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The rigidity of biconservative surfaces in Sol_3

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Abstract
We consider biconservative surfaces in Sol_3, find their local equations, and then show that all biharmonic surfaces in this space are minimal.

Keywords Biconservative surfaces · Biharmonic surfaces · Thurston geometries

Mathematics Subject Classification 53C42

1 Introduction

Almost forty years ago, B.-Y. Chen [5] introduced biharmonic submanifolds of Euclidean spaces $\mathbb{E}^n$, as isometric immersions with harmonic mean curvature vector field. Then, in [6, 18] it was proved that biharmonic surfaces in $\mathbb{E}^3$ are minimal, a result that led to the still open Chen's Conjecture that biharmonic submanifolds of Euclidean spaces are minimal [5].

In the same era, independently and in a more abstract way, biharmonic maps were defined by G.-Y. Jiang [16] as critical points of the L^2-norm of the tension field. This type of a variational problem was suggested back in the 1964 by J. Eells and J. H. Sampson in their seminal paper [10]. We note that the two definitions agree in Euclidean spaces $\mathbb{E}^n$.

Since ambient spaces with non-positive curvatures do not allow for interesting compact examples, most of the research has been done on biharmonic submanifolds of Euclidean spheres and other spaces with convenient curvature properties (for detailed accounts see [12, 30]).

The notion of biconservative submanifolds was derived from the theory of biharmonic submanifolds by only requiring the vanishing of the tangent part of the bitension field. Although a rather new one, this subject is already well established. If we are to illustrate only the literature on biconservative surfaces, we refer to papers like [2, 4, 11, 15, 21, 25–27, 29] to gain a satisfactory (if fairly incomplete) imagine.

Studying the geometry of surfaces in Sol_3 seems to be the most challenging among the eight Thurston geometries. The lack of some powerful tools used in other homogeneous 3-manifolds to describe the geometry of immersed surfaces lead to many difficulties that do not appear in the case of the remaining seven geometries (see [8]). However, in spite of these difficulties, there are important results on the existence and uniqueness of constant mean

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curvature (CMC) spheres [8, 23, 24], on totally umbilic surfaces [32], half-space theorems for minimal surfaces [7], as well as Jenkins-Serrin type results for minimal [28] and CMC surfaces [19]. In a local approach, one classified constant angle surfaces [20]. Also CMC biharmonic surfaces were studied in [31] and it turned out that in Sol_3 they are actually minimal.

In our paper, we first prove that CMC biconservative surfaces in Sol_3 are minimal. Next, we show that non-CMC biconservative surfaces are constant angle surfaces and find their local equations. We conclude with the fact that all biharmonic surfaces in Sol_3 are minimal. This last result is similar to that for biharmonic surfaces in $\mathbb{E}^3$ (see [5, 9]).

Conventions. Throughout the paper surfaces are oriented, we will use the following sign conventions

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \quad \Delta = \operatorname{trace}\nabla^2 = \operatorname{trace}(\nabla\nabla - \nabla_\nabla),$$

and objects on Sol_3 will be indicated by $\overline{(\cdot)}$.

2 Preliminaries

Biharmonic maps $\phi : M^m \to N^n$ between two Riemannian manifolds are critical points of the bienergy functional

$$E_2 : C^\infty(M,N) \to \mathbb{R}, \quad E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv,$$

where $\tau(\phi) = \operatorname{trace}\nabla d\phi$ is the tension field of ϕ. The Euler-Lagrange equation, also called the biharmonic equation, was derived by G.-Y. Jiang [16]

$$\tau_2(\phi) = \Delta\tau(\phi) - \operatorname{trace} R^N(d\phi(\cdot), \tau(\phi))d\phi(\cdot) = 0, \tag{2.1}$$

where $\tau_2(\phi)$ is the bitension field of ϕ.

Since any harmonic map is biharmonic, the important case, from the point of view of biharmonicity, is that of non-harmonic biharmonic ones.

Next, if we consider a fixed map ϕ and let the domain metric vary, one obtains a functional on the set $\mathcal{G}$ of Riemannian metrics on M

$$\mathcal{F}_2 : \mathcal{G} \to \mathbb{R}, \quad \mathcal{F}_2(g) = E_2(\phi).$$

Critical points of this functional are characterized by the vanishing of the stress-energy tensor S_2 of the bienergy (see [21]). This tensor was introduced in [17] as

$$\begin{aligned} S_2(X,Y) = {} & \frac{1}{2}|\tau(\phi)|^2\langle X,Y\rangle + \langle d\phi, \nabla\tau(\phi)\rangle\langle X,Y\rangle - \langle d\phi(X), \nabla_Y\tau(\phi)\rangle \\ & -\langle d\phi(Y), \nabla_X\tau(\phi)\rangle, \end{aligned}$$

and it satisfies

$$\operatorname{div} S_2 = -\langle\tau_2(\phi), d\phi\rangle.$$

For isometric immersions, $(\operatorname{div} S_2)^\sharp = -\tau_2(\phi)^\top$, where $\tau_2(\phi)^\top$ is the tangent part of the bitension field.

Definition 2.1 A submanifold $\phi : M^m \to N^n$ of a Riemannian manifold N^n is called biconservative if $\operatorname{div} S_2 = 0$.

From the definition, it is easy to see that a submanifold is biconservative if and only if the tangent part of the bitension field vanishes.

