (5.1)–(5.3). It is well-established that for a stochastic differential equation to possess a unique global solution (*i.e.*, no explosion in a finite time) for any given initial value, the coefficients of the equation typically need to adhere to both the linear growth condition and the local Lipschitz condition [46]. However, the coefficients of system (5.1)–(5.3) do not conform to the linear growth condition despite being locally Lipschitz continuous. Consequently, there exists the possibility that the system solution (5.1)–(5.3) may explode at a finite time [46].

Theorem 5.1. *For any given initial value $(J_0, A_0, G_0) \in \mathbb{R}^3_+$, there exists a unique positive solution (J_t, A_t, G_t) to system (5.1)–(5.3) on $t \geq 0$ and the solution will remain $\mathbb{R}^3_+$ with probability 1, that is to say, $(J_t, A_t, G_t) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely.*

Proof. Since the coefficients of our SDE equations satisfy the Local Lipschitz condition, then for any initial value $(J_0, A_0, G_0) \in \mathbb{R}^3_+$ there is a unique local solution $(J_t, A_t, G_t) \in \mathbb{R}^3_+$ on $t \in [0, \tau_e)$ where τ_e is the explosion time [46]. To establish this solution is global, it is only necessary to demonstrate that $\tau_e = \infty$ a.s. Firstly, we demonstrate that J_t, A_t, and G_t do not blow up within a finite time. To achieve this, let $l_0 > 0$ be sufficiently large for every element of $(J_0, A_0, G_0) \in \mathbb{R}^3_+$ falling within the interval $[\frac{1}{l_0}, l_0]$. For each integer $l \geq l_0$, define the stopping time as [46]

$$\tau_l = \inf \left\{ t \in [0, \tau_e) : \min \{J_t, A_t, G_t\} \leq \frac{1}{l} \text{ or } \max \{J_t, A_t, G_t\} \geq l \right\},$$

where τ_l is increasing as $l \to \infty$. We specify that in this section, we set $\inf \emptyset = \infty$, where $\emptyset$ represents the empty set. We denote $\tau_\infty = \lim_{l \to \infty} \tau_l$, whence $\tau_\infty \geq \tau_e$ a.s. if $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and for t nonnegative, $(J_t, A_t, G_t) \in \mathbb{R}^3_+$ a.s. To finalize our proof, we must prove that $\tau_\infty = \infty$. To end this, we use the method of proof by contradiction. If this assertion is false, then there is $\epsilon \in (0, 1)$ and $T > 0$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \epsilon,$$

this implies that there is an integer $l_1 \geq l_0$ so that for all $l \geq l_1$

$$\mathbb{P}(\tau_l \leq T) \geq \epsilon. \tag{5.14}$$

We define a nonnegative C^2 function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ by the following expression

$$V(J, A, G) = (J - 1 - \log(J)) + (A - 1 - \log(A)) + (G - 1 - \log(G)),$$

Thanks to Itô's formula, we have

$$\mathrm{d}V(J_t, A_t, G_t) = V_J + V_A + V_G,$$

where,

$$V_J = \Big(r_1 A(1 - \frac{1}{J})(1 - \frac{A}{K_1}) + \omega_1 A(1 - J) + \frac{\omega_3 G}{\theta + J}(1 - J) + \rho(1 - J) + \frac{1}{2}\sigma_1^2 A^2\Big)\mathrm{d}t + \sigma_1 A(J - 1)\mathrm{d}W_1(t),$$

$$V_A = \Big(eJ(1 - \frac{1}{A}) + \omega_2 J(A - 1) + h_1(1 - A) + \frac{1}{2}\sigma_1^2 J^2\Big)\mathrm{d}t + \sigma_1 J(A - 1)\mathrm{d}W_1(t),$$

and

$$V_G = \Big(r_2(G - 1)(1 - \frac{G}{K_2}) + \frac{\omega_4 J}{\theta + J}(G - 1) + h_2(1 - G) + \frac{1}{2}\sigma_2^2\Big)\mathrm{d}t + \sigma_2(G - 1)\mathrm{d}W_2(t).$$

This implies that

$$\mathrm{d}V(J_t, A_t, G_t) = \alpha(J, A, G)\mathrm{d}t + \beta(J, A, G)\mathrm{d}W_1(t) + \gamma(J, A, G)\mathrm{d}W_2(t),$$

where,

$$\alpha(J, A, G) = r_1 A(1 - \frac{1}{J})(1 - \frac{A}{K_1}) + \omega_1 A(1 - J) + \frac{\omega_3 G}{\theta + J}(1 - J) + \rho(1 - J) + \frac{1}{2}\sigma_1^2 A^2$$

$$+eJ(1 - \frac{1}{A}) + \omega_2 J(A - 1) + h_1(1 - A) + \frac{1}{2}\sigma_1^2 J^2 + r_2(G - 1)(1 - \frac{G}{K_2}) + \frac{\omega_4 J}{\theta + J}(G - 1) + h_2(1 - G) + \frac{1}{2}\sigma_2^2,$$

$$\beta(J, A, G) = \sigma_1(2AJ - A - J),$$

and

$$\gamma(J, A, G) = \sigma_2(G - 1).$$

Following calculus analysis, we establish that $\alpha(J, A, G)$ is upper bounded by a positive constant M wherein

$$\mathrm{d}V(J_t, A_t, G_t) \leq M\mathrm{d}t + \beta(J, A, G)\mathrm{d}W_1(t) + \gamma(J, A, G)\mathrm{d}W_2(t).$$

The rest of the proof aligns with the methodology presented by Dalal *et al.* [47], hence taking expectations from 0 to $\tau_l \wedge T$, yields.

$$\mathbb{E}(V(J(\tau_l \wedge T), A(\tau_l \wedge T), G(\tau_l \wedge T))) \leq V(J_0, A_0, G_0) + MT.$$

Setting $S_l := \{\tau_l \leq T\}$ for $l \geq l_1$, using (5.14) this implies that $\mathbb{P}(S_l) \geq \epsilon$. For every $w \in S_l$, there is $J(\tau_l, w)$ or $A(\tau_l, w)$ or $G(\tau_l, w)$ equal to either l or $\frac{1}{l}$. Therefore, $V(J(\tau_l, w), A(\tau_l, w), G(\tau_l, w)$ is not inferior to either

$$l - 1 - log(l),$$

or

$$\frac{1}{l} - 1 + log(l).$$

Hence,

$$V(J(\tau_l, w), A(\tau_l, w), G(\tau_l, w) \geq (l - 1 - log(l)) \wedge (\frac{1}{l} - 1 + log(l).$$

We then have,

$$V(J_0, A_0, G_0) + MT \geq \mathbb{E}(\mathbb{1}_{S_l}(w) V(J(\tau_l \wedge w), A(\tau_l \wedge w), G(\tau_l \wedge w)))$$

$$V(J_0, A_0, G_0) + MT \geq \epsilon\big((l - 1 - log(l)) \wedge (\frac{1}{l} - 1 + log(l))\big),$$

hence, $l \to \infty$ yields to absurdity $\infty > V(J_0, A_0, G_0) + MT = \infty$. Therefore, τ_∞ should be equal to ∞ a.s. □

5.1.2. Reaction-diffusion model (5.7)–(5.9): Dissipation & Persistence

Theorem 5.2. *All non-negative solution (J, A, G) of model (5.7)–(5.9) with initial condition (5.13), satisfies the following inequality*

$$\limsup_{t\to\infty} \max_{z\in\bar{\Omega}} J(z,t) + A(z,t) \leq max\{K_1, \max_{\bar{\Omega}}(J_0(z) + A_0(z))\}, \tag{5.15}$$

$$\limsup_{t\to\infty} \max_{z\in\bar{\Omega}} G(z,t) \leq max\{K_2 \left(\frac{r_2 - h_2 + \omega_4}{r_2}\right), \max_{\bar{\Omega}}(G_0(z))\} \tag{5.16}$$

Proof. By virtue of (5.7) and (5.8) we have the following inequality,

$$\frac{\partial J}{\partial t} \leq D_J \nabla^2 J + r_1 A \left(1 - \frac{A}{K_1}\right) - \rho J - \omega_1 A J, \tag{5.17}$$

$$\frac{\partial A}{\partial t} \leq D_A \nabla^2 A + \rho J + \omega_2 A J. \tag{5.18}$$

