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BOREL SUBALGEBRAS OF LIE ALGEBRAS OF VECTOR FIELDS

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To Leonid Makar-Limanov on occasion of his 80th birthday

Abstract. Let X be an affine variety. In the present paper, we introduce the notion of a locally integrable Borel subalgebra of the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X))$. We show that these are precisely the tangent algebras of the Borel subgroups of $\mathrm{Aut}(X)$. We classify the locally integrable Borel subalgebras in the Lie algebras of the groups $\mathrm{Aut}(S)$ of toric affine surfaces S, notably of the affine plane and its cyclic quotients. See [5] for a description of the Borel subgroups and, more generally, maximal solvable subgroups of $\mathrm{Aut}(S)$.

Contents

1. Introduction 1
2. Preliminaries 4
2.1. Tangent Lie algebras of derivations 4
2.2. Correspondence "ind-subgroups ⇜⇝ tangent subalgebras" 5
3. Correspondence "Borel subgroups ⇜⇝ Borel subalgebras" 9
4. Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$ for small n 11
4.1. Borel subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ 11
4.2. The case of the affine line 11
4.3. The case of the affine plane 13
4.4. The case of $\mathbb{A}^3$ 17
5. Subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ 18
5.1. Generalities 18
5.2. On the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ 19
5.3. The Lie algebra of zero divergence vector fields on $X_{d,e}$ is not simple 24
References 25

1. Introduction

We are working over an algebraically closed field $\Bbbk$ of characteristic zero; $\mathbb{A}^n$ stands for the affine space over $\Bbbk$ of dimension n, and $\mathbb{G}_{\mathrm{a}}$ resp. $\mathbb{G}_{\mathrm{m}}$ for the additive resp. multiplicative group of $\Bbbk$.

Recall that a *Borel subalgebra* of a Lie algebra $\mathfrak{g}$ is a maximal solvable Lie subalgebra of $\mathfrak{g}$, that is, a solvable Lie subalgebra that is maximal by inclusion among the solvable Lie subalgebras of $\mathfrak{g}$. A *Borel subgroup* of an ind-group $\mathfrak{G}$ is a maximal solvable connected subgroup of $\mathfrak{G}$.

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Let G be a connected affine algebraic group over $\Bbbk$ with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. It is well known that the correspondence $B \mapsto \mathrm{Lie}(B)$ yields a bijection between the Borel subgroups of G and the Borel subalgebras of $\mathfrak{g}$. In this case, all Borel subgroups are conjugate, and all Borel subalgebras are Ad-conjugate. The latter is also true for all finite dimensional Lie algebras in characteristic zero: all their Borel subalgebras are conjugate under the action of the automorphism group, see [17, Theorem 16.4]. However, this breaks down for some simple Lie algebras of Cartan type in positive characteristic $p > 0$, as follows from [16, Remark 14.3]. For example, if $p > 3$, then the first Witt Lie algebra W_1 has precisely two conjugacy classes of Borel subalgebras, see [32, Theorem 4.2]. See also [7, Theorem 4.6] for the conjugacy classes of Borel subalgebras of W_n in positive characteristic $p > 3$.

In the following, X denotes an irreducible affine algebraic variety over $\Bbbk$. The automorphism group $\mathrm{Aut}(X)$ has a natural affine ind-group structure, see e.g. [13]. Its Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X))$ is a subalgebra of the Lie algebra of derivations $\mathrm{Der}(\mathcal{O}(X))$. For example, the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^n)$ consists of polynomial vector fields over $\mathbb{A}^n$ with constant divergence, see e.g. [13, Proposition 15.7.2], while $\mathrm{Der}(\Bbbk[x_1, \ldots, x_n]) = \mathrm{Vec}(\mathbb{A}^n)$ consists of all polynomial vector fields over $\mathbb{A}^n$.

The one to one correspondence between Borel subgroups of algebraic groups and Borel subalgebras of their Lie algebras is generally not satisfied for $\mathrm{Aut}(X)$, even in the case $X = \mathbb{A}^2$. Indeed, all Borel subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ are conjugate to the triangular subgroup $\mathrm{JONQ}(\mathbb{A}^2)$, see [8]. Their Lie algebras are Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$; they are pairwise conjugate under the adjoint action of $\mathrm{Aut}(\mathbb{A}^2)$. However, not every Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ corresponds to a Borel subgroup of $\mathbb{A}^2$, see Remark 4.7.

It is known that the triangular subgroup $\mathrm{JONQ}(\mathbb{A}^n)$ is maximal among the solvable subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ for every $n \geq 1$, see [14]. Concerning the maximality of the triangular Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$, we have the following theorem, see Propositions 4.9 and 4.13.

Theorem 1.1. *The Lie subalgebra* $\mathfrak{j}_2 = \mathrm{Lie}(\mathrm{JONQ}(\mathbb{A}^2))$ *of triangular derivations of* $\Bbbk[x, y]$ *is maximal among the solvable subalgebras of* $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$. *By contrast, the Lie subalgebra of triangular derivations* $\mathfrak{j}_3 = \mathrm{Lie}(\mathrm{JONQ}(\mathbb{A}^3))$ *is not maximal among the solvable subalgebras of* $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3))$.