Now, let us consider a surface Σ^2 in a Riemannian manifold N^n, with the unit normal vector field ξ. The Gauss and the Weingarten equations of the surface

$$\nabla^N_X Y = \nabla_X Y + \sigma(X, Y) \quad \text{and} \quad \nabla^N_X \xi = -AX,$$

hold for all vector fields X and Y tangent to the surface, where ∇ is the induced connection on Σ^2, σ is its second fundamental form, and A its shape operator. The mean curvature vector field of Σ^2 is given by $H = f\xi$, where $f = (1/2)\,\text{trace}\, A$ is the mean curvature function.

If f is constant, then Σ^2 is called a constant mean curvature (CMC) surface.

The Codazzi equation of the surface reads, for any vector fields X, Y, Z tangent to Σ^2, and any normal vector field V,

$$\langle \bar{R}(X, Y)Z, V\rangle = \langle (\nabla^\perp_X \sigma)(Y, Z), V\rangle - \langle (\nabla^\perp_Y \sigma)(X, Z), V\rangle, \tag{2.2}$$

where $(\nabla^\perp_X \sigma)(Y, Z) = \nabla^\perp_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$.

The Gauss equation of the surface is

$$\langle R(X, Y)Z, W\rangle = \langle \bar{R}(X, Y)Z, W\rangle + \langle AY, Z\rangle\langle AX, W\rangle - \langle AX, Z\rangle\langle AY, W\rangle, \tag{2.3}$$

for all tangent vector fields X, Y, Z, and W, where R is the curvature tensor of Σ^2.

Specializing a general result in [21] in the case of surfaces, we have the following splitting of biharmonic equation theorem.

Theorem 2.2 [21] *A surface Σ^2 in a Riemannian manifold N^n is biharmonic if and only if*

$$\begin{cases} \Delta f - f|A|^2 - f\langle \text{trace}(R^N(\cdot, \xi)\cdot), \xi\rangle = 0 \\ A(\nabla f) + f\nabla f + f\,\text{trace}(R^N(\cdot, \xi)\cdot)^\top = 0. \end{cases}$$

Corollary 2.3 *A surface Σ^2 in a Riemannian manifold N^n is biconservative if and only if*

$$A(\nabla f) + f\nabla f + f\,\text{trace}(R^N(\cdot, \xi)\cdot)^\top = 0. \tag{2.4}$$

Next, we shall briefly recall some basic facts on Sol_3. This Lie group is $\mathbb{R}^3$ with the Riemannian metric

$$\langle , \rangle = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the canonical coordinates of $\mathbb{R}^3$.

A left-invariant orthonormal frame field $\{E_1, E_2, E_3\}$ with respect to this metric, called the canonical frame, is defined by

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \quad E_2 = e^{z}\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

The Levi-Civita connection of Sol_3 is then the following

$$\begin{aligned} &\bar{\nabla}_{E_1}E_1 = -E_3,\ \bar{\nabla}_{E_1}E_2 = 0, \quad \bar{\nabla}_{E_1}E_3 = E_1 \\ &\bar{\nabla}_{E_2}E_1 = 0, \qquad \bar{\nabla}_{E_2}E_2 = E_3,\ \bar{\nabla}_{E_2}E_3 = -E_2 \\ &\bar{\nabla}_{E_3}E_1 = 0, \qquad \bar{\nabla}_{E_3}E_2 = 0, \quad \bar{\nabla}_{E_3}E_3 = 0. \end{aligned} \tag{2.5}$$

One can see that the vertical vector field E_3 foliates Sol_3 by vertical geodesics. Moreover, we get that the sectional curvatures of the vertical plane fields (E_1, E_3) and (E_2, E_3) are equal to -1, while that of the horizontal plane field (E_1, E_2) is equal to 1.

The only totally geodesic surfaces in Sol_3 are the leaves of the first two canonical foliations

$$\mathcal{F}_1 \equiv \{x = \text{constant}\} \quad \text{and} \quad \mathcal{F}_2 \equiv \{y = \text{constant}\},$$

which are isometric to the hyperbolic plane $\mathbb{H}^2$ (see [32]).

The leaves of the third canonical foliation $\mathcal{F}_3 \equiv \{z = \text{constant}\}$ are isometric to $\mathbb{R}^2$ with its usual flat metric and are minimal.

The curvature tensor $\bar{R}$ of Sol_3 is given by (see [32])

$$\begin{aligned}\bar{R}(X,Y)Z = {} & \langle Y,Z\rangle X - \langle X,Z\rangle Y + 2\langle Z,E_3\rangle(\langle X,E_3\rangle Y - \langle Y,E_3\rangle X) \\ & +2(\langle X,Z\rangle\langle Y,E_3\rangle - \langle Y,Z\rangle\langle X,E_3\rangle)E_3.\end{aligned} \tag{2.6}$$

3 Biconservative surfaces in Sol_3

As we have seen, CMC biharmonic surfaces in Sol_3 are minimal [31]. The following result shows that this stands true when we only keep the weaker biconservative condition.

Proposition 3.1 *A CMC biconservative surface in Sol_3 is minimal.*

Proof From Corrolary 2.3 and the expression (2.6) of the curvature tensor of Sol_3, one can easily see that a CMC surface Σ^2 is biconservative if and only if either the vector field E_3 is either normal or tangent to the surface.