Since system (5.17)–(5.18) is cooperative, therefore by using the comparison principle of parabolic equation [48], one can show that $J(x,t), A(x,t)$ is a subsolution of the problem

$$\frac{d\bar{J}}{dt} = r_1 \bar{A} \left(1 - \frac{\bar{A}}{K_1}\right) - \rho \bar{J} - \omega_1 \bar{A} \bar{J}, \tag{5.19}$$

$$\frac{d\bar{A}}{dt} = \rho \bar{J} + \omega_2 \bar{A} \bar{J}, \tag{5.20}$$

Let $C = \bar{A} + \bar{J}$. Adding (5.19)–(5.20) and performing straightforward computations, we get

$$\frac{d(\bar{J}+\bar{A})}{\mathrm{d}t} = r_1\bar{A}\left(1-\frac{\bar{A}}{K_1}\right) + (\omega_2-\omega_1)\bar{A}\bar{J}, \tag{5.21}$$

$$\frac{d(\bar{J}+\bar{A})}{\mathrm{d}t} \leq r_1(\bar{A}+\bar{J})\left(1-\frac{\bar{A}+\bar{J}}{K_1}\right) + (\omega_2-\omega_1)\bar{A}\bar{J}, \tag{5.22}$$

$$\frac{dC}{\mathrm{d}t} \leq r_1 C\left(1-\frac{C}{K_1}\right), (\text{since } \ \omega_1 > \omega_2) \tag{5.23}$$

Again, using the comparison principle, one can show $C(t)$ is a subsolution of the following problem

$$\frac{d\bar{C}}{\mathrm{d}t} = r_1\bar{C}\left(1-\frac{\bar{C}}{K_1}\right) \tag{5.24}$$

We observe that the positive constant

$$\bar{C} = max\{K_1, \max_{\bar{\Omega}}(J_0(z)+A_0(z))\} \tag{5.25}$$

is the superposition to (5.19)–(5.20). Therefore,

$$\bar{J}+\bar{A} \leq C(t) \leq \bar{C}(t). \tag{5.26}$$

Also, $\lim_{t\to\infty}\bar{C}(t) \leq K_1$. Thus, from a well known comparison principle for parabolic equations, we finally have

$$J(z,t)+A(z,t) \leq \bar{J}(t)+\bar{A}(t) \leq max\{K_1, \max_{\bar{\Omega}}(J_0(z)+A_0(z))\} := \bar{C}(t), \forall z \in \Omega, t \geq 0.$$

Hence, we have

$$J(z,t)+A(z,t) \leq \bar{J}(t)+\bar{A}(t) \leq max\{K_1, \max_{\bar{\Omega}}(J_0(z)+A_0(z))\} := \bar{C}(t), \forall z \in \Omega, t \geq 0.$$

This implies

$$\limsup_{t\to\infty}\max_{z\in\bar{\Omega}} J(z,t)+A(z,t) \leq max\{K_1, \max_{\bar{\Omega}}(J_0(z)+A_0(z))\} = \bar{U}_1. \tag{5.27}$$

Next we will consider (5.9)

$$\frac{\partial G}{\partial t} = D_G\nabla^2 G + r_2 G\left(1-\frac{G}{K_2}\right) + \frac{\omega_4 GJ}{\theta+J} - h_2 G, \tag{5.28}$$

$$\frac{\partial G}{\partial t} \leq D_G\nabla^2 G + r_2 G\left(1-\frac{G}{K_2}\right) + \omega_4 G - h_2 G. \tag{5.29}$$

By the comparison principle of the parabolic equation, one can show that $G(z,t)$ is a subsolution of the following problem,

$$\frac{d\bar{G}}{dt} = r_2\bar{G}\left(1-\frac{\bar{G}}{K_2}\right) + (\omega_4-h_2)\bar{G} \tag{5.30}$$

We observe that the positive constant

$$\bar{G} = max\{K_2\left(\frac{r_2 - h_2 + \omega_4}{r_2}\right), \max_{\bar{\Omega}}(G_0(z))\}, \tag{5.31}$$

Also, $lim_{t\to\infty}\bar{G}(t) \leq K_2\left(\frac{r_2-h_2+\omega_4}{r_2}\right)$. Thus, from a well-known comparison principle for parabolic equations, we finally have $G(z,t) \leq \bar{G}(t) \leq K_2\left(\frac{r_2-h_2+\omega_4}{r_2}\right), \forall z \in \Omega, t \geq 0$. This implies

$$\limsup_{t\to\infty} \max_{z\in\bar{\Omega}} G(z,t) \leq max\{K_2\left(\frac{r_2 - h_2 + \omega_4}{r_2}\right), \max_{\bar{\Omega}}(G_0(z))\} = \bar{U}_2. \tag{5.32}$$

□

Definition [49]: The spatial model (5.7)–(5.9) is said to have the persistence property if for any nonnegative initial data $(J_0(z), A_0(z), G_0(z))$, there exists positive constants $\epsilon_1 = \epsilon_i(J_0(z), A_0(z), G_0(z))$ for $i = 1, 2, 3$ such that corresponding solution, (J, A, G) of model (5.7)–(5.9) satisfies,

$$\liminf_{t\to\infty} \min_{z\in\bar{\Omega}} J(z,t) \geq \epsilon_1, \liminf_{t\to\infty} \min_{z\in\bar{\Omega}} A(z,t) \geq \epsilon_2, \liminf_{t\to\infty} \min_{z\in\bar{\Omega}} G(z,t) \geq \epsilon_3. \tag{5.33}$$

Theorem 5.3. *Assume that if*

$$r_2 > h_2, \omega_2 > h_1, K_1 > \omega_2 - h_1 \tag{5.34}$$

holds, then system (5.7)–(5.9) *has the persistence property.*

Proof. From equation (5.9), we have

$$\frac{\partial G}{\partial t} \geq D_G\nabla^2 G + G\left(r_2\left(1 - \frac{G}{K_2}\right) - h_2\right) \tag{5.35}$$

Since (5.34) holds, then for small enough $\epsilon > 0$ chosen

$$K_2\left(\frac{r_2 - h_2}{r_2}\right) - \epsilon > 0 \tag{5.36}$$

Hence, there exists $t_2 > t_1$ such that for any $t > t_2$,

$$G(x,t) \geq \underline{L_1},$$

where $\underline{L_1} = K_2\left(\frac{r_2 - h_2}{r_2}\right) - \epsilon$.
Note that from (5.8) we can have the following inequality,

$$\begin{aligned} &\frac{\partial A}{\partial t} - D_A\nabla^2 A \geq \omega_2 A - h_1 A - A^2, \\ &\frac{\partial A}{\partial t} - D_A\nabla^2 A \geq (\omega_2 - h_1 - A)A. \end{aligned} \tag{5.37}$$

Then there exists $t_3 > t_2$ such that for any $t > t_3$,

$$A(x,t) \geq \underline{L_2},$$

where

$$\underline{L_2} = \omega_2 - h_1.$$

Now from (5.7) we have

$$\begin{aligned} \frac{\partial J}{\partial t} - D_J \nabla^2 J &\geq r_1 A \left(1 - \frac{A}{K_1}\right) - (\omega_1 A + \omega_3 G + \rho) J, \\ \frac{\partial J}{\partial t} - D_J \nabla^2 J &\geq r_1 \underline{L_2} \left(1 - \frac{\underline{L_2}}{K_1}\right) - (\omega_1 \bar{U_1} + \omega_3 \bar{U_2} + \rho) J \end{aligned} \tag{5.38}$$

$$\implies J(x,t) \geq \frac{r_1 \underline{L_2} \left(1 - \frac{\underline{L_2}}{K_1}\right)}{\omega_1 \bar{U_1} + \omega_3 \bar{U_2} + \rho}$$

Summarizing, we have the following

$$\begin{aligned} \liminf_{t\to\infty} \min_{\bar{\Omega}} G(x,t) &\geq \underline{L_1}. \\ \liminf_{t\to\infty} \min_{\bar{\Omega}} A(x,t) &\geq \underline{L_2}. \\ \liminf_{t\to\infty} \min_{\bar{\Omega}} J(x,t) &\geq \underline{L_3}. \end{aligned}$$

According to the definition of persistence for species and the above results, we can conclude that system (5.7)–(5.9) is persistent. The proof is completed. □

Note: The analysis of the boundedness and persistence of equations (5.4)–(5.6) and (5.10)–(5.12) follows the same approach as mentioned earlier and is therefore excluded to prevent unnecessary lengthening of the paper. Additionally, the complete solution of the optimal control problem for stochastic and reaction–diffusion models is beyond the scope of the paper and will be considered for future work.