Thus, the triangular Lie subalgebra $\mathfrak{j}_3$ is not a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3))$. In Section 3 we provide a criterion for determining whether a subalgebra of the Lie algebra of $\mathrm{Aut}(X)$ corresponds to a Borel subgroup of $\mathrm{Aut}(X)$. We use the following terminology.

Definitions 1.2. Recall that a set S of derivations of the structure algebra $\mathcal{O}(X)$ is called *locally finite* if, for every $f \in \mathcal{O}(X)$, the vector subspace $A(S)(f) \subset \mathcal{O}(X)$ is finite dimensional, where $A(S)$ stands for the associative subalgebra of $\mathrm{End}(\mathcal{O}(X))$ generated by S. Since $\mathrm{Lie}(\mathrm{Aut}(X)) \subset \mathrm{Der}(\mathcal{O}(X))$, we can speak about locally finite Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$.

A Lie subalgebra $\mathfrak{h}$ of $\mathrm{Der}(\mathcal{O}(X))$ will be called *integrable* if $\mathfrak{h} \subset \mathrm{Lie}(G)$ for a connected algebraic subgroup $G \subset \mathrm{Aut}(X)$. By [22, Theorem E], $\mathfrak{h}$ is integrable if and only if it is locally finite; in the latter case, $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}(X))$.

A Lie subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(\mathrm{Aut}(X))$ will be called *locally integrable* if it is filtered by a sequence of locally finite Lie subalgebras. If a solvable $\mathfrak{h}$ is locally integrable and maximal among the solvable locally integrable subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$, then we say that $\mathfrak{h}$ is a *locally integrable Borel subalgebra* of $\mathrm{Lie}(\mathrm{Aut}(X))$.

Note that a Lie subalgebra $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is locally integrable if and only if $\mathfrak{h} \subset \mathrm{Lie}(G)$, where G is a connected nested subgroup of $\mathrm{Aut}(X)$, cf. [22]. The following result is central to our paper, see Corollary 3.6.

Theorem 1.3. *The correspondence $G \mapsto \mathrm{Lie}(G)$ leads to a one to one correspondence between the conjugacy classes of the Borel subgroups of $\mathrm{Aut}(X)$ and the Ad-conjugacy classes of the locally integrable Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$.*

The proof explores several recent results concerning the structure of unipotent and solvable subgroups of $\mathrm{Aut}(X)$, see [10, 21, 22, 27]; see also Theorem 2.13 and Lemma 3.1 below.

Recall that every Borel subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is nested and conjugate to the triangular subgroup $\mathrm{JONQ}(\mathbb{A}^2)$ ([8]). The same applies to Borel subgroups of the group $\mathfrak{U}_2$ of automorphisms of $\mathbb{A}^2$ whose Jacobian is equal to 1 ([8]) and to Borel subgroups of the group $\mathrm{Aut}_0(\mathbb{A}^2)$ of automorphisms fixing the origin ([5]).

As follows from Theorem 1.3, any locally integrable Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ is Ad-conjugate to the triangular Borel subalgebra $\mathfrak{j}_2$ of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ (which is maximal among the solvable subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$, see Theorem 1.1).

The group $\mathrm{Aut}(\mathbb{A}^3)$ has non-conjugate Borel subgroups, see [29]. Hence, by Theorem 1.3, it also has locally integrable Borel subalgebras that are not Ad-conjugate to the subalgebra $\mathfrak{j}_3 \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3))$ of the triangular derivations.

Given a primitive root of unity ζ of order $d > 1$ and an integer number e with $1 \leq e < d$ and $\gcd(d, e) = 1$, we denote by $G_{d,e} \subset \mathrm{Aut}(\mathbb{A}^2)$ the cyclic group generated by $(x, y) \mapsto (\zeta^e x, \zeta y)$. The quotient $X_{d,e} = \mathbb{A}^2/G_{d,e}$ is a normal affine toric surface. Any normal affine toric surface is isomorphic to some $X_{d,e}$ except for $\mathbb{A}^2$, $\mathbb{A}^1 \times \mathbb{A}^1_*$ and $(\mathbb{A}^1_*)^2$, where $\mathbb{A}^1_* := \mathbb{A}^1 \setminus \{0\}$.

The groups $\mathrm{Aut}(X_{d,e})$ are amalgams, see [4]. Any Borel subgroup of $\mathrm{Aut}(X_{d,e})$ is isomorphic to a triangular one. However, the lower and upper triangular subgroups of $\mathrm{Aut}(X_{d,e})$ are conjugate if and only if $e^2 \equiv 1 \mod d$. Otherwise, there exist exactly two conjugacy classes of Borel subgroups of $\mathrm{Aut}(X_{d,e})$, see [5, Theorem 1.1] or Theorem 5.4 below. The next corollary follows immediately from this last result and from Theorem 1.3.

