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SUPERIORIZATION ON SOLUTION SETS OF COMMON FIXED POINT PROBLEMS WITH COUNTABLE FAMILIES OF MAPS

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Dedicated to Professor R. T. Rockafellar on the occasion of his 90th birthday

Abstract: In this work using subgradient algorithm we study a minimization problem with a convex objective function on a domain, which is the solution set of a common fixed point problem with a countable family of quasi-nonexpansive mappings. Our algorithm generates a sequence of iterates which are approximate solutions of the corresponding fixed point problem. Additionally, also either this sequence converges to a solution of our minimization problem or the sequence is strictly Fejer monotone regarding the solution set of the common fixed point problem.

Key words: *complete metric space, convergence analysis, fixed point, Hilbert space, subgradient*

Mathematics Subject Classification: *90C25, 90C26*

1 Introduction

The starting point of the fixed point theory of nonlinear operators is Banach's famous work [1] where it was established the existence of a unique fixed point of a strict contraction. Since that many important results were established in this area [11, 12, 14–16, 19, 20, 23–25, 28, 29] which includes the investigation of the asymptotic behavior of iterates of a nonlinear mappings. It also includes the studies of feasibility, common fixed points, iterative methods and variational inequalities and their applications in engineering, medical and the natural sciences [4, 6–8, 10, 13, 24, 25, 28–30]. In the study of common fixed point problems the goal is to find a common fixed point of a family of operators while for the feasibility problem the goal is to find a point which belongs to the intersection of a given family of sets. Usually in these studies the families of operators and sets are finite although there are interesting exceptions [2, 3, 17, 18, 21, 26, 27, 31, 32].

Assume that (X, ρ) is a metric space. A mapping $T : X \to X$ is called a strict contraction if there exists $\lambda \in (0, 1)$ such that

$$\rho(T(u), T(v)) \le \lambda \rho(u, v)$$

for each $u, v \in X$.

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According to the Banach's celebrated theorem [1] a strict contraction T has a fixed point $x_T \in X$ for which

$$T(x_T) = x_T$$

and which attracts every sequence of iterates of T. Moreover it is known that this convergence of iterates of T is uniform on all bounded subsets of X.

In [22] A. M. Ostrowski investigated the influence of computational errors on the behavior of iterates of the strict contraction T. He proved that every sequence $\{u_i\}_{i=0}^{\infty} \subset X$ for which

$$\sum_{i=0}^{\infty} \rho(u_{i+1}, T(u_i)) < \infty$$

converges and its limit is the fixed point of T.

A different approach was applied in [5] in order to generalize the result of [22] for a map $T : X \to X$ which is merely nonexpansive. We assumed that

$$\rho(T(v), T(u)) \le \rho(v, u)$$

for all pairs of points $v, u \in X$ and showed that if all sequences of exact iterates of T converge, then all the sequences of its inexact iterates with summable errors converges too. This result has many applications and is an essential ingredient in superiorization and perturbation resilience of algorithms [4, 6, 7]. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(X, \|\cdot\|)$ is a Banach space, $\rho(x, y) = \|x - y\|$ for all $x, y \in X$, a mapping $A : X \to X$ is nonexpansive and that for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. $x_0 \in X$, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty,$$

$\{v_k\}_{k=0}^{\infty} \subset X$ is a norm bounded sequence and that for any integer $k \ge 0$,

$$x_{k+1} = A(x_k + \beta_k v_k).$$

Then it follows from the result of [5] that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case the mapping A is called bounded perturbations resilient (see [6] and Definition 10 of [9]). In other words, if exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too. Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function. The superiorization technique was applied in [10,30] where it was investigated an optimization problem with a convex objective function and with a feasible region, which is the intersection of a finite family of closed convex constraint sets. In this work we investigate a minimization problem with a convex objective function on a domain, which is the solution set of a common fixed point problem with an infinite family of quasi-nonexpansive mappings. We use a subgradient algorithm which generates a sequence of iterates which converges to a solution of the corresponding fixed point problem. Additionally, also either this sequence converges to a solution of our minimization problem or the sequence is strictly Fejer monotone regarding the solution set of the common fixed point problem.

2 Preliminaries

Assume that (X, ρ) is a complete metric space endowed with a metric ρ. For every $x \in X$ and every $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

For each $x \in X$ and each nonempty set $D \subset X$ define

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Fix

$$\theta \in X.$$

Denote by $\mathrm{Card}(E)$ the cardinality of a set E. We assume that the sum over empty set is zero. For each set $C \subset X$ denote by $cl(C)$ its closure.