6. Numerical simulations

6.1. Numerical simulation for ODE model (2.1)–(2.3): Without Allee effect

Runge-Kutta's fourth-order method has been used to solve the ODE system (2.1)–(2.3) with non-zero favorable initial conditions. We observe that for the following set of parameters given in Table 2, stable dynamics are observed (*cf.* Fig. 4). There is enough literature on Atlantic cod, even though basic biological parameters are still missing. However, we tried our best to utilize parameter values in the literature. References and ranges are shown in Table 2.

6.1.1. Dynamical behavior for the equilibrium points

For the parameter values given in example 3.2, we found the existence of two positive equilibrium $E_1^* = (0.48,\ 100.1547,\ 43.1580)$ and $E_2^* = (0.4799,\ 18.2116,\ 43.1579)$ as shown previously. Here, we numerically illustrate the local stability of two equilibrium E_1^* and E_2^*. For this, we took several initial values $I_1 = (10, 80, 10)$, $I_2 = (50, 80, 50)$, $I_3 = (50, 150, 5)$ and $I_4 = (10, 150, 60)$ (*cf.* Fig. 5). We observe that all the solution converges to the equilibrium E_1^* but not to E_2^* because of its unstable nature at these parametric

TABLE 2. Parameter values used for simulating model (2.1)–(2.3).

Parameters	Values	Range	References
r_1	0.49	0.24–0.50	*
r_2	1.2	1.1–2.1	**
h_1	0.12	0.1 – −5.479	[50]
h_2	0.4		
K_1	1000		Assumed
K_2	100		Assumed
β	−0.9	0–1	
ω_1	0.9		
ω_2	0.25	$< \omega_1$	
ω_3	1.1	0.8–1.2	[51]
ω_4	0.95	$< \omega_3$	
θ	25		Assumed
ρ	0.00039	Adult age 7year	[50]

*- (https://www.sararegistry.gc.ca/virtual_sara/files/cosewic/sr_Atlantic%20Cod_0810_e1.pdf).
**- (https://www.dfo-mpo.gc.ca/species-especes/profiles-profils/greyseal-phoquesgris-eng.html).

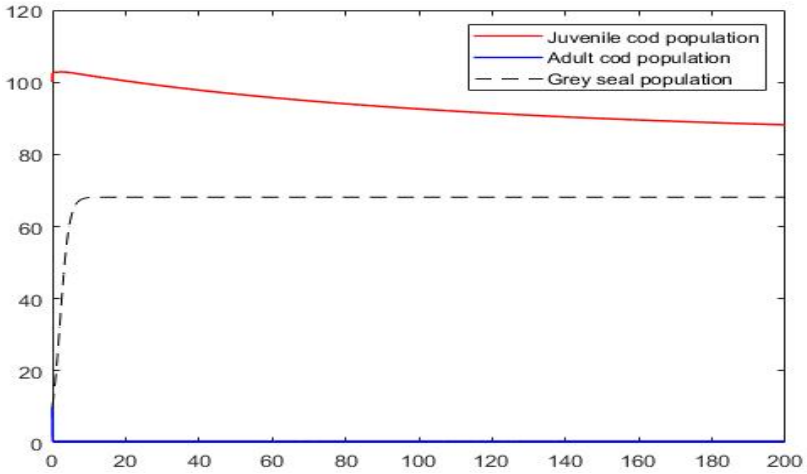


FIGURE 4. Time series solution of system (2.1)–(2.3) taking parameters given in Table 2 and with initial conditions [10, 8, 60].

values (the eigenvalues of variational matrix corresponding to E_1^* are $-91.9559, -0.519, -0.0109$ (all negative) and E_2^* are $-18.2271, -0.5197, 0.0099$ (not all negative) respectively). The boundary equilibrium points, E_0, E_1, and E_2 also exists but are not stable (*cf.* Fig 5). This analysis shows that even though many equilibrium point exists for a given set of parameter values, only E_1^* is acting as a point attractor. Simulation results are compatible with the analysis done in the previous section.

6.1.2. Persistence and Extinction dynamics

We also performed the simulation experiments and demonstrated the global stability of the model system (2.1)–(2.3) about three equilibrium points E_1, E_2, and E^*. A cod-free equilibrium point E_1 is undesirable as both juvenile and adult cod populations are absent. By numerical simulation, we observe that with the parameter values given in Table 2 and with the initial condition as [10, 8, 60], the Atlantic cod (juvenile and adult) population is extinct (see Fig. 6a). While if we slightly vary r_1 from 0.49 to 0.6, we see that cod population

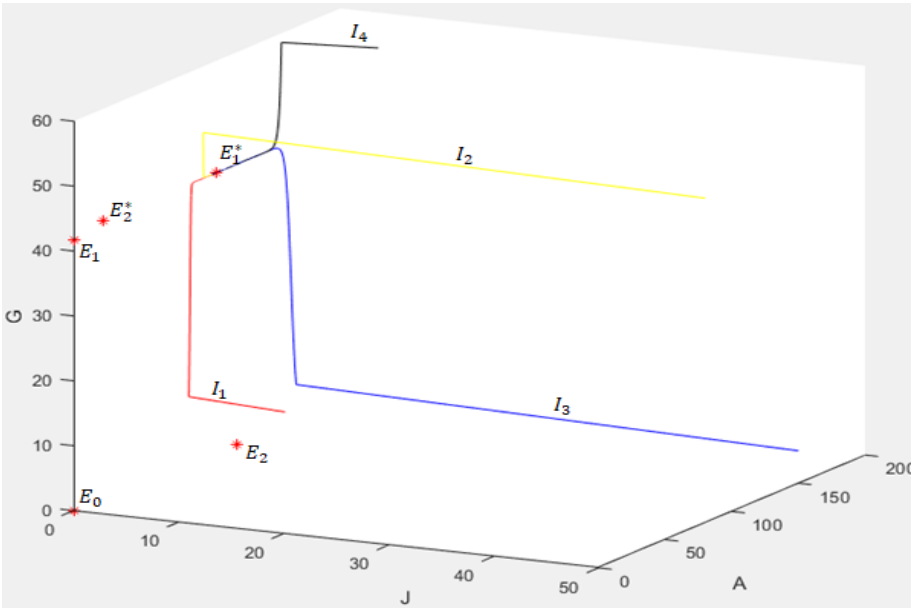


FIGURE 5. Dynamical behaviour of the solution of system (2.1)–(2.3) taking different initial conditions.

persists (see Fig. 6b). Taking the same parameters while increasing the cannibalism rate from 0.9 to 1.1 can again lead juvenile cod to extinction. We illustrated this numerically (see Fig. 6c). Now, taking the same initial population and parameters as in Table 2, while increasing the culling/harvesting rate h_2 from 0.4 to 1.22, the grey seal extinct (see Fig. 6d) permanently providing cod a better place to survive.

From this section, we conclude that increasing the value of r_1 beyond a threshold value cod population will persist forever. Therefore, to save the cod population from becoming extinct, it is suggested to increase the parameter r_1 by artificial and captive breeding. On the other hand, we also note that harvesting grey seals (h_2) above a threshold value can be a measure to control the rapid decline of cod population and thus preserve the remnant population.