Corollary 1.4. *If $e^2 \equiv 1 \mod d$, then there exists only one Ad-conjugacy class of locally integrable Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$. Otherwise, there exist exactly two such classes.*

Recall that the Lie algebra $\mathrm{Der}(X)$ is simple if and only if X is smooth, see [19] and [31]; see also [9] and [15] for alternative proofs. The Lie algebra $\mathrm{Vec}^0(\mathbb{A}^n)$ of zero divergence vector fields on $\mathbb{A}^n$ also is simple and is the unique proper ideal of the Lie algebra $\mathrm{Vec}^c(\mathbb{A}^n) = \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$ of constant divergence vector fields, see [30, Lemma 3]. For the affine toric surfaces $X_{d,e}$ (which are singular for $d > 1$), we establish the following results.

Theorem 1.5.

(a) *The Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ can be naturally identified with a subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$.*

(b) *Suppose that $e|(d+1)$. Then $\mathrm{Lie}(\mathrm{Aut}(X_{d,e})) = \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$.*

(c) *The subalgebra $\mathfrak{g}_{d,e} \subset \mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ of zero divergence vector fields is not simple.*

See Lemma 5.8 and Corollary 5.10 for (a), Example 5.13 (where $e = 1$) and Proposition 5.16 (where $e > 1$) for (b), and Proposition 5.20 for (c). We do not know whether (b) holds for all pairs (d, e).

The content of this paper is as follows. Section 2 contains the necessary preliminaries. Theorem 1.3 is proved in Section 3, Theorem 1.1 in Section 4, and Theorem 1.5 in Section 5.

2. Preliminaries

2.1. Tangent Lie algebras of derivations.

Definition 2.1 (cf. [22, Definition 4.1.1]). A Lie algebra is called *nested* if it has an ascending filtration by a sequence of finite-dimensional Lie subalgebras.

For an affine variety X, the automorphism group $\mathrm{Aut}(X)$ is an affine ind-group, see e.g. [13]. The Lie algebra of derivations $\mathrm{Der}(\mathcal{O}(X))$ coincides with the Lie algebra of vector fields $\mathrm{Vec}(X)$. The tangent Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X))$ is a subalgebra of $\mathrm{Der}(\mathcal{O}(X))$. For example, $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$ consists of derivations in $\mathrm{Der}(\Bbbk[x_1,\dots,x_n])$ with constant divergence, see [13, Proposition 15.7.2]. For $n \geq 2$, $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$ is a maximal proper Lie subalgebra of $\mathrm{Der}(\Bbbk[x_1,\dots,x_n])$, see [6, Proposition 2.21].

Note that the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(X))$ viewed as a vector space is of countable dimension. Indeed, let us choose a dense open subset $U \subset X$ such that $TX|_U$ is a trivial vector bundle of rank $n = \dim(X)$. The vector space $\mathrm{Vec}(X)$ is clearly a subspace of $\mathrm{Vec}(U) \simeq \mathcal{O}(U)^n$. Since $\mathcal{O}(U)$ is of countable dimension, the same is true for $\mathrm{Vec}(X)$ and $\mathrm{Lie}(\mathrm{Aut}(X))$.

The following theorem gives a uniform estimate for the derived lengths of solvable Lie subalgebras of derivations, cf. also [25, Theorems 1.1–1.3].

Theorem 2.2 ([24, Corollary 4]). *Let L be a nilpotent (solvable, respectively) subalgebra of $\mathrm{Der}(\mathcal{O}(X))$. Then the derived length of L is at most n ($2n$, respectively).*

We need the following simple lemma. Its second part follows from Proposition 3.10 in [14] and its proof.

Lemma 2.3. *Let $\mathfrak{h}$ be a Lie algebra. An ascending chain of solvable subalgebras $\mathfrak{h}_\alpha$ of $\mathfrak{h}$, whose derived lengths are uniformly bounded, has a solvable union whose derived length satisfies the same bound. An analogous statement holds for an ascending chain of solvable subgroups of a given group, whose derived lengths are uniformly bounded.*

Proof. We present the proof for the case of a Lie algebra; it adapts mutatis mutandis to the case of a group. We write $\mathfrak{h}^{(0)} = \mathfrak{h}$, $\mathfrak{h}^{(r+1)} = [\mathfrak{h}^{(r)}, \mathfrak{h}^{(r)}]$.

By our assumption, $\mathfrak{h}_\alpha^{(d)} = 0$ for some $d > 0$ and for every index α. Let $\mathfrak{h}_0 = \bigcup_\alpha \mathfrak{h}_\alpha$. Any element $w \in \mathfrak{h}_0^{(d)}$ is a Lie word formed from a finite set of elements $h_1,\dots,h_n$ in $\mathfrak{h}_0$. Choose α such that $h_i \in \mathfrak{h}_\alpha$ for $i = 1,\dots,n$. Then $w \in \mathfrak{h}_\alpha^{(d)} = 0$, which implies that $\mathfrak{h}_0^{(d)} = 0$. □

Corollary 2.4. *Any solvable Lie subalgebra $L \subset \mathrm{Der}(\mathcal{O}(X))$ is contained in a Borel subalgebra. The same is true for the Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$.*

Proof. This follows from Zorn's Lemma due to Lemma 2.3 and the uniform boundedness of derived lengths in totally ordered chains of Lie subalgebras, see Theorem 2.2. □