For each map $S : X \to X$ denote by S^0 the identity map, $S^0(x) = x$, $x \in X$, set $S^{i+1} = S \circ S^i$ for each integer $i \geq 0$ and define

$$\mathrm{Fix}(S) = \{z \in X : \ S(z) = z\}.$$

Assume that $T_i : X \to X$, $i = 1, 2, \dots$ are continuous mappings such that for each $i \in \{1, 2, \dots\}$,

$$\mathrm{Fix}(T_i) \neq \emptyset$$

and that for each $i \in \{1, 2, \dots\}$, each $z \in \mathrm{Fix}(T_i)$ and each $x \in X$,

$$\rho(z, T_i(x)) \leq \rho(z, x). \tag{2.1}$$

In the literature the following assumptions are used [28, 29].

(A1) There exists $\bar{c} \in (0, 1)$ such that for each integer $i \geq 1$, each $x \in X$ and each $z \in \mathrm{Fix}(T_i)$,

$$\rho(z, x)^2 \geq \rho(z, T_i(x))^2 + \bar{c}\rho(x, T_i(x))^2.$$

(A2) For each $M, \gamma > 0$ there exists $\delta > 0$ such that for each integer $i \geq 1$, each $x \in B(\theta, M)$ satisfying $\rho(x, T_i(x)) \geq \gamma$ and each $z \in \mathrm{Fix}(T_i) \cap B(\theta, M)$ the inequality

$$\rho(z, x) - \delta \geq \rho(z, T_i(x))$$

holds.

(A3) For each $M, \gamma > 0$ there exists $\delta > 0$ such that for each integer $i \geq 1$, each $x \in B(\theta, M)$ satisfying $\rho(x, \mathrm{Fix}(T_i)) \geq \gamma$ and each $z \in \mathrm{Fix}(T_i) \cap B(\theta, M)$ the inequality

$$\rho(z, x) - \delta \geq \rho(z, T_i(x))$$

holds.

Note that assumptions (A1) holds for projections on convex, closed sets in a Hilbert space (see Proposition 10.1 of [28]).

It is known [32] that (A1) implies (A2) and (A3) implies (A2).

3 The Common Fixed Point Problem

We continue to use the notation and definitions of Section 2. Throughout the paper we use the following notation.

Assume that for each integer $i \geq 0$,

$$\Omega_i \subset \{1, 2, \dots\}$$

is nonempty,

$$\Omega_i \subset \Omega_{i+1},\ \cup_{i=0}^{\infty}\Omega_i = \{1, 2, \dots\} \tag{3.1}$$

and there exists $s_0 \in \Omega_0$ such that

$$cl(T_{s_0}(X)) \cap B(\theta, r) \text{ is compact for every } r > 0. \tag{3.2}$$

Suppose that

$$F := \cap_{i=1}^{\infty}\mathrm{Fix}(T_i) \neq \emptyset. \tag{3.3}$$

Our goal is to find an approximate solution of the inclusion

$$z \in F.$$

We consider the following algorithm. Assume that $\{\Delta_i\}_{i=0}^{\infty} \subset (0, 1]$.

Initialization: choose any element $x_0 \in X$.

Iterative step: given a current iterate x_t calculate x_{t+1} such that

$$B(x_{t+1}, \Delta_t) \cap \{T_j(x_t) :\ j \in \Omega_t,\ \rho(x_t, T_j(x_t)) \geq \rho(x_t, T_i(x_t)),\ i \in \Omega_t\} \neq \emptyset.$$

Clearly, our algorithm is well-defined if all the sets Ω_i, $i = 0, 1, \dots$ are finite. This is the case which happens in practice.

The convergence of this algorithm was studied in [32] when $\sum_{i=0}^{\infty}\Delta_i < \infty$. According to our algorithm, given a current iterate x_t we first find $j \in \Omega_t$ satisfying

$$\rho(x_t, T_j(x_t)) \geq \rho(x_t, T_i(x_t)),\ i \in \Omega_t$$

and then choose $x_{t+1} \in B(T_j(x_t), \Delta_t)$. We do this according to the result of [5].

The following auxiliary result was proved in [32].

Lemma 3.1. *Let $M, \epsilon > 0$. Then there exist $\delta > 0$ and an integer $t \geq 0$ such that for each $x \in B(\theta, M)$ satisfying*

$$\rho(x, T_i(x)) \leq \delta,\ i \in \Omega_t$$

the inequality $\rho(x, F) \leq \epsilon$ holds.

The following theorem in one of the main results of [32].

Theorem 3.2. *Assume that (A2) holds, $M > 1$, $\epsilon \in (0, 1]$, $\{\Delta_i\}_{i=0}^{\infty} \subset (0, 1]$,*

$$\sum_{i=0}^{\infty}\Delta_i < \infty,$$

$$F \cap B(\theta, M) \neq \emptyset.$$

Then there exists an integer $q \geq 1$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying

$$\rho(x_0, \theta) \leq M$$

and such that for each integer $t \geq 0$,

$$B(x_{t+1}, \Delta_t) \cap \{T_j(x_t) : \ j \in \Omega_t, \ \rho(x_t, T_j(x_t)) \geq \rho(x_t, T_i(x_t)), \ i \in \Omega_t\} \neq \emptyset$$

there exist

$$z_0 \in F$$

such that

$$\rho(z_0, x_t) \leq \epsilon \text{ for all integers } t \geq q$$

and the sequence $\{x_t\}_{t=0}^{\infty}$ *converges to a point of* F.