6.2. Numerical simulation for ODE model (2.4)–(2.6): With Allee effect

Now, we simulate our model (2.4)–(2.6) with the same parameter as given in Table 2 and then compare our result with the simulation of model (2.1)–(2.3). In Figure 7a, we observe that in the absence of the Allee effect and with initial condition (1, 0.1, 1), the cod population increases following the sigmoid shape. However, with the same set of parameters and initial condition but under the influence of strong Allee effect ($\beta = -0.9$) (*cf.* Fig. 7b), cod species goes to extinction. Now, we change the initial condition to (1, 1, 1) and observe that in the absence of the Allee effect, the cod population persists and approaches 600 (*cf.* Fig. 7c). Now, with the same set of parameters and initial condition but under the influence of the strong Allee effect ($\beta = -0.9$), we find that cod species increases rapidly to its carrying capacity of 1000. This simulation result is in agreement with [23]. They showed that a cannibalistic population could survive under the strong Allee effect when a non-cannibalistic population cannot. These results indicate that Allee's effects outweigh the negative impact and boost the cannibalistic population's growth rate if the initial value exceeds the threshold component. However, in a small population or 'under-crowding,' when the size of the breeding population drops below a certain threshold number, breeding fails, and the population becomes extinct, resulting in inverse density dependence. In other words, a population demonstrating a strong Allee effect will have a critical population size or density below which the population growth rate becomes negative. Therefore, when the population density or size strikes a number

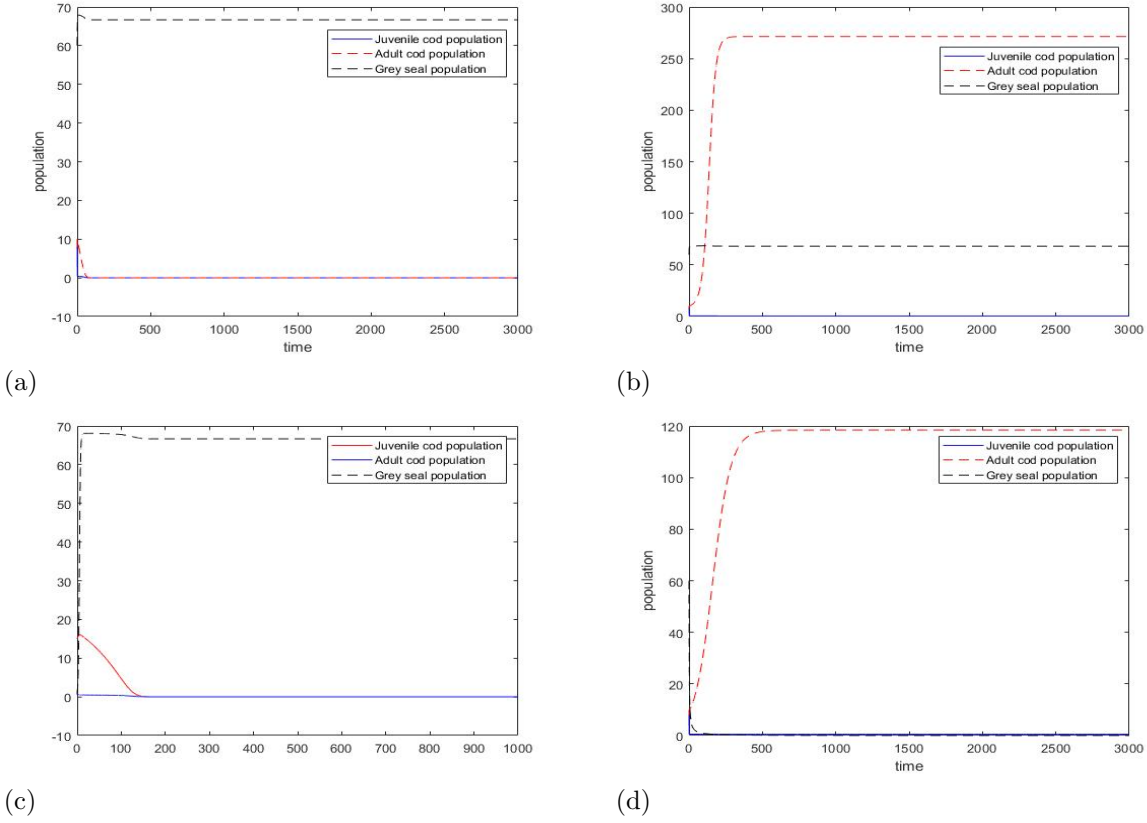


FIGURE 6. (a) Extinction of cod for $r_1 = 0.49$, (b) Persistance of cod for $r_1 = 0.6$, (c) Extinction of Juvenile at $\omega_1 = 1.1$, and (d) Extinction of greay seal at $h_2 = 1.22$. Other parameters are same as given in Table 2.

TABLE 3. Comparison of population densities with/without Allee effect corresponding to Figure 7.

	Without Allee effect and IC= $[1, 0.1, 1]$	With Allee effect and IC= $[1, 0.1, 1]$	Without Allee effect and IC= $[1, 1, 1]$	With Allee effect and IC= $[1, 1, 1]$
J	≈ 0.1276	≈ 0	≈ 0.1277	≈ 0.1276
A	≈ 599.0706	≈ 0	≈ 599.0706	≈ 999.5991
G	≈ 67.0689	≈ 66.6667	≈ 67.0689	≈ 67.0689

below this threshold, the population will be intended for extinction without any supplementary assistance. The tabular comparison has been summarised in Table 3.

6.2.1. Global sensitivity analysis

To determine how best we can reduce Atlantic cod morbidity and mortality due to the Allee effect, cannibalism, and predation by grey seals, we must know the relative importance of different parameters responsible for its death. Cod extinction and persistence can be related to the primary reproduction number, R_0^{J+A}. We calculate the sensitivity indices of the reproductive number, R_0^{J+A}, to the parameters in the model. Here, we

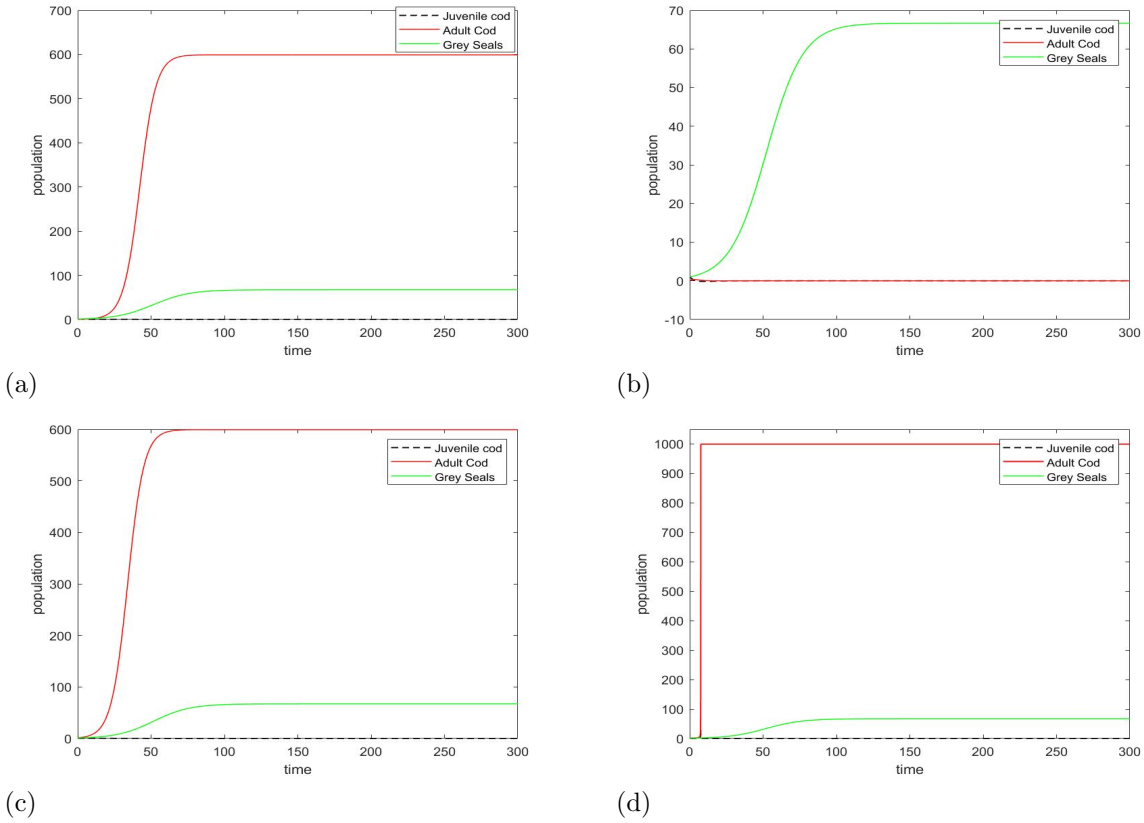


FIGURE 7. (a) Time series solution for the system (2.1)–(2.3) without Allee effect and with the initial condition [1, 0.1, 1]. (b) Time series solution for the system (2.4)–(2.6) with Allee effect and initial condition [1, 0.1, 1]. (c) Time series solution for the system (2.1)–(2.3) without Allee effect and with the initial condition [1, 1, 1]. (d) Time series solution for the system (2.4)–(2.6) with Allee effect and initial condition [1, 1, 1]. Other parameters are given in Table 2.

use the partial rank correlation coefficient (PRCC) method to find the parameters that highly impact R_0^{J+A} and should be targeted by intervention strategies.