Remark 2.5. The Lie algebra $\mathrm{Der}(\mathcal{O}(X))$ may be unsolvable even if $\mathrm{Aut}(X)$ is a solvable algebraic group. For example, this is the case for $X = \mathbb{A}^1$. Indeed, $\mathrm{Aff}(\mathbb{A}^1) = \mathrm{Aut}(\mathbb{A}^1) \cong \mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}$ is a metabelian affine algebraic group, while $\mathrm{Der}(\mathcal{O}(\mathbb{A}^1))$ contains a simple Lie subalgebra $\mathfrak{h} \simeq \mathfrak{sl}(2,\Bbbk)$ generated by

$$e = x^2\partial/\partial x, \quad f = -\partial/\partial x, \quad \text{and} \quad h = 2x\partial/\partial x,$$

see [3, Proposition 3]. The solvable Lie subalgebra

$$\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^1)) = \Bbbk\partial/\partial x \oplus \Bbbk x\partial/\partial x \subset \mathrm{Der}(\mathcal{O}(\mathbb{A}^1))$$

consists of vector fields on $\mathbb{A}^1$ with constant divergence.

2.2. **Correspondence "ind-subgroups $\leftrightsquigarrow$ tangent subalgebras".** In this subsection we mainly follow [22]. We begin with the following remarks.

Remarks 2.6.

1. A Lie subalgebra $\mathfrak{h} \subset \mathrm{Der}(\mathcal{O}(X))$ is locally finite (see Definitions 1.2 in the Introduction) if and only if every finite dimensional subspace $V \subset \mathcal{O}(X)$ is contained in an $\mathfrak{h}$-invariant finite dimensional subspace $\tilde{V} \subset \mathcal{O}(X)$. A locally finite Lie subalgebra $\mathfrak{h} \subset \mathrm{Der}(\mathcal{O}(X))$ is finite dimensional, see [22, Lemma 1.6.2]. Indeed, let us choose an $\mathfrak{h}$-invariant finite dimensional subspace $E \subset \mathcal{O}(X)$ that contains a system of generators of the algebra $\mathcal{O}(X)$. Then the linear restriction map $\mathfrak{h} \to \mathfrak{h}|_E \subset \mathrm{End}(E)$ is injective.

2. Let $G \subset \mathrm{Aut}(X)$ be an algebraic subgroup. Then $\mathrm{Lie}(G) \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is a locally finite Lie subalgebra. However, a locally finite Lie subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$ is not necessarily the tangent algebra of an algebraic subgroup of $\mathrm{Aut}(X)$.

If $G \subset \mathrm{Aut}(X)$ is a nested ind-subgroup, then $\mathrm{Lie}(G) \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is a nested Lie algebra filtered by locally finite Lie subalgebras. A partial converse of this is also true.

Proposition 2.7 ([22, Proposition 4.2.1][1])**.** *Let $\mathfrak{h} = \varinjlim \mathfrak{h}_i \subset \mathrm{Lie}(\mathrm{Aut}(X))$ be a nested Lie subalgebra. Then the following hold.*

(1) *If $\mathfrak{h}$ is generated by locally nilpotent derivations, then there exists a nested subgroup $G = \varinjlim G_i \subset \mathrm{Aut}(X)$ such that $\mathfrak{h} = \mathrm{Lie}(G) = \varinjlim \mathrm{Lie}(G_i)$.*

(2) *Suppose that all $\mathfrak{h}_i$ are locally finite. Then there exists a unique smallest connected nested ind-subgroup $G = G_{\min}(\mathfrak{h}) \subset \mathrm{Aut}(X)$ such that $\mathfrak{h} \subset \mathfrak{g} := \mathrm{Lie}(G)$. Moreover, a closed ind-subgroup $\hat{G} \subset \mathrm{Aut}(X)$ contains G if and only if $\mathfrak{h} \subset \mathrm{Lie}(\hat{G})$.*

The following is a version of [22, Theorem 5.1.1]. [2]

Theorem 2.8. *Let $G \subset \mathrm{Aut}(X)$ be a solvable subgroup generated by a family of connected algebraic subgroups $H_i \subset G$, $i \in I$.*

(1) *If I is finite, then G is a solvable algebraic subgroup with Lie algebra $\mathrm{Lie}(G) = \langle \mathrm{Lie}(H_i) \,|\, i \in I \rangle_{\mathrm{Lie}}$.*

(2) *In general, G is nested and is of the form $G = U_G \ltimes T$, where $T \subset \mathrm{Aut}(X)$ is a torus and U_G is a nested unipotent group.*

(3) *If G is generated by a family of unipotent algebraic subgroups, then $G = U_G$ is a nested unipotent group.*

The proof relies on the following recent and important result, kindly communicated by Hanspeter Kraft to the second author. It confirms a conjecture in [23, Conjecture 6.6]. See also an earlier version [11, Theorem B] concerning connected abelian subgroups of $\mathrm{Aut}(X)$.

[1] The adjective "unipotent" is misleading in part (1) of [22, Proposition 4.2.1]. By deleting it, the proposition and its proof remain valid.