4 The First Main Result

From now we assume that our space X is a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the complete norm

$$\|x\| = \langle x, x \rangle^{1/2}, \ x \in X$$

and that $\rho(x, y) = \|x - y\|$, $x, y \in X$. We use all the definitions and notation introduced in Sections 2, 3 and assume all their assumptions assumed there. In particular, we assume that (A2) holds. Assume that $f : X \to R^1$ is a convex continuous function such that

$$\lim_{\|x\| \to \infty} f(x) = \infty. \tag{4.1}$$

For each $x \in X$,

$$\partial f(x) = \{l \in X : \ \langle l, y - x \rangle \leq f(y) - f(x), \ y \in X\}$$

is the subdifferential of the function f at x. For each nonempty set $C \subset X$ set

$$\inf(f; C) = \inf\{f(x) : \ x \in C\}, \ \operatorname{argmin}(f; C) = \{x \in C : \ f(x) = \inf(f; C)\}.$$

We consider the minimization problem

$$f(x) \to \min, \ x \in F.$$

Since for each integer $i \geq 1$ the operator T_i is continuous, the set $\operatorname{Fix}(T_i)$ is closed. Therefore the set F is closed too. In view of (3.2), the following property holds:

(i) For each $r > 0$, the set $F \cap B(0, r)$ is compact.

By (4.1), $\inf(f; F)$ is finite and the set

$$\{\xi \in F : \ f(\xi) \leq \inf(f; F) + 4\}$$

is compact. Assume that $\theta = 0$, $M > 1$,

$$\{\xi \in X : \ f(\xi) \leq \inf(f; F) + 4\} \subset B(0, M), \tag{4.2}$$

$L \geq 1$,

$$|f(\xi_1) - f(\xi_2)| \leq L\|\xi_1 - \xi_2\|, \ \xi_1, \xi_2 \in B(0, 3M + 2). \tag{4.3}$$

We consider the following subgradient algorithm. Assume that

$$\{\alpha_i\}_{i=0}^{\infty} \subset (0, L^{-2}], \tag{4.4}$$

$$\sum_{i=0}^{\infty} \alpha_i < \infty. \tag{4.5}$$

Initialization: choose any element $x_0 \in B(0, M)$.
Iterative step: given a current iterate $x_t \in X$ calculate

$$u_t \in \partial f(x_t) \tag{4.6}$$

and

$$j(t) \in \Omega_t \tag{4.7}$$

such that

$$\|x_t - \alpha_t u_t - T_{j(t)}(x_t - \alpha_t u_t)\| \geq \|x_t - \alpha_t u_t - T_i(x_t - \alpha_t u_t)\|,\ i \in \Omega_t \tag{4.8}$$

and set

$$x_{t+1} = T_{j(t)}(x_t - \alpha_t u_t). \tag{4.9}$$

Clearly, our algorithm is well-defined if all the sets Ω_i, $i = 0, 1, \ldots$ are finite. This is the case which happens in practice. It follows from the compactness of the set $F \cap B(0, M)$ and (4.2) that

$$\operatorname{argmin}(f; F) \neq \emptyset,\ \operatorname{argmin}(f; F) \subset B(0, M).$$

We prove our first main result.

Theorem 4.1. *The sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X to $x_* \in F$ and one of the following cases holds:*

(i) $f(x_*) = \inf(f; F)$;

(ii) $f(x_*) > \inf(f; F)$ *and there exist a natural number k_0 and $\Delta_0 > 0$ such that for each $x \in argmin(f; F)$ and each integer $k \geq k_0$,*

$$\|x_k - x\|^2 \leq \|x_{k-1} - x\|^2 - \Delta_0 \alpha_{k-1}.$$

5 An Auxiliary Lemma

We use all the definitions, notation and assume all the assumptions from Section 4.

Lemma 5.1. *Assume that*

$$z \in argmin(f, F), \tag{5.1}$$

$$\alpha \in (0, L^{-2}], \tag{5.2}$$

$$x \in B(z, 2M + 1), \tag{5.3}$$

$P : X \to X$,

$$\|z - P(\xi)\| \leq \|z - \xi\|,\ \xi \in X, \tag{5.4}$$

$x \in X$,

$$u \in \partial f(x), \tag{5.5}$$

$$y = P(x - \alpha u). \tag{5.6}$$

Then the following assertions hold.

$$\|y - z\|^2 \leq \|x - z\|^2 + \alpha^2 L^2 + 2\alpha(f(z) - f(x)),$$
$$\|y - z\| \leq 2M + 1.$$