For assessing the monotonicity amongst input and output variables while accounting for correlation between input parameters, the PRCC sensitivity technique has been widely utilized ([52–54]). Downing *et al.* [55] delineated that the PRCC is more functional for stipulating the sensitivity of parameters holding a nonlinear but monotonic relationship. PRCC, as a great mathematical work, answers questions such as how the output acts if we increase (or decrease) a certain parameter. Thus, by calculating PRCC, we can assert what parameters we can take to get the desired output value. We begin the uncertainty inspection by initiating the values of parameters and their ranges. Parameters behave as random variables with corresponding probability density function (PDF) if the estimation is undetermined. If no prior (or specific) information about the distribution (of parameters) is given, we proceed by assigning each parameter with some uniform probability function (*cf.* Tab. 4).

The PRCC sensitivity results are calculated for R_0^{J+A}, they are listed in Table 5 and also illustrated using bar charts in Figure 8. The sign of PRCC indicates the direction of association between the input and output factors. PRCCs for all the parameters are significantly different from zero. A value of $+1$ indicates a perfect positive linear relationship, a value of -1 indicates a perfect negative linear relationship, and 0 indicates no relationship. The resulting sensitivity indices of R_0^{J+A} to the 9 different parameters in the model are shown in

TABLE 4. Distributions and ranges of the input parameters.

Parameters	Distribution	Range
h_1	Uniform	[0.001,0.49]
K_2	Uniform	[1,500]
r_2	Uniform	[0.01,0.7]
h_2	Uniform	[0.001,0.9]
ω_3	Uniform	[0.8,1.2]
θ	Uniform	[1,40]
ρ	Uniform	$[10^{-5}, 0.1]$
r_1	Uniform	[0.24, 0.5]
β	Uniform	$[10^{-5}, 2]$

TABLE 5. The PRCC sensitivity indices of the reproduction number (R_0^{J+A}), to the system parameters, p_j.

Parameters (p_j)	PRCC Index
h_1	0.0105
K_2	0.1462
r_2	–0.6999
h_2	0.7455
ω_3	0.0306
θ	–0.1216
ρ	–0.0746
r_1	–0.0328
β	–0.0402

Table 5 (see Fig. 8). From the results in Table 5 we found that the most influential parameter for R_0^{J+A} is h_2 followed by r_2. The most influential parameter is the harvesting rate of seals, h_2. Since, the sensitivity index of h_2 is 0.7455, increasing (decreasing) h_2 by 10% increases (or decreases) the value of R_0^{J+A} by approximately 7.45%. The most negatively influential parameter is the growth rate of seals, r_2. Since, the sensitivity index of r_2 is -0.6999, increasing (decreasing) r_2 by 10% decreases (or increases) the value of R_0^{J+A} by approximately 6.99%. We also found that other parameters are less sensitive for R_0^{J+A}.

The identification of these parameters is vital in formulating control strategies that save the cod population. The result of sensitivity analysis suggests that an approach that reduces the grey seal birth rate (r_2) and increases its harvesting rate (h_2) would be pretty practical in restricting the decline of the cod population. Increasing seal populations worldwide have created resource competition and conflicts between the seals and local commercial fisheries, leading to culling programs with uncertain benefits [56]. In light of the current study, we propose selective culling directed toward the adult male seals that are found to be the most specialized in their foraging tactics. Climate warming may have delayed effects on the reproductive rate of Baltic grey seals, which, in turn, influences the growth rate of the population [57].

6.2.2. Optimal control simulations

In this section, state system (4.1)–(4.3), adjoint system (4.4) and optimal control (4.5) are simulated using MATLAB program. As in optimality, we have the initial condition of the state system as $J(0)$, $A(0)$ and $G(0)$ and terminal condition for adjoint variable $\prod(T) = 0$ in interval $[0, T]$.

Initially we postulate controls u_1, u_2 and u_3. Then, we solve the state system using the Runge-Kutta method forward in time. After that, we solve the adjoint system backward in time, taking terminal conditions and state

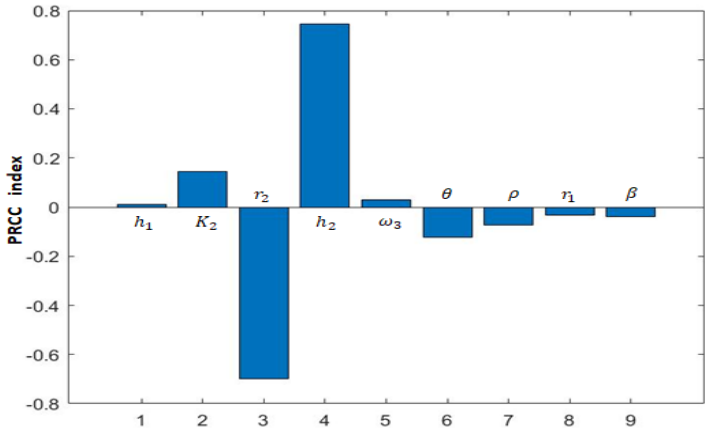


FIGURE 8. Graphical representation of PRCC index for R_0^{J+A}.

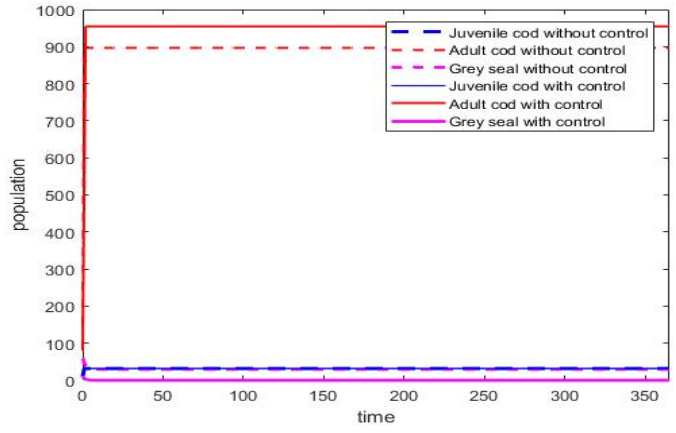


FIGURE 9. The evolution and comparison (with and without control measure) of all species.

system solution at time T, which we just calculated. Through the ongoing process, $u_i's, i = 1, 2, 3$ are updated at each iteration using (4.5). We will continue the above steps until successive iteration convergence is attained. The effect of control compared to the model without control has been shown in Figure 9. We observe that the cod population increases on implementing control. Thus, if we maintain the optimal value of the three discussed controls, (i) efforts to increase artificial breeding, (ii) the effort to isolate juvenile and adult cod populations, and (iii) the effort to increase grey seal harvesting, we can effectively decelerate adult cod death at unusually high rates. Biologically, we can do this by keeping apart juvenile and adult cod by planting more and more eelgrass [58]. Eelgrass is a genus (*Zostera*) of a marine plant with long, grass-like leaves, which grow in coastal waters and brackish areas worldwide. It provides the foundation for entire ecosystems, just as corals do for a coral reef ecosystem. The density of juvenile Atlantic cod was found to be higher in eelgrass meadows than in alternative habitats [58]. Our work mathematically validates that isolating juveniles and adults by any means, like eelgrass meadows, is essential to contributing to Atlantic cod stocks. These findings are of major significance given the continued threats to these systems.

TABLE 6. Parameter values used for simulating model (5.1)–(5.3) and (5.7)–(5.9).

Parameters	Values
r_1	0.49
r_2	0.12
h_1	0.12
h_2	0.04
K_1	1000
K_2	100
β	-0.5
ω_1	1.534
ω_2	0.94
ω_3	1.1
ω_4	0.095
θ	25
ρ	0.00039

6.3. Numerical simulation for stochastic model: without and with Allee effect

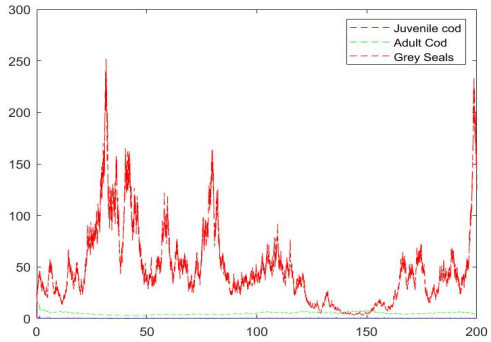
In this section, we numerically simulate system (5.1)–(5.3) and (5.4)–(5.6). We used Euler's Maruyama method [59] to numerically show the effect of the environmental stochasticity at all trophic levels. We took parameters from Table 6 for which all population coexists in deterministic case (*cf.* Fig. 14a).