[2] In the original formulation, Theorem 5.1.1 in [22] is applied to an arbitrary affine ind-group, not necessarily of the form $\mathrm{Aut}(X)$. The proof of this theorem in [22] is based on Lemma 5.1.6 in [22]. However, this lemma turns our to be false, and so Theorem 5.1.1 therefore requires a proof. In the present restricted version, Theorem 2.8 is due to Hanspeter Kraft and the second author.

Theorem 2.9 ([10, Theorem A]; see also [21, Theorem 5]). *Let X be a quasi-affine algebraic variety, A be an irreducible algebraic variety, and $f\colon A \to \mathrm{Aut}(X)$ be a morphism such that $\mathrm{id}_X \in f(A)$. Let G be the subgroup of $\mathrm{Aut}(X)$ generated by the image $f(A)$. If G is solvable, then G is a connected algebraic subgroup of $\mathrm{Aut}(X)$.*

Proof of Theorem 2.8. For the implications (1)$\Longrightarrow$(2) and (2)$\Longrightarrow$(3), see the proof of Theorem 5.1.1 in [22]. To prove (1), we note that $G = \langle H_1, \dots, H_n\rangle$ is generated by the product $H_1 \cdots H_n$. Let, in Theorem 2.9, $A = H_1 \times \cdots \times H_n$ and $f\colon A \to \mathrm{Aut}(X)$ be the multiplication map. By this theorem, we conclude that $G = \langle f(A)\rangle \subset \mathrm{Aut}(X)$ is an algebraic subgroup. □

Corollary 2.10. *Any solvable subgroup $G \subset \mathrm{Aut}(X)$ generated by connected algebraic subgroups is nested of derived length $\le \dim(X) + 1$.*

Proof. By Theorem 2.8(2), G is nested of the form $G = U_G \ltimes T$. A nested unipotent subgroup $U \subset \mathrm{Aut}(X)$ is solvable with derived length $\le \max\{\dim(Ux) \,|\, x \in X\} \le \dim(X)$, see [22, Theorem 5.3.1]. Therefore, G is of derived length $\le \dim(X) + 1$. □

Definition 2.11. Abusing the language, we say that an ind-group G is *nested by connected algebraic subgroups*, if it admits a countable ascending filtration by connected algebraic subgroups.

Remarks 2.12.

- If a subgroup $G \subset \mathrm{Aut}(X)$ is nested by connected algebraic subgroups, then it is algebraically generated in the sense of [22] and $\mathrm{Lie}(G)$ is locally integrable.
- If $\mathfrak{g} \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is a locally integrable subalgebra, then it consists of locally finite elements.

Corollary 2.10 can be strengthen as follows.

Theorem 2.13 ([10, Theorem A]; see also [21, P. 29, Corollary B]). *Let $G \subset \mathrm{Aut}(X)$ be a solvable closed connected subgroup. Then G is nested by a sequence of connected algebraic subgroups G_i and has a derived length $\le \dim(X) + 1$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is locally integrable and solvable with derived length $\le \dim(X) + 1$.*

For the reader's convenience, we provide an argument.

Proof. The closed ind-subgroup $G \subset \mathrm{Aut}(X)$ has an ascending filtration by irreducible algebraic subsets A_i equivalent to the original filtration, see [13, Proposition 1.6.3]. We can replace any A_i by A_i' such that A_i' is an irreducible algebraic subset containing id_X and the A_i' form an equivalent filtration again. Indeed, let us take an irreducible curve $C \subset G$ that contains id_X and an element a_i^{-1} where $a_i \in A_i$. The image CA_i of $C \times A_i$ under the multiplication morphism $C \times A_i \to \mathrm{Aut}(X)$, $(c, a) \mapsto ca$, contains A_i and is an irreducible constructible subset of some $A_{n(i)}$. Now replace A_i by the closure $A_i' = \overline{CA_i}$. Then $G_i = \langle A_i'\rangle$ is a solvable algebraic subgroup of G, see Theorem 2.9. Therefore, $G = \varinjlim G_i$ is nested by the connected algebraic subgroups G_i. By Corollary 2.10, the derived length of G is $\le \dim(X) + 1$. This proves the first assertion. Now, the second assertion follows by applying [22, Lemma 5.1.4(2)] and its proof. □

Let us also mention the following theorem.

Theorem 2.14 ([27, Theorem 1.4]). *Any connected nested subgroup $G \subset \mathrm{Aut}(X)$ is closed.*

Remark 2.15. A connected ind-group $\mathfrak{G}$ is commutative if $\mathrm{Lie}(\mathfrak{G})$ is, see [13, Corollary 7.5.3]. However, we do not know whether a connected ind-group $\mathfrak{G}$ is solvable if $\mathrm{Lie}(\mathfrak{G})$ is, see [22, Question 5].

The following partial result is implicit in the proof of [22, Proposition 4.2.1] (see Proposition 2.7). For the reader's convenience, we provide a proof.