If E_3 is normal, then Σ^2 is minimal and therefore we only have to study the case when E_3 is tangent. In this situation, we consider an orthonormal frame field $\{X_1 = E_3, X_2\}$ on the surface and, since $\bar{\nabla}_{E_3}E_3 = 0$, it follows that $AX_1 = 0$ and $AX_2 = 2fX_2$. Also, we can write X_2 as

$$X_2 = \cos\alpha E_1 + \sin\alpha E_2,$$

and then

$$E_1 = \cos\alpha X_2 + \sin\alpha\xi, \qquad E_2 = \sin\alpha X_2 - \cos\alpha\xi,$$

along the surface, where $\xi = \sin\alpha E_1 - \cos\alpha E_2$ is a unit vector field normal to Σ^2.

Next, using formulas (2.5) and the Gauss and Weingarten equations of the surface, one obtains

$$\begin{aligned}\bar{\nabla}_{X_2}E_1 &= -\cos\alpha E_3 \\ &= X_2(\cos\alpha)X_2 + \cos\alpha\nabla_{X_2}X_2 + 2f\cos\alpha\xi + X_2(\sin\alpha)\xi - 2f\sin\alpha X_2,\end{aligned}$$

and

$$\begin{aligned}\bar{\nabla}_{X_2}E_2 &= \sin\alpha E_3 \\ &= X_2(\sin\alpha)X_2 + \sin\alpha\nabla_{X_2}X_2 + 2f\sin\alpha\xi - X_2(\cos\alpha)\xi + 2f\cos\alpha X_2.\end{aligned}$$

Taking the inner product with X_1 in both equations, we get

$$\cos\alpha\langle\nabla_{X_2}X_2, X_1\rangle = -\cos\alpha \quad \text{and} \quad \sin\alpha\langle\nabla_{X_2}X_2, X_1\rangle = \sin\alpha,$$

which leads to $\sin\alpha\cos\alpha = 0$. This means that either E_1 or E_2 is normal to Σ_2 and therefore the surface is totally geodesic.

Remark 3.2 In contrast to our situation, any CMC surface in a 3-dimensional space form is biconservative [4] and also in the other 3-dimensional BCV-spaces (that are local models for six of the Thurston geometries, the exceptions being $\mathbb{H}^3$ and Sol_3) non-minimal CMC biconservative surfaces do exist [27].

We will henceforth focus on non-CMC biconservative surfaces. Let Σ^2 be a biconservative surface in Sol_3 such that $\nabla f \neq 0$ at a point $p \in \Sigma^2$. Therefore, there exists a neighborhood V of p with $\nabla f \neq 0$ on V. Since f cannot vanish on V, there exists an open subset U of V such that $f \neq 0$ at each point of U. Moreover, we can assume that $f > 0$ on U. Next, we consider an orthonormal frame field $\{X_1, X_2\}$ on U, where

$$X_1 = \frac{\nabla f}{|\nabla f|}, \quad X_2 \perp X_1, \quad |X_2| = 1.$$

We now have

$$\nabla f = X_1(f)X_1 + X_2(f)X_2,$$

which leads to

$$X_1(f) = |\nabla f| > 0 \quad \text{and} \quad X_2(f) = 0. \tag{3.1}$$

Next, let $H = f\xi$ be the mean curvature vector field of Σ^2, where ξ is a unit vector field normal to the surface. By a straightforward computation, using isothermal coordinates (x, y) on the surface and since Σ^2 is biconservative, one obtains (see also [22]),

$$E = \left(\nabla_{\frac{\partial}{\partial x}} A_H\right) \frac{\partial}{\partial y} - \left(\nabla_{\frac{\partial}{\partial y}} A_H\right) \frac{\partial}{\partial x} = 3f \left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right),$$

which shows that $X_1 \perp E$ on U, i.e., E and X_2 are collinear.

On the other hand, from the Codazzi equation (2.2) and the expression of the curvature tensor (2.6), we get

$$E = \left(\nabla_{\frac{\partial}{\partial x}} A_H\right) \frac{\partial}{\partial y} - \left(\nabla_{\frac{\partial}{\partial y}} A_H\right) \frac{\partial}{\partial x} = 2f\langle \xi, E_3\rangle \left(\left\langle \frac{\partial}{\partial y}, E_3\right\rangle \frac{\partial}{\partial x} - \left\langle \frac{\partial}{\partial x}, E_3\right\rangle \frac{\partial}{\partial y}\right),$$

and this readily shows that E is orthogonal to E_3. Therefore, the two vector fields X_2 and E_3 are orthogonal and, along U, we can write

$$\begin{cases} E_1 = \sin\beta \sin\theta X_1 + \cos\beta X_2 - \sin\beta\cos\theta\xi \\ E_2 = -\cos\beta\sin\theta X_1 + \sin\beta X_2 + \cos\beta\cos\theta\xi \\ E_3 = \cos\theta X_1 + \sin\theta\xi, \end{cases} \tag{3.2}$$

or, equivalently,

$$\begin{cases} X_1 = \sin\beta\sin\theta E_1 - \cos\beta\sin\theta E_2 + \cos\theta E_3 \\ X_2 = \cos\beta E_1 + \sin\beta E_2 \\ \xi = -\sin\beta\cos\theta E_1 + \cos\beta\cos\theta E_2 + \sin\theta E_3, \end{cases} \tag{3.3}$$

where β and θ are real valued functions on U.

From Corrolary 2.3 and formula (2.6) we easily get that the biconservative condition is equivalent to

$$AX_1 = -\left(f + \frac{f}{|\nabla f|}\sin(2\theta)\right) X_1 = \lambda_1 X_1, \tag{3.4}$$

and therefore X_1 is a principal direction with the corresponding eigenfunction λ_1. It follows that also X_2 is a principal direction and

$$AX_2 = \left(3f + \frac{f}{|\nabla f|}\sin(2\theta)\right) X_2 = \lambda_2 X_2. \tag{3.5}$$

In the following, working on U, we will prove a sequence of three lemmas that will eventually lead to our first main result.