We took the initial condition where the cod population is above the minimum threshold density and compared the results with or without the Allee effect. From Figure 10, we observed that without Allee, when the environmental stochasticity experienced in Adult and juvenile cod is too high, both cod populations tend to extinction. However, incorporating the Allee effect, we observe more fluctuations among all species, especially the cod population. Moreover, it requires more stochasticity for the extinction of the cod population (*cf.* Fig. 11). Now, without the Allee effect, if the environmental stochasticity experienced by cod is low while the grey population experiences too much environmental stochasticity, then cod will persist. Seals themselves will tend to extinction (Fig. 12). Here also, we observe more fluctuations among all the species in the presence of the Allee effect (*cf.* Fig. 13). This result shows that the persistence and extinction of cod and grey seal species depends on the demographic impacts of the environmental stochasticity of its population. This analysis indicates that both adult and juvenile survival can benefit from the Allee effect and can provide survival even when stochasticity is high. These results strengthen our hypothesis that the Allee effect is beneficial among the cannibalized population like the Atlantic cod with obviously some minimum support of the cod population.

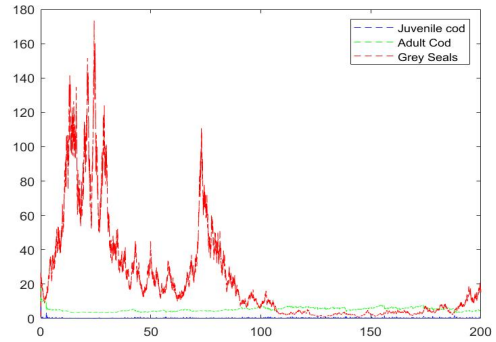
We now compare the stochastic and corresponding deterministic models' results. The main results are given as follows: (i) Taking parameter as in Table 6 and for small environmental stochasticity $\sigma_1 = 0.00009, \sigma_2 = 0.00009$ we found that the density of the adult cod reduced significantly due to deteriorating environment. Moreover, the stable point in the deterministic case remains stable; rather, it fluctuates randomly around some trivial average value (*cf.* Fig. 14b). It is well known that most natural phenomena do not strictly follow deterministic laws but rather oscillate randomly around some average value so that the deterministic equilibrium is no longer a fixed state [60–62].

6.4. Numerical simulation for reaction–diffusion model: without and with Allee effect

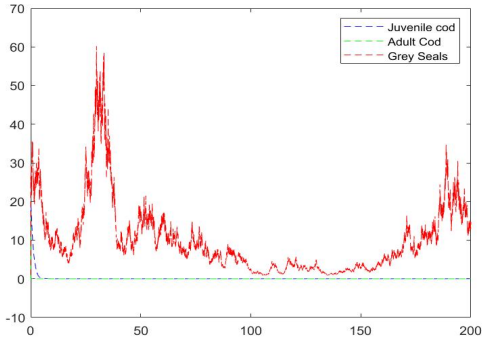
In this section, we present pattern formation for the model system (5.7)–(5.9) in two dimensions. These patterns are computed using a finite difference numerical method for spatial discretization with explicit time-stepping. The initial conditions for all of our calculations are based on small random perturbations of the positive uniform steady state E^*, and homogeneous Neumann (zero-flux) boundary conditions are imposed. All simulations are carried out on 200×200 square domain using a mesh with $\Delta x = 0.25$ and a 10^{-3} time-step.



(a) $\sigma_1 = 0.00009$



(b) $\sigma_1 = 0.0001$



(c) $\sigma_1 = 0.001$

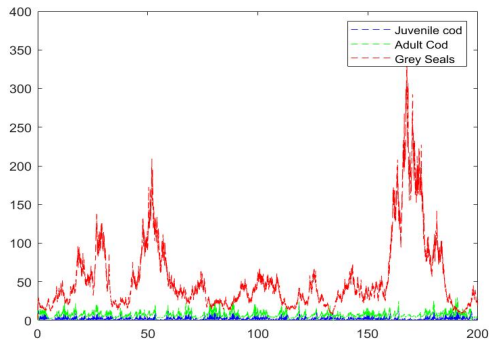
FIGURE 10. Time series of model (5.1)–(5.3) without Allee effect for the parameters given in Table 6 and $\sigma_2 = 0.00009$ (a) $\sigma_1 = 0.00009$, (b) $\sigma_1 = 0.0001$, (c) $\sigma_1 = 0.001$ with x-axis as time interval (in days) and y-axis as the population sizes of each species.

Figures 15–18 shows the evolution of the spatial pattern of all species at iterations 1000, 10000, and 100000. We took parameters from Table 6 and with diffusion coefficient $D_J = 0.00004$, $D_A = 0.00004$, $D_G = 0.00001$. Additionally, we considered the periodic growth rate in grey seals as $r_2 = r_2 * sin(2\pi t/365) * cos(2\pi t/365)$. This is considered to investigate the impact of periodicity in birth time (max in September to December, min: January to November) among Grey seals.

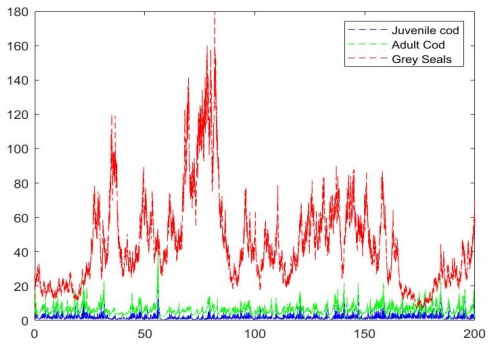
Without Allee effect: Numerical simulation of (5.7)–(5.9) were performed for increasing time. As time increases from $t = 1$ to 100 days, we observe strip and spot patterns (*cf.* Figs. 15–16). At the time $t = 1$ day, we observe that cod and seals are randomly distributed in the entire domain. A close examination shows that the number of grey seals and cods decreases as time increases. However, a significant change in number is not observed in the scenario, and it seems like the population is tending towards a steady state.

With Allee effect: We now consider the numerical simulation of (5.10)–(5.12) (with Allee effect in the cod population). Here, we find that as time increases from $t = 1$ to $t = 100$, the grey seal population decreases, as in the above case. However, it is observed that cod population density recovers due to the positive Allee effect as observed in the non-spatial case (*cf.* Figs. 17–18).

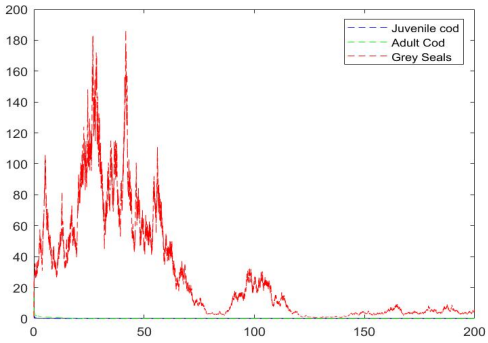
All simulations performed for the ODE, stochastic, and reaction–diffusion model indicate that if a critical size of Atlantic cod is maintained, the Allee effect may not cause extinction even for a diffusive system. This



(a) $\sigma_1 = 0.00009$



(b) $\sigma_1 = 0.0001$



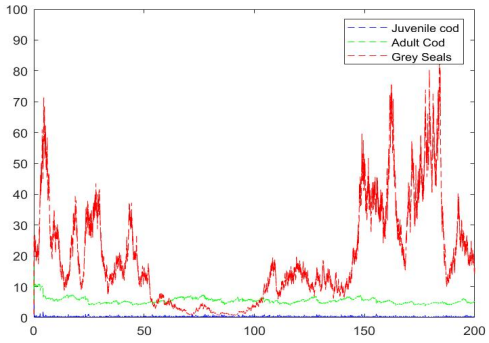
(b) $\sigma_1 = 0.001$

FIGURE 11. Time series of model (5.4)–(5.6) with Allee effect for the parameters given in Table 6 and $\sigma_2 = 0.00009$, (a) $\sigma_1 = 0.00009$, (b) $\sigma_1 = 0.0001$, (c) $\sigma_1 = 0.001$ with x-axis as time interval (in days) and y-axis as the population sizes of each species.