Proof. In view of (5.5),

$$\langle u, z - x\rangle \le f(z) - f(x). \tag{5.7}$$

By (4.2), (5.1), (5.3) and (5.5),

$$\|x\| \le 3M + 1,\ |u\| \le L. \tag{5.8}$$

It follows from (5.4), (5.6), (5.7) and (5.8) that

$$\begin{aligned}\|y - z\|^2 = \|P(x - \alpha u) - z\|^2 &\le \|x - \alpha u - z\|^2\\ &\le \|x - z\|^2 + \alpha^2\|u\|^2 - 2\alpha\langle u, x - z\rangle\\ &\le \|x - z\|^2 + \alpha^2\|u\|^2 + 2\alpha(f(z) - f(x))\\ &\le \|x - z\|^2 + \alpha^2 L^2 + 2\alpha(f(z) - f(x)).\end{aligned} \tag{5.9}$$

There are two cases:

$$f(x) \le f(z) + 4, \tag{5.10}$$

$$f(x) > f(z) + 4. \tag{5.11}$$

Assume that (5.10) holds. It follows from (5.1) and (5.10) that

$$\|x\| \le M. \tag{5.12}$$

By (4.2), (5.2), (5.4), (5.6), (5.8), (5.12),

$$\begin{aligned}\|y - z\| = \|P(x - \alpha u) - z\| &\le \|x - \alpha u - z\|\\ &\le \|x - z\| + \alpha\|u\| \le 2M + \alpha L \le 2M + 1.\end{aligned}$$

Assume that (5.11) holds. By (5.2), (5.9) and (5.11),

$$\|y - z\|^2 \le \|x - z\|^2 + \alpha^2 L^2 - 4\alpha L \le \|x - z\|^2,$$

$$\|y - z\| \le \|x - z\| \le 2M + 1.$$

Thus in both cases

$$\|y - z\| \le 2M + 11.$$

This completes the proof of Lemma 5.1 itself. □

6 Proof of Theorem 4.1

Fix

$$z \in \operatorname{argmin}(f; F). \tag{6.1}$$

In view of (4.2) and (6.1),

$$\|z\| \le M. \tag{6.2}$$

We show that for each integer $t \ge 0$,

$$\|x_t - z\| \le 2M + 1. \tag{6.3}$$

By (6.2) and the relation $\|x_0\| \le M$, inequality (6.3) holds for $t = 0$.

Assume that $t \geq 0$ is an integer and (6.3) is true. In view of (4.9),

$$x_{t+1} = T_{j(t)}(x_t - \alpha_t u_t). \tag{6.4}$$

Equations (4.4), (4.6), (6.1), (6.3), (6.4) and Lemma 5.1 applied with $P = T_{j(t)}$, $x = x_t$, $u = u_t$, $\alpha = \alpha_t$, $y = x_{t+1}$ imply that

$$\|x_{t+1} - z\| \leq 2M + 1,$$

$$\|x_{t+1} - z\|^2 \leq \|x_t - z\|^2 + \alpha_t^2 L^2 + 2\alpha_t(f(z) - f(x_t)). \tag{6.5}$$

Thus we showed by induction that (6.3) and (6.5) hold for each integer $t \geq 0$. In view of (4.3), (4.6), (6.2) and (6.3),

$$\|u_t\| \leq L, \ t = 0, 1, \ldots. \tag{6.6}$$

For each integer $t \geq 0$, set

$$y_t = x_t - \alpha_t u_t. \tag{6.7}$$

By (4.8) and (6.7), for each integer $t \geq 0$ and each $i \in \Omega_t$,

$$\|T_{j(t)}(y_t) - y_t\| = \|T_{j(t)}(x_t - \alpha_t u_t) - (x_t - \alpha_t u_t)\|$$

$$\geq \|T_i(x_t - \alpha_t u_t) - (x_t - \alpha_t u_t)\| = \|T_i(y_t) - y_t\|, \tag{6.8}$$

$$y_{t+1} = x_{t+1} - \alpha_{t+1} u_{t+1} \tag{6.9}$$

and in view of (6.4), (6.6),

$$\|y_{t+1} - T_{j(t)}(y_t)\| = \|y_{t+1} - x_{t+1}\| \leq \alpha_{t+1} L. \tag{6.10}$$

In view of (6.6) and (6.7),

$$\|y_0\| \leq \|x_0\| + \alpha_0 L \leq M + 1.$$

Equations (4.4), (6.1), (6.2), (6.8), (6.10) and Theorem 3.2 applied with $x_t = y_t$, $\Delta_t = \alpha_{t+1} L$, $t = 0, 1, \ldots$ imply that there exists

$$x_* \in F$$

such that

$$\lim_{t \to \infty} \|y_t - x_*\| = 0. \tag{6.11}$$

It follows from (6.6), (6.7) and (6.11) that

$$\lim_{t \to \infty} \|x_t - x_*\| \leq \lim_{t \to \infty} \|x_t - y_t\| + \lim_{t \to \infty} \|y_t - x_*\| = 0. \tag{6.12}$$