Lemma 3.3 *On the set U the following identities hold:*

(1) $X_1(\theta) = -\lambda_1 + \cos(2\beta)\sin\theta$;
(2) $X_2(\theta) = -\sin(2\beta)$;
(3) $\cos\theta\langle\nabla_{X_1}X_1, X_2\rangle = \sin(2\beta)\sin\theta$;
(4) $\cos\theta\langle\nabla_{X_2}X_1, X_2\rangle = \lambda_2\sin\theta + \cos(2\beta)$;
(5) $(X_1(\beta) - \langle\nabla_{X_1}X_1, X_2\rangle\sin\theta)\sin\beta = 0$;
(6) $X_1(\beta)\sin\beta\cos\theta = 2\sin^2\beta\cos\beta\sin^2\theta$;
(7) $X_2(\beta)\cos\theta = \lambda_2 + \cos(2\beta)\sin\theta$;
(8) $X_2(\beta)\sin\theta - \langle\nabla_{X_2}X_1, X_2\rangle = -\cos(2\beta)\cos\theta$.

Proof From (2.5), (3.2), and (3.3), we have

$$\begin{aligned}\bar{\nabla}_{X_1}E_3 &= \sin\beta\sin\theta E_1 + \cos\beta\sin\theta E_2\\ &= -X_1(\theta)\sin\theta X_1 + \cos\theta\nabla_{X_1}X_1 + \cos\theta\sigma(X_1, X_1)\\ &\quad + X_1(\theta)\cos\theta\xi - \sin\theta\lambda_1 X_1,\end{aligned}$$

and

$$\begin{aligned}\bar{\nabla}_{X_2}E_3 &= \cos\beta E_1 - \sin\beta E_2\\ &= -X_2(\theta)\sin\theta X_1 + \cos\theta\nabla_{X_2}X_1 + X_2(\theta)\cos\theta\xi - \sin\theta\lambda_2 X_2.\end{aligned}$$

Considering the inner product with X_1, X_2, and ξ, one obtains the first four equations.

Next, again using (2.5), (3.2), and (3.3), it follows

$$\begin{aligned}\bar{\nabla}_{X_1}E_1 &= -\sin\beta\sin\theta E_3\\ &= X_1(\theta)\sin\beta\cos\theta X_1 + X_1(\beta)\cos\beta\sin\theta X_1 + \sin\beta\sin\theta\nabla_{X_1}X_1\\ &\quad + \sin\beta\sin\theta\sigma(X_1, X_1) - X_1(\beta)\sin\beta X_2 + \cos\beta\nabla_{X_1}X_2\\ &\quad + X_1(\theta)\sin\beta\sin\theta\xi - X_1(\beta)\cos\beta\cos\theta\xi + \sin\beta\cos\theta\lambda_1 X_1.\end{aligned}$$

We take the inner product with X_2 and obtain the fifth identity. From this and the third identity, the sixth one follows immediately.

Finally, we have

$$\begin{aligned}\bar{\nabla}_{X_2}E_1 &= -\cos\beta E_3\\ &= X_2(\theta)\sin\beta\cos\theta X_1 + X_2(\beta)\cos\beta\sin\theta X_1 + \sin\beta\sin\theta\nabla_{X_2}X_1\\ &\quad + \cos\beta\sigma(X_2, X_2) - X_2(\beta)\sin\beta X_2 + \cos\beta\nabla_{X_2}X_2\\ &\quad + X_2(\theta)\sin\beta\sin\theta\xi - X_2(\beta)\cos\beta\cos\theta\xi + \sin\beta\cos\theta\lambda_2 X_2,\end{aligned}$$

and, taking the inner product with each X_1, X_2, and ξ, also using the third and the fourth identities, we prove the last two items of the lemma.

Remark 3.4 Any two of the third, fifth, and sixth items of Lemma 3.3 imply the other one. This also happens with the fourth, seventh, and eighth items. Therefore only six of the identities are independent. It is easy to verify that this is the maximum number of independent equations of this type that can be derived from the expressions of $\bar{\nabla}_{X_i}E_j$, $i, j \in \{1, 2, 3\}$.

The next result shows that one of the two vector fields E_1 or E_2 is tangent to the surface Σ^2 at any point of U.

Lemma 3.5 *On the set U we have* $\sin(2\beta) = 0$.

Proof From the Codazzi equations (2.2) and (2.6), one obtains

$$\left(\nabla_{X_1} A\right) X_2 - \left(\nabla_{X_2} A\right) X_1 = -\sin(2\theta) X_2,$$

which can be written as

$$\begin{cases} X_1(\lambda_2) + (\lambda_2 - \lambda_1)\langle \nabla_{X_2} X_1, X_2\rangle + \sin(2\theta) = 0 \\ X_2(\lambda_1) - (\lambda_2 - \lambda_1)\langle \nabla_{X_1} X_2, X_1\rangle = 0. \end{cases} \tag{3.6}$$

Using Lemma 3.3, the second equation become

$$X_2(\lambda_1)\cos\theta + (\lambda_2 - \lambda_1)\sin(2\beta)\sin\theta = 0. \tag{3.7}$$