Proposition 2.16. *Let $\mathfrak{h} = \varinjlim \mathfrak{h}_i \subset \mathrm{Lie}(\mathrm{Aut}(X))$ be a solvable locally integrable Lie subalgebra, where all $\mathfrak{h}_i$ are locally finite (see Definitions 1.2). Let $G = G_{\min}(\mathfrak{h}) \subset \mathrm{Aut}(X)$ be the unique smallest connected nested ind-subgroup such that $\mathfrak{h} \subset \mathrm{Lie}(G)$, see Proposition 2.7(2). Then the following hold.*

1. *G is solvable and closed with derived length $\leq \dim(X) + 1$.*
2. *If $\mathfrak{h}$ is generated by locally nilpotent derivations, then it consists of locally nilpotent derivations, the group G is unipotent and $\mathfrak{h} = \mathrm{Lie}(G)$.*

Before moving on to the proof, let us first recall some definitions and Lemmas 2.18–2.20 borrowed in [22], then deduce an additional Lemma 2.21.

Definitions 2.17.

- A finite-dimensional Lie subalgebra $\mathfrak{t} \subset \mathrm{Der}(\mathcal{O}(X))$ is called *toral* if it consists of semisimple (locally finite) elements.
- A locally finite Lie subalgebra $\mathfrak{g} \subset \mathrm{Der}(\mathcal{O}(X))$ is called *J-saturated* if it contains, with each element μ, the semisimple and the locally nilpotent parts μ_{s} and μ_{n}.

A toral subalgebra $\mathfrak{t}$ is abelian and therefore locally finite, see [17, Section 8.1] and [22, Section 1.8].

Lemma 2.18 ([22, Lemma 1.8.2])**.** *For a toral Lie subalgebra $\mathfrak{t} \subset \mathrm{Der}(\mathcal{O}(X))$, there exists a unique smallest torus $T \subset \mathrm{Aut}(X)$ such that $\mathfrak{t} \subset \mathrm{Lie}(T)$. Furthermore, if a subspace $E \subset \mathrm{Der}(\mathcal{O}(X))$ is invariant under the adjoint action of $\mathfrak{t}$, then E is invariant under T.*

Lemma 2.19 (see [22, Lemmas 1.6.3 and 3.1.3])**.** *Let $\mathfrak{g} \subset \mathrm{Der}(\mathcal{O}(X))$ be a solvable locally finite Lie subalgebra and $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{g}$ is J-saturated. Then the locally nilpotent elements from $\mathfrak{g}$ form a Lie ideal $\mathfrak{g}_{\mathrm{n}}$ of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{t} \ltimes \mathfrak{g}_{\mathrm{n}}$.*

Lemma 2.20 ([22, Lemma 3.2.1(1)])**.** *Let $\mathfrak{h} \subset \mathrm{Vec}(X)$ be a solvable locally finite Lie subalgebra. Then, the smallest J-saturated Lie subalgebra $\mathfrak{g} = \mathfrak{g}_{\min}(\mathfrak{h}) \subset \mathrm{Vec}(X)$ containing $\mathfrak{h}$ is solvable and locally finite.*

In the proof of Proposition 2.16, we use the following lemma (cf. the proof of [22, Proposition 4.2.1]).

Lemma 2.21. *Let $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}(X))$ be a locally finite solvable Lie subalgebra. Then there exists a unique minimal solvable connected algebraic subgroup $G = G_{\min}(\mathfrak{h}) \subset \mathrm{Aut}(X)$ with derived length $\leq \dim(X) + 1$ such that $\mathfrak{h} \subset \mathrm{Lie}(G)$.*

Proof. Let $\mathfrak{g} = \mathfrak{g}_{\min}(\mathfrak{h})$ be the smallest locally finite solvable J-saturated Lie subalgebra of $\mathrm{Der}(\mathcal{O}(X))$ that contains $\mathfrak{h}$, see Lemma 2.20. We have a semidirect product decomposition $\mathfrak{g} = \mathfrak{t} \ltimes \mathfrak{g}_{\mathrm{n}}$, where $\mathfrak{t}$ is a maximal toral Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{n}}$ is a nilpotent ideal consisting of locally nilpotent elements, see Lemma 2.19. Moreover, there exists a minimal torus $T \subset \mathrm{Aut}(X)$ with $\mathfrak{t} \subset \mathrm{Lie}(T)$ such that T normalizes $\mathfrak{g}_{\mathrm{n}}$, see Lemma 2.18. The algebraic

subgroup $U := \exp(\mathfrak{g}_n) \subset \mathrm{Aut}(X)$ is unipotent with Lie algebra $\mathrm{Lie}(U) = \mathfrak{g}_n$. The product $G = TU = UT \simeq T \ltimes U$ is a solvable connected algebraic subgroup of $\mathrm{Aut}(X)$ such that $\mathfrak{h} \subset \mathfrak{g} \subset \mathrm{Lie}(G)$. By construction, G is the smallest subgroup possessing this property. By Corollary 2.10, the derived length of G is bounded above by $\dim(X) + 1$. □

Proof of Proposition 2.16. By Lemmas 2.20 and 2.21, we have $\mathfrak{h}_i \subset \mathfrak{g}_{\min}(\mathfrak{h}_i) \subset \mathrm{Lie}(G_{\min}(\mathfrak{h}_i))$. By the uniqueness of subalgebras $\mathfrak{g}_i := \mathfrak{g}_{\min}(\mathfrak{h}_i)$ and solvable connected algebraic subgroups $G_i := G_{\min}(\mathfrak{h}_i)$ we conclude that $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ and $G_i \subset G_{i+1}$.