Assume that the case (i) does not hold. This implies that there exists $\Delta_0 > 0$ such that

$$f(x_*) > f(z) + 2\Delta_0. \tag{6.13}$$

By (4.5), (6.12) and (6.13), there exists a natural number k_0 such that for all integers $k \geq k_0$,

$$\alpha_{k-1} \leq L^{-2} \Delta_0$$

and

$$f(x_{k-1}) > f(z) + 2\Delta_0.$$

Let $k \geq k_0$ be an integer. By (6.5) and the relations above,

$$\|x_k - z\|^2 \leq \|x_{k-1} - z\|^2 + \alpha_{k-1}^2 L^2 - 2\alpha_{k-1}\Delta_0$$

$$\leq \|x_{k-1} - z\|^2 - \alpha_{k-1}(2\Delta_0 - \alpha_{k-1} L^2)$$

$$\leq \|x_{k-1} - z\|^2 - \alpha_{k-1}\Delta_0.$$

Theorem 4.1 is proved.

7 The Second Main Result

We use all the definitions, notation and assume all the assumptions from Section 4.

Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. For each $x, y \in X$, $\rho(x, y) = \|x - y\|$.

It is well-known that the following proposition holds (see Proposition 10.1 of [28]).

Proposition 7.1. *Let C be a nonempty, closed convex subset of X. Then for every point $x \in X$ there exists a unique point $P_C(x) \in C$ which satisfies*

$$\|x - P_C(x)\| = \rho(x, C).$$

Moreover

$$\|P_C(x) - P_C(y)\| \le \|x - y\| \text{ for all } x, y \in X$$

and for every point $x \in X$ and every point $z \in C$,

$$\langle z - P_C(x), x - P_C(x) \rangle \le 0,$$

$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2.$$

Let $f : X \to R^1$ be a continuous convex function such that

$$\{x \in X : \ f(x) \le 0\} \ne \emptyset. \tag{7.1}$$

Clearly, for every $y_0 \in X$ and $l \in \partial f(y_0)$,

$$\{x \in X : \ f(x) \le 0\} \subset \{x \in X : \ f(y_0) + \langle l, x - y_0 \rangle \le 0\}. \tag{7.2}$$

It is well-known that the following lemma holds (see Lemma 10.2 of [28]).

Lemma 7.2. *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D = \{x \in X : \ f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

Then $l \ne 0$ and

$$P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l. \tag{7.3}$$

Let $X = R^n$ and for each integer $i \ge 1$, $f_i : R^n \to R^1$ be a convex continuous function,

$$C_i = \{x \in X : \ f_i(x) \le 0\}, \tag{7.4}$$

$$T_i = P_{C_i} \tag{7.5}$$

and

$$C = \cap_{i=1}^{\infty} C_i. \tag{7.6}$$

It is easy to see that for each integer $i \ge 1$, the set C_i is closed, the function f_i is Lipschitz on all bounded sets and the set C is closed too. Clearly, (3.2) holds. Our goal is to find an approximate solution of the inclusion

$$z \in C.$$

Recall that for each integer $i \ge 0$,

$$\Omega_i \subset \{1, 2, \dots\}$$

is nonempty,

$$\Omega_i \subset \Omega_{i+1}, \ \cup_{i=0}^{\infty} \Omega_i = \{1, 2, \dots\}.$$

The following result was obtained in [32].

Theorem 7.3. *Assume that* $\{\Delta_i\}_{i=0}^{\infty},\ \{\gamma_i\}_{i=0}^{\infty} \subset (0,1],\ M, L \geq 1,$

$$\sum_{i=0}^{\infty} \Delta_i < \infty,\ \sum_{i=0}^{\infty} \gamma_i < \infty,$$

$$|f_i(u) - f_i(v)| \leq L\|u - v\|,\ u, v \in B(0, 3M + \sum_{j=0}^{\infty} \Delta_j + 1),\ i = 1, 2, \ldots$$

and

$$B(0, M) \cap C \neq \emptyset,$$

Then for each sequence $\{x_t\}_{t=0}^{\infty} \subset X$, *each sequence* $\{l_t\}_{t=0}^{\infty} \subset X$ *and each sequence of natural numbers* $\{j(t)\}_{t=0}^{\infty}$ *which satisfy*

$$\|x_0\| \leq M,$$

and that for each $t \geq 0$,

$$j(t) \in \Omega_t,$$

$$f_{j(t)}(x_t) \geq f_i(x_t),\ i \in \Omega_t,$$

if

$$f_{j(t)}(x_t) \leq \gamma_t,\ \textit{then}\ \|x_{t+1} - x_t\| \leq \Delta_t,\ l_t = 0,$$

if

$$f_{j(t)}(x_t) > \gamma_t,\ \textit{then}\ l_t \in \partial f_{j(t)}(x_t),$$

$$\|x_{t+1} - (x_t - f_{j(t)}(x_t)\|l_t\|^{-2} l_t)\| \leq \Delta_t$$

the inequality

$$\|x_t\| \leq 3M + \sum_{i=0}^{\infty} \Delta_i,\ t = 0, 1, \ldots$$

holds and the sequence $\{x_i\}_{i=0}^{\infty}$ *converges to a point of* C.