In the following, we will compute $X_2(\lambda_1)$. Since $X_2(f) = 0$ and $X_1(f) = |\nabla f|$, we have, also using Lemma 3.3,

$$\begin{aligned} X_2(\lambda_1) &= -X_2\left(f + \frac{f}{|\nabla f|}\sin(2\theta)\right) \\ &= \frac{f}{|\nabla f|}\left(\frac{X_2(X_1(f))}{|\nabla f|}\sin(2\theta) + 2\sin(2\beta)\cos(2\theta)\right). \end{aligned} \tag{3.8}$$

Now, since $X_2(X_1(f)) = X_1(X_2(f)) - [X_1, X_2](f) = -[X_1, X_2](f)$ and, from Lemma 3.3, we know that

$$\begin{aligned} \cos\theta [X_1, X_2] &= \cos\theta(\nabla_{X_1} X_2 - \nabla_{X_2} X_1) \\ &= -\sin(2\beta)\sin\theta X_1 - (\lambda_2 \sin\theta + \cos(2\beta)) X_2, \end{aligned}$$

one obtains

$$\cos\theta X_2(X_1(f)) = \sin(2\beta)\sin\theta X_1(f) = \sin(2\beta)\sin\theta|\nabla f|. \tag{3.9}$$

Replacing in (3.8) we get

$$X_2(\lambda_1) = \frac{2f}{|\nabla f|}\sin(2\beta)\cos^2\theta,$$

and then, from (3.7), since

$$\lambda_2 - \lambda_1 = 4f + \frac{2f}{|\nabla f|}\sin(2\theta),$$

one obtains

$$\frac{2\sin(2\beta)}{|\nabla f|}\left(2|\nabla f|\sin\theta + \cos\theta\left(1 + \sin^2\theta\right)\right) = 0.$$

Therefore, at each point of U, either

$$\sin(2\beta) = 0 \quad \text{or} \quad 2|\nabla f|\sin\theta + \cos\theta\left(1 + \sin^2\theta\right) = 0.$$

Assume that $\sin(2\beta) \neq 0$ at a point $q \in U$. Then, there exists a neighborhood $W \subset U$ of q such that $\sin(2\beta) \neq 0$ at all points of W, which means that

$$2|\nabla f|\sin\theta + \cos\theta\left(1 + \sin^2\theta\right) = 0 \tag{3.10}$$

on W. From here, it follows that

$$2X_2(X_1(f))\sin\theta + (2|\nabla f|\cos\theta - \sin\theta - \sin^3\theta + 2\sin\theta\cos^2\theta)X_2(\theta) = 0,$$

on W. We then multiply by $\cos\theta$, use (3.9), Lemma 3.3, and again (3.10), to obtain, after a straightforward computation, that $\cos\theta = 0$ on the set W. But, from (3.10), this implies that also $\nabla f = 0$ on W, which is a contradiction and we conclude.

Lemma 3.6 *Throughout the set U we have*

$$\cos\theta \neq 0, \quad \sin\theta \neq 0, \quad X_1(\theta) = -2f, \quad X_2(\theta) = 0, \quad \nabla_{X_1}X_1 = 0, \quad X_2(X_1(f)) = 0,$$

and either

(1) $\nabla_{X_2}X_1 = \cos\theta X_2$;
(2) $\lambda_2 = -\sin\theta$,

or

(1) $\nabla_{X_2}X_1 = -\cos\theta X_2$;
(2) $\lambda_2 = \sin\theta$,

as $\beta = 0$ or $\beta = \pi/2$ on U.

Proof The claims on $\nabla_{X_2}X_1$, $X_2(\theta)$, and λ_2 follow directly from Lemmas 3.3 and 3.5. Then, using the expression of λ_2 and again the two previous lemmas, one obtains that $X_1(\theta) = -2f$.

If $\sin\theta = 0$ at a point $q \in U$, then, at this point, $\lambda_2 = 0$ and $\lambda_1 = 2f$. But, from the biconservative condition (3.4), it follows that $\lambda_1 = -f$ which implies that $f = 0$, at q. Therefore, $\sin\theta \neq 0$ at any poiny of U.

Next, from the first equation of (3.6), one can see that, if we assume $\cos\theta = 0$ at a point $q \in U$, then, at that point, we have $X_1(\lambda_2) = 0$. But, in general, on U,

$$X_1(\lambda_2) = 3X_1(f) + X_1\left(\frac{f}{|\nabla f|}\right)\sin(2\theta) + \frac{2f}{|\nabla f|}X_1(\theta)\cos(2\theta),$$

and therefore, at q,

$$X_1(\lambda_2) = 3|\nabla f| + \frac{4f^2}{|\nabla f|},$$

as $X_1(\theta) = -2f$. Since $f > 0$ and $|\nabla f| \neq 0$ on U, this is a contradiction, i.e., $\cos\theta \neq 0$ throughout U. To conclude, we note that $\nabla_{X_1}X_1 = 0$ now follows immediately from Lemma 3.3, and $X_2(X_1(f)) = 0$ from Eq. (3.9).

Now, we can state the rigidity of biconservative surfaces result.