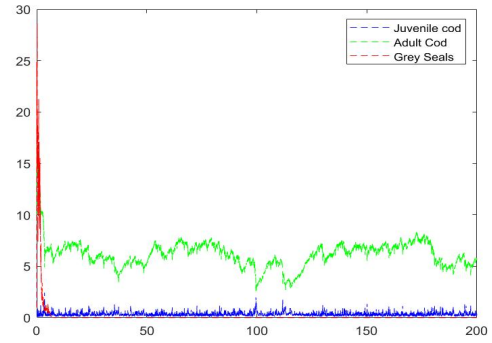
is biological, as densities above the minimum threshold density will have favorable growth rates, boosting the population away from the threshold density.

7. Conclusion

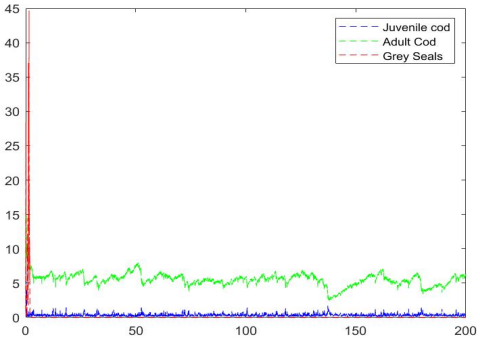
The protection of biodiversity and nature conservation has become a primary public concern. Understanding food chains can be beneficial for developing conservation policies for endangered species. This presents a valuable chance to comprehend the process of extinction in a dynamic environment, using the Atlantic cod population as an illustrative example. Cannibalism and predation by grey seals pose the greatest threat to cod. The paper presents and analyzes a food-chain model that includes juvenile cod, adult cod, and grey seal populations. We also incorporated the Allee effect in our basic model, which has been reported for the Southern Gulf of St Lawrence cod population [63]. The system without the Allee effect has four equilibrium points. Equilibrium states were determined and analyzed for their stability. The computation and theoretical analysis results show that the model exhibits rich dynamics. A threshold value for our proposed food chain model incorporating the Allee effect is calculated. It is found that cod survival is possible even when $R_0^{J+A} < 1$. Furthermore, if the harvesting rate of grey seals exceeds its intrinsic growth rate, cod species will have a greater chance of survival. Our numerical simulation of the ODE model (2.4)–(2.6) shows that the Atlantic cod population continues to exist if critical population size is maintained. Stochastic model simulation shows that when the environmental stochasticity experienced in adult and juvenile cod is too high, both populations tend to extinction.



(a) $\sigma_2 = 0.00009$



(b) $\sigma_2 = 0.0009$

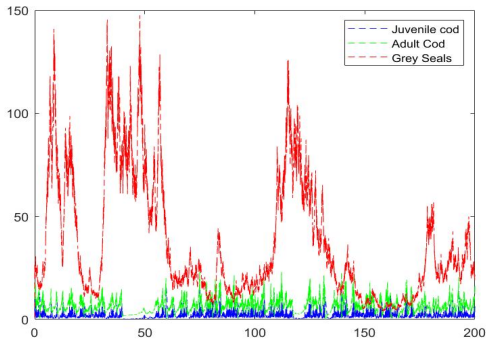


(c) $\sigma_2 = 0.009$

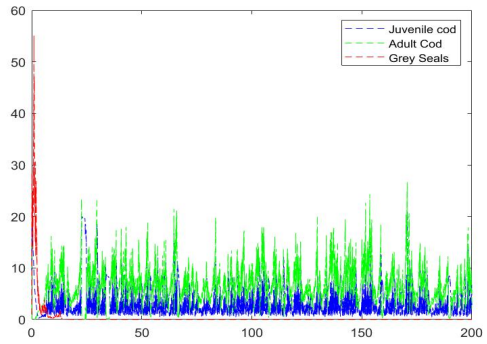
FIGURE 12. Time series of model (5.1)–(5.3) without Allee effect for the parameters given in Table 6 and $\sigma_1 = 0.0001$, (a) $\sigma_2 = 0.00009$, (b) $\sigma_2 = 0.0009$, (c) $\sigma_2 = 0.009$ with x-axis as time interval (in days) and y-axis as the population sizes of each species.

Most studies concluded that Allee's effects made the species extinction-prone. Another issue we addressed in our paper is to confirm the positive feedback of the Allee effect on the cannibalistic population for the continuous-time model as documented by [23] in the discrete-time model. We got mutually aligned results from non-spatial, stochastic, and spatial models, strengthening our hypothesis that the allee effect positively influences the cannibalistic population. The synergy between spatial and non-spatial models allows for a more holistic assessment of complex systems, leading to better-informed decisions and more effective solutions. It enables conservation scientists to consider what is happening, where, and why, resulting in more targeted and impactful interventions. In conclusion, non-spatial and spatial models provide unique insights into complex systems. Integrating these approaches can deepen our comprehension of diverse phenomena, facilitate informed decision-making, and contribute to more productive solutions to species extinction problems and environmental science. Our main objective is to provide a long-term justification for cod persistence by evaluating Allee's response to cod persistence. If we maintain the cod population above a threshold level, the Allee effect may boost the growth rate and be beneficial due to the cannibalistic nature of the cod.

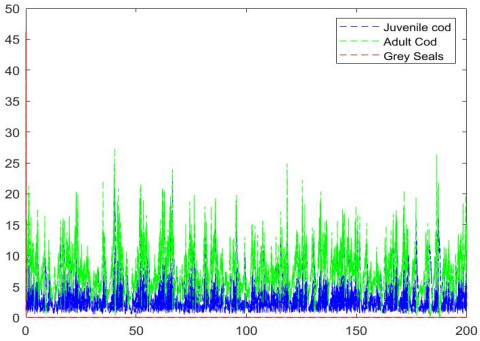
To minimize the expenses associated with preserving the cod population, the formulation in equations (2.4)–(2.6) was transformed into an optimal control problem. We also numerically simulated the optimal control model. These simulations confirmed that controlled cod harvesting and captive reproduction could prevent cod from extinction. From our optimal control results, we found that an effort to isolate the juvenile and cod populations may be helpful. This can be achieved by planting more eel grass and other seagrasses in the area resided by Atlantic cod, which must be implemented. This observation was biologically made by [58]. They



(a) $\sigma_2 = 0.00009$



(b) $\sigma_2 = 0.0009$

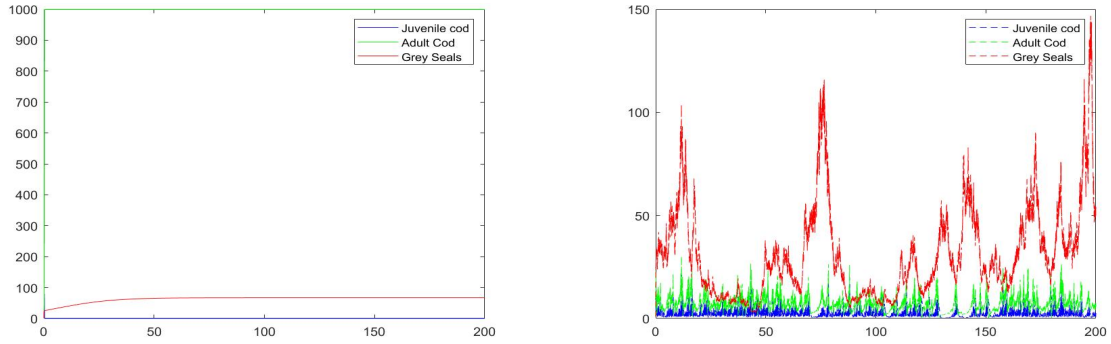


(c) $\sigma_2 = 0.009$

FIGURE 13. Time series of model (5.4)–(5.6) with Allee effect for the parameters given in Table 6 and $\sigma_1 = 0.0001$, (a) $\sigma_2 = 0.00009$, (b) $\sigma_2 = 0.0009$, (c) $\sigma_2 = 0.009$ with x-axis as time interval (in days) and y-axis as the population sizes of each species.