Assume that $f : X \to R^1$ is a convex continuous function such that

$$\lim_{\|x\| \to \infty} f(x) = \infty. \tag{7.7}$$

We consider the minimization problem

$$f(x) \to \min,\ x \in C.$$

Clearly, $\inf(f; C)$ is finite and the set

$$\{\xi \in F :\ f(\xi) \leq \inf(f; C) + 4\} \text{ is compact.} \tag{7.8}$$

Assume that $M \geq 1$,

$$\{\xi \in X :\ f(\xi) \leq \inf(f; C) + 4\} \subset B(0, M), \tag{7.9}$$

$L \geq 1$,

$$|f(\xi_1) - f(\xi_2)| \leq L\|\xi_1 - \xi_2\|,\ \xi_1, \xi_2 \in B(0, 3M + 2). \tag{7.10}$$

We consider the following subgradient algorithm. Assume that

$$\{\alpha_i\}_{i=0}^{\infty} \subset (0, L^{-2}], \tag{7.11}$$

$\{\gamma_i\}_{i=0}^{\infty} \subset (0,1]$ satisfy

$$\sum_{i=0}^{\infty} \Delta_i < \infty, \ \sum_{i=0}^{\infty} \gamma_i < \infty. \tag{7.12}$$

Initialization: choose any element

$$x_0 \in B(0, M). \tag{7.13}$$

Iterative step: for each integer $t \geq 0$, given a current iterate $x_t \in X$ calculate

$$u_t \in \partial f(x_t) \tag{7.14}$$

and

$$j(t) \in \Omega_t \tag{7.15}$$

such that

$$f_{j(t)}(x_t - \alpha_t u_t) \geq f_i(x_t - \alpha_t u_t), \ i \in \Omega_t, \tag{7.16}$$

if

$$f_{j(t)}(x_t - \alpha_t u_t) \leq \gamma_t, \text{ then}$$
$$l_t = 0, \ x_{t+1} = x_t - \alpha_t u_t, \tag{7.17}$$

if

$$f_{j(t)}(x_t - \alpha_t u_t) > \gamma_t, \text{ then calculate}$$
$$l_t \in \partial f_{j(t)}(x_t), \tag{7.18}$$
$$x_{t+1} = x_t - \alpha_t u_t - f_{j(t)}(x_t - \alpha_t u_t)\|l_t\|^{-2} l_t. \tag{7.19}$$

We prove the following result.

Theorem 7.4. *The sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X to $x_* \in C$ and one of the following cases holds:*

(i) $f(x_*) = \inf(f; C)$;

(ii) $f(x_*) > \inf(f; C)$ and there exist a natural number k_0 and $\Delta_0 > 0$ such that for each $x \in \operatorname{argmin}(f; C)$ and each integer $k \geq k_0$,

$$\|x_k - x\|^2 \leq \|x_{k-1} - x\|^2 - \Delta_0 \alpha_{k-1}.$$

Proof. Fix

$$z \in \operatorname{argmin}(f; C). \tag{7.20}$$

In view of (7.9), (7.13) and (7.20),

$$\|z\| \leq M, \ \|x_0 - z\| \leq 2M. \tag{7.21}$$

We show that for each integer $t \geq 0$,

$$\|x_t - z\| \leq 2M + 1. \tag{7.22}$$

By (7.21), inequality (7.22) holds for $t = 0$.

Assume that $t \geq 0$ is an integer and (7.22) is true. There are two cases:

$$f_{j(t)}(x_t - \alpha_t u_t) \leq \gamma_t; \tag{7.23}$$

$$f_{j(t)}(x_t - \alpha_t u_t) > \gamma_t. \tag{7.24}$$

Assume that (7.23) holds. By (7.17),

$$x_{t+1} = x_t - \alpha_t u_t.$$

Equations (7.9), (7.10), (7.11), (7.14), (7.20), (7.22), the equality above and Lemma 5.1 applied with the identity operator P, $x = x_t$, $u = u_t$, $\alpha = \alpha_t$, $y = x_t - \alpha_t u_t$ imply that

$$\|x_{t+1} - z\| \leq 2M + 1, \tag{7.25}$$

$$\|x_{t+1} - z\|^2 \leq \|x_t - z\|^2 + \alpha_t^2 L^2 + 2\alpha_t(f(z) - f(x_t)). \tag{7.26}$$