Theorem 3.7 *Let Σ^2 be a biconservative surface in Sol_3 with $f > 0$ and $\nabla f \neq 0$ at any point. Then, locally, Σ^2 can be parametrized as either*

$$x_1(u, v) = ve^{\Psi(u)}E_1 + \Phi_1(u)e^{-\Psi(u)}E_2 + \Psi(u)E_3 = (v, \Phi_1(u), \Psi(u)), \tag{3.11}$$

or

$$x_2(u, v) = \Phi_2(u)e^{-\Psi(u)}E_1 + ve^{-\Psi(u)}E_2 + \Psi(u)E_3 = (\Phi_2(u), v, \Psi(u)), \tag{3.12}$$

where

$$\Psi(u) = \int_{u_0}^{u}\cos\theta(s)ds, \Phi_1(u) = -\int_{u_0}^{u}\sin\theta(s)e^{\Psi(s)}ds, \Phi_2(u) = \int_{u_0}^{u}\sin\theta(s)e^{-\Psi(s)}ds.$$

The function $\theta = \theta(u)$ is given either by

$$\theta(u) = 2\arctan e^{(1-\sqrt{13})u/3}, \quad u < 0, \tag{3.13}$$

or by the following first order ODE

$$(\theta' + 2a_1\sin\theta)^{6a_2} = c(\theta' + 2a_2\sin\theta)^{6a_1}, \tag{3.14}$$

such that $\theta' < 0$ and $\theta'' < 0$, where $a_{1,2} = (-1 \pm \sqrt{13})/6$ and $c > 0$ is a positive constant.

Proof From Lemma 3.5, we know that either E_1 or E_2 is tangent to our surface and therefore Σ^2 is a constant angle surface as defined in [20], where Eqs. (3.11) and (3.12) of such surfaces were found.

We will consider only the case of the first surface, that with our notations correspond to $\beta = 0$, meaning that $X_2 = E_1$. The computations in the second case are similar and therefore we will omit them.

In [20], the local coordinates (u, v) were chosen such that

$$X_1 = \frac{\partial}{\partial u} \quad \text{and} \quad X_2 = \alpha(u, v)\frac{\partial}{\partial v},$$

for some function α, which is allowed by the fact that

$$[X_1, X_2] = -\nabla_{X_2} X_1 = -\cos\theta X_2.$$

Therefore, in this setting, $X_2(f) = 0$ and $X_2(\theta) = 0$ imply that $f = f(u)$ and $\theta = \theta(u)$. It is now straightforward to verify that surfaces defined by (3.11) and (3.12) satisfy all identities in Lemma 3.6.

Thus, we only have to find the function $\theta = \theta(u)$ by solving the biconservative equation, which, for the first surface, taking into account Lemma 3.6, can be written as

$$\lambda_1 = -\left(f + \frac{f}{f'}\sin(2\theta)\right) = 2f + \sin\theta,$$

or, equivalently,

$$3ff' + f'\sin\theta + f\sin(2\theta) = 0, \tag{3.15}$$

where $\theta' = -2f$. Denote $y = \sin\theta$ and then we can see that

$$2yy' = 2\theta'\sin\theta\cos\theta = -2f\sin(2\theta),$$

and Eq. (3.15) becomes

$$3ff' + yf' - yy' = 0.$$

We can write $f' = y'(df/dy)$ and, replacing in the last equation, we have

$$3fy'\frac{df}{dy} + yy'\frac{df}{dy} - yy' = 0.$$

Since from Lemma 3.6 we know that $\cos\theta \neq 0$ and we also have $\theta' = -2f \neq 0$, it follows that $y' = \theta'\cos\theta \neq 0$, which shows that the above equation is equivalent to

$$3f\frac{df}{dy} + y\frac{df}{dy} - y = 0.$$

We denote $f = yg$ and, after a straightforward computation, this equation reads as

$$y(3g + 1)\frac{dg}{dy} + 3g^2 + g - 1 = 0. \tag{3.16}$$

We now have two cases as $3g^2 + g - 1 = 0$ or $3g^2 + g - 1 \neq 0$. In the first one, $g = a_{1,2} = (-1 \pm \sqrt{13})/6$ is a constant and $f = a\sin\theta$, where a stands either for a_1 or a_2. Since $\theta' = -2f$, it follows that

$$\theta(u) = 2\arctan e^{-2au}, \quad f(u) = \frac{2ae^{-2au}}{1 + e^{-4au}}, \quad \text{and} \quad f'(u) = \frac{4a^2e^{-2au}(e^{-4au} - 1)}{(1 + e^{-4au})^2}.$$

Since $f > 0$, we see that $a = (\sqrt{13} - 1)/6$, and, from $f'(u) = X_1(f) = |\nabla f| > 0$, it follows that u must be negative.

In the second case, a simple computation shows that Eq. (3.16) has the implicit solution

$$(y(g - a_1))^{6a_2} = c(y(g - a_2))^{6a_1}, \tag{3.17}$$

where $c \in \mathbb{R}$ is a positive constant. Hence, we have

$$(f - a_1 \sin\theta)^{6a_2} = c(f - a_2 \sin\theta)^{6a_1},$$

which is just Eq. (3.14).

Remark 3.8 In the case when $\theta = \theta(u)$ is given by (3.13), one can compute the explicit expression of the function $\Psi = \Psi(u)$ as

$$\Psi(u) = \int_{u_0}^{u} \cos\theta(s)ds = \frac{1}{2a} \ln\left(e^{-4au} + 1\right) + u + c_0,$$

with $a = (-1 + \sqrt{13})/6$, $c_0 = \text{constant} \in \mathbb{R}$.

In this case, we can also compute, using Eq. (2.3), the Gaussian curvature of the surface

$$K = -\cos^2\theta - 2f \sin\theta = -\frac{4ae^{-4au} + (1 - e^{-4au})^2}{(1 + e^{-4au})^2} < 0.$$

Remark 3.9 The only possible solution of the biconservative Eq. (3.15) of the form $f = a \sin\theta$, where $a \in \mathbb{R}$ is a constant, is the one with $a = (-1 + \sqrt{13})/6$. Otherwise, Eq. (3.14) implies that θ is a constant and therefore $f = 0$.