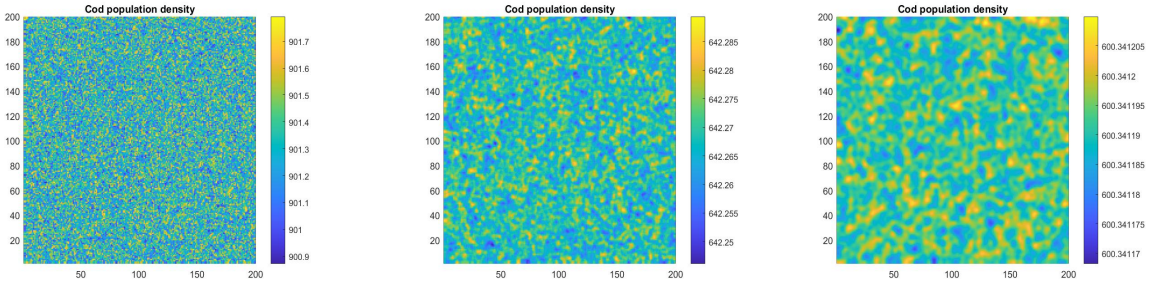
found that predation was generally higher in unvegetated than eelgrass sites during the day and dusk. Some other control measures that can be used to preserve the cod population and to promote the recovery of native wildlife species under threat include (i) artificial breeding, (ii) habitat restoration, and (iii) harvesting grey seals. Methods for artificial breeding are well described in [64]. Another way is by catching adult cod, hatching and releasing yolk sac larvae of cod, and then releasing them into the wild. The focus should be placed on optimizing the growth of cod larvae. Our optimal control result mathematically validates the methods for increasing initial growth rates of cod larvae, for example, by natural zooplankton as reported by Busch *et al.* [65] may be a helpful strategy feed for juvenile cods. The main advantage of using natural zooplankton as a starter feed is cod larvae's high initial growth rates [65]. Early growth may significantly increase the long-term development of cods by preventing them from cannibalism. In summary, our study employs mathematical simulations to underscore the importance of safeguarding crucial habitats like seagrass in coastal seas, emphasizing their role in supporting biodiversity and food security. The utilization of natural zooplankton in intensive systems shows promising potential, and optimizing factors like larval stocking and prey densities is likely to enhance larval production from such systems significantly. Educating the public on cod extinction issues, assessing cod population densities and distribution, and developing alternative traps to control grey seals can save the cod population.

Our model suffers from certain limitations and could be considered for future work. It does not show the effect of diseases such as Vibriosis, Viral hemorrhagic septicemia virus, Infectious pancreatic necrosis virus, *etc.* These may also be a significant cause of death in cod populations [66]. Some researchers also showed that cods with high predation pressure might respond by losing weight to allow them to move faster, increasing the probability



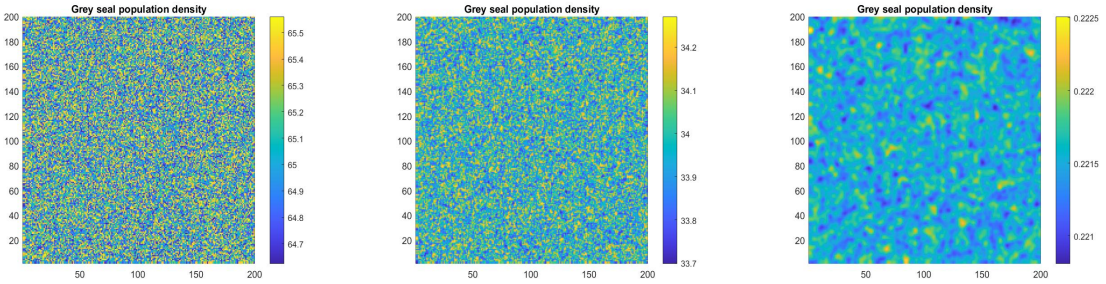
(a) Time Series of model (1)-(3) (b) Time series of stochastic model (20)-(22)

FIGURE 14. (a) Time series of the model (2.1)–(2.3) where the x-axis represents a time interval (in days) and the y-axis represents the population sizes of each species. (b) Time series of the model (5.1)–(5.3) where the x-axis represents the time interval (in days) and the y-axis as the population sizes of each species.



(a) days= 1 (b) days $t = 10$ (c) days $t = 100$

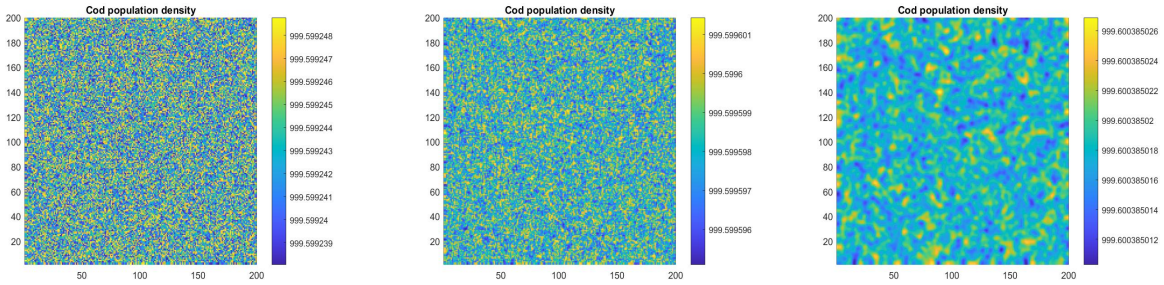
FIGURE 15. Adult Atlantic cod population density without Allee effect when $D_J = 0.00004$, $D_A = 0.00005$ and $D_G = 0.00001$ for days $t = 1,\ t = 10$ and $t = 100$.



(a) days $t = 1$ (b) days $t = 500$ (c) days $t = 1000$

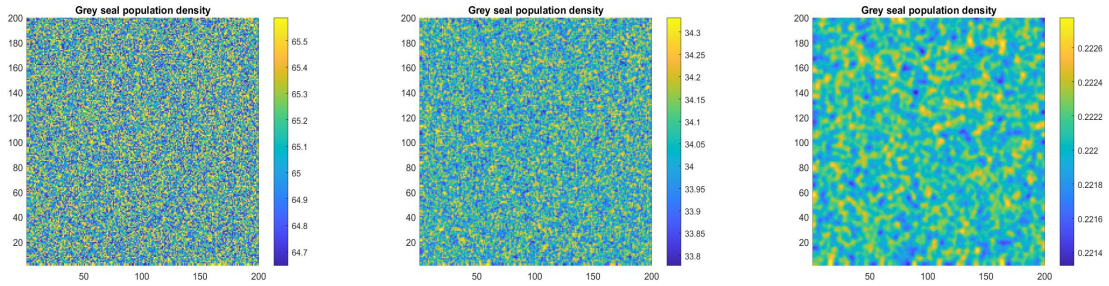
FIGURE 16. Grey seal population density without Allee effect when $D_J = 0.00004$, $D_A = 0.00005$ and $D_G = 0.00001$ for days $t = 1,\ t = 10$ and $t = 100$.

of escaping the attack. More mathematical complications must be implemented to relate dynamical models more to reality. This article aims to study the impact of environmental fluctuations and self-diffusion for the designed



(a) days= 1 (b) days $t = 10$ (c) days $t = 100$

FIGURE 17. Adult Atlantic cod population density with Allee effect when $D_J = 0.00004$, $D_A = 0.00005$ and $D_G = 0.00001$ for days $t = 1,\ t = 10$ and $t = 100$.



(a) days $t = 1$ (b) days $t = 10$ (c) days $t = 100$

FIGURE 18. Grey seal population density with Allee effect when $D_J = 0.00004$, $D_A = 0.00005$ and $D_G = 0.00001$ for days $t = 1,\ t = 10$ and $t = 100$.

model system separately. However, in the future, complex models might integrate environmental fluctuations and self-diffusion terms to capture the intricate interplay between external factors and inherent system dynamics. These models depict how systems respond to internal processes (like self-diffusion) and external influences (like environmental fluctuations). The optimal control of the models with environmental stochasticity and self-diffusion terms could evolve into an intriguing avenue of research in the near future. Further work is encouraged to study the effect of cannibalism and the Allee effect exhibited by cod populations. Hence, an original benefaction to science is uplifting the conservation management strategies for holding up the cod population and assisting the biodiversity managers.

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Conflicts of interest. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author contribution statement. Parimita Roy: Model Building, Validation, Development of Computational Skills, Preparing the research paper.

Sanjoli Jain: Analysis, Development of Coding and Mathematical Techniques, Simulation.
Mohamed Maama: Modelling, Mathematical Analysis, Writing.

Ethics approval. The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

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