Assume that (7.24) holds. Lemma 7.2 and equations (7.18), (7.19), (7.24) imply that

$$x_{t+1} = x_t - \alpha_t u_t - f_{j(t)}(x_t - \alpha_t u_t)\|l_t\|^{-2} l_t$$

$$= P_D(x_t - \alpha_t u_t) \tag{7.27}$$

where

$$D = \{x \in X : \ f_{j(t)}(x_t - \alpha_t u_t) + \langle l_t, x - (x_t - \alpha_t u_t)\rangle \leq 0\} \supset C_{j(t)}. \tag{7.28}$$

Equations (7.9)-(7.11), (7.14), (7.20), (7.22), (7.27), (7.28) and Lemma 5.1 applied with $P = P_D$, $x = x_t$, $u = u_t$, $\alpha = \alpha_t$, $y = x_{t+1}$ imply that

$$\|x_{t+1} - z\| \leq 2M + 1,$$

$$\|x_{t+1} - z\|^2 \leq \|x_t - z\|^2 + \alpha_t^2 L^2 + 2\alpha_t(f(z) - f(x_t)).$$

Thus in both cases (7.25) and (7.26) hold and we showed by induction that (7.25) and (7.26) hold for each integer $t \geq 0$. In view of (7.11), (7.14), (7.21) and (7.25),

$$\|u_t\| \leq L, \ t = 0, 1, \ldots. \tag{7.29}$$

For each integer $t \geq 0$, set

$$y_t = x_t - \alpha_t u_t. \tag{7.30}$$

By (7.29) and (7.30), for each integer $t \geq 0$,

$$\|y_t - x_t\| \leq \alpha_t L. \tag{7.31}$$

By (7.16), (7.17), (7.30) and (7.31), for each integer $t \geq 0$,

$$f_{j(t)}(y_t) \geq f_i(y_t), \ i \in \Omega_t, \tag{7.32}$$

if

$$f_{j(t)}(y_t) \leq \gamma_t,$$

then

$$\|y_{t+1} - y_t\| \leq \|y_{t+1} - x_{t+1}\| + \|x_{t+1} - y_t\| \leq \alpha_{t+1} L. \tag{7.33}$$

Assume that

$$f_{j(t)}(y_t) > \gamma_t.$$

Then by (7.18), (7.19), (7.30) and (7.31),

$$l_t \in \partial f_{j(t)}(y_t), \tag{7.34}$$

$$\begin{aligned}\|y_{t+1} - (y_t - f_{j(t)}(y_t)\|l_t\|^{-2}l_t)\| \\ \leq \|y_{t+1} - x_{t+1}\| + \|x_{t+1} - (x_t - \alpha_t u_t - f_{j(t)}(x_t - \alpha_t u_t)\|l_t\|^{-2}l_t)\| \\ \leq \|y_{t+1} - x_{t+1}\| \leq \alpha_{t+1}L.\end{aligned} \tag{7.35}$$

In view of (7.21),

$$\|y_0\| \leq M + 1.$$

Equations (7.32)-(7.35) and Theorem 7.3 applied with $x_t = y_t$, $\Delta_t = \alpha_{t+1}L$, $t = 0, 1, \ldots$ imply that there exists

$$x_* \in C$$

such that

$$\lim_{t\to\infty} \|y_t - x_*\| = 0.$$

Together with (7.3) this implies that

$$\lim_{t\to\infty} \|x_t - x_*\| = 0.$$

Assume that the case (i) does not hold. This implies that there exist $\Delta_0 > 0$ such that

$$f(x_*) > f(z) + 2\Delta_0$$

and a natural number k_0 such that for all integers $k \geq k_0$,

$$\alpha_{k-1} \leq L^{-2}\Delta_0,$$

$$f(x_{k-1}) > f(z) + 2\Delta_0$$

and together with (7.26) this implies that

$$\begin{aligned}\|x_k - z\|^2 \leq \|x_{k-1} - z\|^2 + \alpha_{k-1}^2 L^2 - 2\alpha_{k-1}\Delta_0 \\ \leq \|x_{k-1} - z\|^2 - \alpha_{k-1}(2\Delta_0 - \alpha_{k-1}L^2) \\ \leq \|x_{k-1} - z\|^2 - \alpha_{k-1}\Delta_0.\end{aligned}$$

Thus (ii) holds and Theorem 7.4 is proved. □

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fund. Math.* 3 (1922) 133–181.

[2] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Mathematische Zeitschrift* 100 (1967) 201–225.

[3] D. Butnariu, Y. Censor and S. Reich, Iterative averaging of entropic projections for solving stochastic convex feasibility problems, *Comp. Optim. Appl.* 8 (1997) 21–39.

[4] D. Butnariu, R. Davidi, G.T. Herman and I.G. Kazantsev, Stable convergence behavior under summable perturbations of a class of projection methods for convex feasibility and optimization problems, *IEEE J. Selected Topics Signal Processing* 1 (2007) 540–547.