Theorem 3.10 *Any biharmonic surface in Sol_3 is minimal.*

Proof The case of CMC biharmonic surfaces was studied in [31] so we only have to show that if Σ^2 is a biharmonic surface in Sol_3 such that there exists a point $p \in \Sigma^2$ at which $\nabla f \neq 0$, then the surface is minimal.

As before, we consider an open neighborhood $U \subset \Sigma^2$ of the point p such that $\nabla f \neq 0$ and $f > 0$ on U. As in the proof of Theorem 3.7, suffices treating only the case of a surface given by (3.11), the other one being similar. We will also use the same setting and notations as in the proof of Theorem 3.7.

From Theorem 2.2, we know that the normal part of the biharmonic equation is

$$\Delta f - f|A|^2 - f\langle \mathrm{trace}(\bar{R}(\cdot, \xi)\cdot), \xi\rangle = 0.$$

A direct computation, using Lemma 3.6, shows that

$$|A|^2 = 4f^2 + 4f \sin\theta + 2\sin^2\theta.$$

From formula (2.6) we find

$$\langle \mathrm{trace}(\bar{R}(\cdot, \xi)\cdot), \xi\rangle = 2\sin^2\theta.$$

It follows that the biconservative surface Σ^2 given by (3.11) is biharmonic if and only if

$$\Delta f = 4f(f^2 + f \sin\theta + \sin^2\theta) > 0. \tag{3.18}$$

Now, let us assume that θ is given by (3.13) and, therefore,

$$f(u) = \frac{2ae^{-2au}}{1 + e^{-4au}}, \quad a = \frac{-1 + \sqrt{13}}{6}.$$

From Lemma 3.6, we have that

$$\Delta f = f'' + \cos\theta\, f',$$

and then, a direct computation leads to

$$\Delta f = \frac{4e^{-6au}\left(e^{-4au}(2a^3 - a^2) + e^{4au}(2a^3 - a^2) + 2a^2 - 12a^3\right)}{\left(1 + e^{-4au}\right)^3}.$$

It is easy to verify that $\Delta f < 0$, in this case, which is a contradiction.

Next, let the function θ be a solution of Eq. (3.14) and again denote $g = f/\sin\theta$, as in the proof of Theorem 3.7. From Remark 3.9, we have that $g \neq 0$ throughout U.

The first derivative of f is given by the biconservative equation as

$$f' = -\frac{f\sin(2\theta)}{3f + \sin\theta},$$

where we have also used $\theta' = -2f$. Taking into account Remark 3.9 (and possibly restricting to an open subset $V \subset U$) we can see that f' is well defined. Using this formula, and again $\theta' = -2f$, it is straightforward to compute f'' and then

$$\begin{aligned}\Delta f = \frac{f}{(3f + \sin\theta)^3}\Big(&36f^3(1 - 2\sin^2\theta) - f^2(6\sin\theta + 18\sin^3\theta)\\ &- f(12\sin^2\theta - 8\sin^4\theta) + 2\sin^3\theta - 2\sin^5\theta\Big).\end{aligned}$$

Replacing in (3.18), and taking into account that $\sin\theta \neq 0$ on U, one obtains the following equation

$$\sin^2\theta\, P_1(g) - P_2(g) = 0, \tag{3.19}$$

where $P_1(g)$ and $P_2(g)$ are two functions of u, given by

$$P_1(g) = 100\, g^5 + 216\, g^4 + 324\, g^3 + 166\, g^2 + 32\, g + 6$$

and

$$P_2(g) = 36\, g^3 - 6\, g^2 - 12\, g + 2.$$

Since g satisfies Eq. (3.17), we can write

$$\sin^2\theta = \frac{(g - a_2)^{b_1}}{c(g - a_1)^{b_2}},$$

where $b_{1,2} = (6a_{1,2})/\sqrt{13}$ and $c \in \mathbb{R}$ is a positive constant. Then, Eq. (3.19) can be written as

$$(g - a_2)^{b_1} P_1(g) = c(g - a_1)^{b_2} P_2(g).$$

Differentiating with respect to u and using that $g'(u) \neq 0$ at any point of U, we obtain

$$(g-a_2)^{b_1}(g-a_1)\left(b_1 P_1 + (g - a_2)\frac{dP_1}{dg}\right) = c(g-a_1)^{b_1}(g-a_2)\left(b_2 P_2 + (g - a_1)\frac{dP_2}{dg}\right)$$

The last two equations imply

$$2(3\, g + 1)P_1 P_2 + (3\, g^2 + g - 1)\left(P_1\frac{dP_2}{dg} - P_2\frac{dP_1}{dg}\right) = 0.$$

After a long but simple computation, this polynomial equation reads as

$$25128\,g^8+92760\,g^7+85632\,g^6+15840\,g^5-19352\,g^4-13224\,g^3-1872\,g^2+656\,g+160=0,$$

and shows that g is a constant, which is a contradiction. We conclude that there are no biharmonic surfaces in Sol_3 other than the minimal ones.

Remark 3.11 This last result is similar to that in [6, 9, 18] on biharmonic surfaces of the Euclidean space $\mathbb{E}^3$ and even to the one in [3] about biharmonic surfaces in the Euclidean sphere $\mathbb{S}^3$. Although in this last case non-minimal biharmonic surfaces do exist, they are also CMC, and, therefore, there are no non-CMC such surfaces in $\mathbb{S}^3$. We note that similar results were also proved for hypersurfaces in space forms, with additional geometric hypotheses (see, for example, [1, 13, 14]).