Let $\hat{G} = \varinjlim(G_i)$. Certainly, $\hat{G}$ is a solvable connected nested subgroup of $\mathrm{Aut}(X)$ with derived length $\le \dim(X)+1$ and $\mathfrak{h} \subset \mathrm{Lie}(\hat{G}) = \varinjlim \mathrm{Lie}(G_i)$. Since G and $\hat{G}$ are connected and nested, they are closed, see Theorem 2.14. By Proposition 2.7(2), we have $G = G_{\min}(\mathfrak{h}) \subset \hat{G}$. Therefore, G is solvable and closed with derived length $\le \dim(X) + 1$. This proves part (1).

To prove part (2), suppose that $\mathfrak{h}$ is generated by locally nilpotent derivations ∂_α, $\alpha \in A$. By Proposition 2.7(1), there exists a nested subgroup $G' = \varinjlim G'_i \subset \mathrm{Aut}(X)$ such that $\mathfrak{h} = \varinjlim \mathrm{Lie}(G'_i)$. We can assume that G' and all G'_i are connected. Since all $\mathrm{Lie}(G'_i)$ are solvable, the connected algebraic subgroups G'_i are also solvable.

Now consider the unipotent one-parameter subgroups $U_\alpha = \exp(\Bbbk\partial_\alpha)$. Since $\partial_\alpha \in \mathrm{Lie}(G'_i)$ for some $i = i(\alpha)$, we have $U_\alpha \subset G'_i$. For a finite set of indices $\eta = \{\alpha_1, \dots, \alpha_n\} \subset A$ we can find $i = i(\eta)$ such that $\mathcal{U}_\eta := \langle U_{\alpha_1}, \dots, U_{\alpha_n}\rangle \subset G'_i$. It follows that $\mathcal{U}_\eta$ is a solvable connected affine algebraic group generated by unipotent subgroups. Certainly, such a group is unipotent.

On the other hand, since by our assumption $\mathfrak{h} = \langle \partial_\alpha \,|\, \alpha \in A\rangle_{\mathrm{Lie}}$, for every i we can find $\eta_i = \{\alpha_1^{(i)}, \dots, \alpha_{n_i}^{(i)}\} \subset A$ such that

$$\mathrm{Lie}(G'_i) \subset \langle \partial_{\alpha_1^{(i)}}, \dots, \partial_{\alpha_{n_i}^{(i)}}\rangle_{\mathrm{Lie}}.$$

It follows that $G'_i \subset U_{\eta_i}$ (see [13, Remark 17.3.3]) is a unipotent algebraic group. Thus, $G' = \varinjlim G'_i = \varinjlim U_{\eta_i}$ is a nested unipotent ind-group, and $\mathfrak{h} = \varinjlim \mathrm{Lie}(U_{\eta_i})$ consists of locally nilpotent derivations.

By Proposition 2.7(2), we have $G \subset G'$. Hence, G is also a nested unipotent ind-group. As shown, $\mathfrak{h} \subset \mathrm{Lie}(G) \subset \mathrm{Lie}(G') = \mathfrak{h}$, therefore $\mathrm{Lie}(G) = \mathfrak{h}$, as stated. □

The following question arises naturally in view of Theorem 2.8(1) and Propositions 2.7 and 2.16. Cf. also closely related Questions 2 and 3 in [22].

Question 1. *Let $\mathfrak{h} = \langle \mathfrak{h}_1, \dots, \mathfrak{h}_n\rangle_{\mathrm{Lie}}$ be a solvable Lie subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$ generated by the locally finite Lie subalgebras $\mathfrak{h}_i \subset \mathrm{Lie}(\mathrm{Aut}(X))$. Is it true that $\mathfrak{h}$ is locally finite, and therefore also finite dimensional?*

See [33] for some partial results.

The following example shows that the local finiteness assumption in Question 1 is essential.

Example 2.22. Let $X = \mathbb{A}^1 \times \mathbb{A}^1_* = \mathrm{Spec}\,\Bbbk[x, y, y^{-1}]$. Consider the metabelian Lie subalgebra $\mathfrak{h} \subset \mathrm{Der}(\Bbbk[x, y, y^{-1}])$ generated by the derivations

$$\delta = \partial/\partial y \quad \text{and} \quad \partial = y^{-1}\partial/\partial x.$$

The vector space $\mathfrak{h}$ is infinite dimensional with a basis $\delta, \partial_1 = \partial, \partial_2, \dots$, where $\partial_i = y^{-i}\partial/\partial x$. Notice that $\delta \notin \mathrm{Lie}(\mathrm{Aut}(X))$, and it is not locally finite.

3. Correspondence "Borel subgroups ⇔ Borel subalgebras"

In this section we prove Theorem 1.3. Let us start with the following simple lemma.

Lemma 3.1. *Every Borel subgroup $\mathcal{B} \subset \mathrm{Aut}(X)$ is a closed subgroup of derived length $\leq \dim(X)+1$ nested by connected algebraic subgroups.*

Proof. The closure of a solvable subgroup is also a solvable subgroup, see [22, Lemma 5.1.3(2)]. Therefore, $\mathcal{B}$ is closed. Theorem 2.13 now applies and gives the result. □

We have the following analogue of Corollary 2.4 (see Definition 1.2).