[5] D. Butnariu, S. Reich and A.J. Zaslavski, Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces, in: Proceedings of Fixed Point Theory and its Applications, Mexico, Yokahama Publishers, 2006, pp. 11–32.

[6] Y. Censor, R. Davidi and G. T. Herman, Perturbation resilience and superiorization of iterative algorithms, *Inverse Problems* **26** (2010): 65008.

[7] Y. Censor, R. Davidi, G.T. Herman, R.W. Schulte and L. Tetruashvili, Projected subgradient minimization versus superiorization, J. Optim. Theory Appl. 160 (2014) 730–747.

[8] Y. Censor, T. Elfving and G.T. Herman, Averaging strings of sequential iterations for convex feasibility problems, in: Inherently Parallel Algorithms in Feasi- bility and Optimization and their Applications, D. Butnariu, Y. Censor and S. Reich (eds.), North-Holland, Amsterdam, 2001, pp. 101–113.

[9] Y. Censor and A. J. Zaslavski, Convergence and perturbation resilience of dynamic string-averaging projection methods, *Comp. Optim. Appl.* 54 (2013) 65–76.

[10] Y. Censor and A. J. Zaslavski, Strict Fejér monotonicity by superiorization of feasibility-seeking projection methods, *J. Optim. Theory Appl.* 165 (2015) 172–187.

[11] B. Djafari-Rouhani, K.R. Kazmi, S. Moradi, R. Ali and S.A. Khan, *Solving the split equality hierarchical fixed point problem*, *Fixed Point Theory* **23** (2022) 351–369.

[12] W.-S. Du, Some generalizations of fixed point theorems of Caristi type and Mizoguchi–Takahashi type under relaxed conditions, *Bull. Braz. Math. Soc. (N.S.)* 50 (2019) 603–624.

[13] A. Gibali, A new split inverse problem and an application to least intensity feasible solutions, *Pure Appl. Funct. Anal.* 2 (2017) 243–258.

[14] A. Gibali, S. Reich and R. Zalas, Outer approximation methods for solving variational inequalities in Hilbert space, *Optim.* 66 (2017) 417–437.

[15] K. Goebel and W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[16] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[17] V.I. Kolobov, S. Reich and R. Zalas, Finitely convergent deterministic and stochastic iterative methods for solving convex feasibility problems, *Math. Program.* 194 (2022) 1163–1183.

[18] T.Y. Kong , H. Pajoohesh and G.T. Herman, String-averaging algorithms for convex feasibility with infinitely many sets, *Inverse Problems* 35 (2019): 045011.

[19] E. Kopecká and S. Reich, A note on alternating projections in Hilbert space, *J. Fixed Point Theory Appl.* 12 (2012) 41–47.

[20] F. Lenzen, F. Becker, J. Lellmann, S. Petra and C. Schnörr, A class of quasi-variational inequalities for adaptive image denoising and decomposition, *Comput. Optim. Appl.* 54 (2013) 371–398.

[21] W. Nilsrakoo and S. Saejung, Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings, *Fixed Point Theory Appl.* 2008 (2008): Art. ID 312454, 19.

[22] A.M. Ostrowski, The round-off stability of iterations, *Z. Angew. Math. Mech.* 47 (1967) 77–81.

[23] S. Reich and A. J. Zaslavski, *Genericity in Nonlinear Analysis*, Developments in Mathematics, vol. 34, Springer, New York, 2014.

[24] W. Takahashi, The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces, *Pure Appl. Funct. Anal.* 2 (2017) 685–699.

[25] W. Takahashi, A general iterative method for split common fixed point problems in Hilbert spaces and applications, *Pure Appl. Funct. Anal.* 3 (2018) 349–369.

[26] S. Wang, L. Yu and B. Guo, An implicit iterative scheme for an infinite countable family of asymptotically nonexpansive mappings in Banach spaces, *Fixed Point Theory Appl.* 2008 (2008): Art. ID 350483, 10.

[27] R. Wangkeeree, An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings, *Fixed Point Theory Appl.* 2008 (2008): Art. ID 134148, 17.

[28] A.J. Zaslavski, *Approximate solutions of common fixed point problems*, Springer Optimization and Its Applications, Springer, Cham, 2016.

[29] A.J. Zaslavski, *Algorithms for Solving Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, Cham, 2018.

[30] A.J. Zaslavski, Superiorization with a projected subgradient method on bounded sets, *Communications Applied Nonlinear Anal.* 29 (2022) 63–72.

[31] A.J. Zaslavski, Solving feasibility problems with infinitely many sets, *Axioms* 12 (2023): 273.

[32] A.J. Zaslavski, Solving common fixed point problems with a countable family of operators, *Vietnam Journal of Mathematics*, (2024). https://doi.org/10.1007/s10013-024-00681-3.

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