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References

1. Andronic, Ş., Fu, Y., Oniciuc, C.: On the biharmonic hypersurfaces with three distinct principal curvatures in space forms. Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 68(116), no. 1, 35–58 (2025)
2. Andronic, Ş., Kayhan, A.: Rigidity results for compact biconservative hypersurfaces in space forms, 212, Paper No. 105460. arxiv:2409.18617 (2025)
3. Caddeo, R., Montaldo, S., Oniciuc, C.: Biharmonic submanifolds of $\mathbb{S}^3$. Int. J. Math. **12**, 867–876 (2001)
4. Caddeo, R., Montaldo, S., Oniciuc, C., Piu, P.: Surfaces in three-dimensional space forms with divergence-free stress-bienergy tensor. Ann. Mat. Pura Appl. **193**, 529–550 (2014)
5. Chen, B.Y.: Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. **17**, 169–188 (1991)
6. Chen, B.Y., Ishikawa, S.: Biharmonic surfaces in pseudo-Euclidean spaces. Mem. Fac. Sci. Kyushu Univ. Ser. A **45**, 323–347 (1991)
7. Daniel, B., Meeks, W.H., III., Rosenberg, H.: Half-space theorems for minimal surfaces in Nil_3 and Sol_3. J. Differ. Geom. **88**, 41–59 (2011)
8. Daniel, B., Mira, P.: Existence and uniqueness of constant mean curvature spheres in Sol_3. J. Reine Angew. Math. **685**, 1–32 (2013)
9. Dimitrić, I.: Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector. Bull. Inst. Math. Acad. Sinica **20**, 53–65 (1992)
10. Eells, J., Sampson, J.H.: Harmonic mappings of Riemannian manifolds. Am. J. Math. **86**, 109–160 (1964)
11. Fetcu, D., Nistor, S., Oniciuc, C.: On biconservative surfaces in 3-dimensional space forms. Comm. Anal. Geom. **24**, 1027–1045 (2016)
12. Fetcu, D., Oniciuc, C.: Biharmonic and biconservative hypersurfaces in space forms, Differential geometry and global analysis–in honor of Tadashi Nagano, 65–90, Contemp. Math. 777, Amer. Math. Soc., [Providence], RI (2022)
13. Fu, Y.: Biharmonic hypersurfaces with three distinct principal curvatures in spheres. Math. Nachr. **288**, 763–774 (2015)
14. Fu, Y.: Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space. Tohoku Math. J. **67**, 465–479 (2015)
15. Hasanis, T., Vlachos, T.: Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Math. Nachr. **172**, 145–169 (1995)
16. Jiang, G.Y.: 2-harmonic maps and their first and second variational formulas. Chin. Ann. Math. Ser. A **7**, 389–402 (1986)

17. Jiang, G.Y.: The conservation law for 2-harmonic maps between Riemannian manifolds. Acta Math. Sinica **30**, 220–225 (1987)
18. Jiang, G.Y.: Some nonexistence theorems on 2-harmonic and isometric immersions in Euclidean space. Chin. Ann. Math. Ser. B **8**, 377–383 (1987)
19. Klaser, P., Menezes, A.: The Dirichlet problem for the constant mean curvature equation in Sol_3. Illinois J. Math. **63**, 259–297 (2019)
20. López, R., Munteanu, M.I.: On the geometry of constant angle surfaces in Sol_3. Kyushu J. Math. **65**, 237–249 (2011)
21. Loubeau, E., Montaldo, S., Oniciuc, C.: The stress-energy tensor for biharmonic maps. Math. Z. **259**, 503–524 (2008)
22. Loubeau, E., Oniciuc, C.: Biharmonic surfaces of constant mean curvature. Pacific J. Math. **271**, 213–230 (2014)
23. Meeks, W.H., III.: Constant mean curvature spheres in Sol_3. Am. J. Math. **135**, 763–775 (2013)
24. Meeks, W.H., III., Mira, P., Pérez, J., Ros, A.: Constant mean curvature spheres in homogeneous three-manifolds. Invent. Math. **224**, 147–244 (2021)
25. Montaldo, S., Oniciuc, C., Ratto, A.: Proper biconservative immersions into the Euclidean space. Ann. Mat. Pura Appl. **195**, 403–422 (2016)
26. Montaldo, S., Oniciuc, C., Ratto, A.: Biconservative surfaces. J. Geom. Anal. **26**, 313–329 (2016)
27. Montaldo, S., Onnis, I.I., Passamani, A.P.: Biconservative surfaces in BCV-spaces. Math. Nachr. **290**, 2661–2672 (2017)
28. Nguyen, M.H.: The Dirichlet problem for the minimal surface equation in Sol_3, with possible infinite boundary data. Illinois J. Math. **58**, 891–937 (2014)
29. Nistor, S.: On biconservative surfaces. Differ. Geom. Appl. **54**, 490–502 (2017)
30. Ou, Y.L., Chen, B.Y.: Biharmonic submanifolds and biharmonic maps in Riemannian geometry. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2020)
31. Ou, Y.L., Wang, Z.P.: Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries. J. Geom. Phys. **61**, 1845–1853 (2011)
32. Souam, R., Toubiana, E.: Totally umbilic surfaces in homogeneous 3-manifolds. Comment. Math. Helv. **84**, 673–704 (2009)

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