Lemma 3.2. *Any solvable locally integrable subalgebra $\mathfrak{h}$ of $\mathrm{Lie}(\mathrm{Aut}(X))$ is contained in a locally integrable Borel subalgebra $\mathfrak{b} \subset \mathrm{Lie}(\mathrm{Aut}(X))$.*

Proof. Let $\{\mathfrak{h}_\alpha \,|\, \alpha \in A\}$ be a totally ordered ascending chain of Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$ that contain $\mathfrak{h}$ and are filtered by solvable locally finite subalgebras. Let $G_\alpha = G_{\min}(\mathfrak{h}_\alpha)$, see Proposition 2.16. Then $\{G_\alpha \,|\, \alpha \in A\}$ is a totally ordered ascending chain of solvable connected nested subgroups of $\mathrm{Aut}(X)$. By Lemma 2.3 and Theorem 2.13, the closure $\bar{G}$ of $G := \varinjlim G_\alpha$ is a solvable connected subgroup of $\mathrm{Aut}(X)$ of derived length $\leq \dim(X)+1$ filtered by an ascending sequence of connected algebraic subgroups $\bar{G}_i$. Then $\mathfrak{g} := \mathrm{Lie}(\bar{G}) = \varinjlim \mathrm{Lie}(\bar{G}_i) \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is a solvable Lie subalgebra filtered by locally finite Lie subalgebras $\mathrm{Lie}(\bar{G}_i)$. Since $\mathfrak{h}_\alpha \subset \mathrm{Lie}(\bar{G})$ for all $\alpha \in A$, $\mathrm{Lie}(\bar{G})$ is an upper bound for the chain $\{\mathfrak{h}_\alpha\}$. By Zorn's Lemma, the set of solvable Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X))$ that contain $\mathfrak{h}$ and are filtered by locally finite subalgebras has a maximal element $\mathfrak{b}$. Thus, $\mathfrak{b}$ is a locally integrable Borel subalgebra which contains $\mathfrak{h}$. □

Lemma 3.3. *Let $\mathcal{B} \subset \mathrm{Aut}(X)$ be a Borel subgroup. Then $\mathfrak{b} := \mathrm{Lie}(\mathcal{B})$ is a locally integrable Borel subalgebra.*

Proof. We have $\mathcal{B} = \varinjlim \mathcal{B}_i$, where the $\mathcal{B}_i$ are connected algebraic subgroups, see Lemma 3.1. Any vector $v \in \mathfrak{b} = T_e\mathcal{B}$ is tangent to $\mathcal{B}_i$ for $i \gg 1$. Therefore, $\mathfrak{b} = \varinjlim \mathfrak{b}_i$, where the $\mathfrak{b}_i = \mathrm{Lie}(\mathcal{B}_i)$ are locally finite, see [22, Section 1.6, Corollary].

Let $\mathfrak{h} = \varinjlim \mathfrak{h}_i \subset \mathrm{Lie}(\mathrm{Aut}(X))$ be a solvable locally integrable subalgebra that contains $\mathfrak{b}$, where the $\mathfrak{h}_i$ are locally finite. Let also $G = G_{\min}(\mathfrak{h}) \subset \mathrm{Aut}(X)$ be the unique smallest solvable connected nested ind-subgroup such that $\mathfrak{h} \subset \mathrm{Lie}(G)$, see Proposition 2.16. We have $G = \varinjlim G_i$, where $G_i = G_{\min}(\mathfrak{h}_i) \subset \mathrm{Aut}(X)$ is the minimal connected algebraic subgroup containing $\mathfrak{h}_i$, see Lemma 2.21. For every i, there exists j such that

$$\mathfrak{b}_i = \mathrm{Lie}(\mathcal{B}_i) \subset \mathfrak{h}_j \subset \mathrm{Lie}(G_{\min}(\mathfrak{h}_j)).$$

An embedding of algebraic Lie subalgebras implies an embedding of the corresponding connected algebraic subgroups, see [18, Theorem 13.1]. Therefore, $\mathfrak{B}_i \subset G_j = G_{\min}(\mathfrak{h}_j)$, and so $\mathfrak{B} \subset G$. Since G is solvable, $G = \mathcal{B}$ by the maximality of $\mathcal{B}$. Thus, $\mathfrak{b} = \mathfrak{h}$, which proves the claimed maximality of $\mathfrak{b}$. □

The main result of this section is the following theorem.

Theorem 3.4. *Let $\mathfrak{b} = \varinjlim \mathfrak{b}_i \subset \mathrm{Lie}(\mathrm{Aut}(X))$ be a locally integrable Borel subalgebra. Then there exists a unique Borel subgroup $\mathcal{B} \subset \mathrm{Aut}(X)$ such that $\mathfrak{b} = \mathrm{Lie}(\mathcal{B})$. Moreover, any solvable closed connected subgroup $H \subset \mathrm{Aut}(X)$ with $\mathrm{Lie}(H) = \mathfrak{b}$ coincides with $\mathcal{B}$.*

Remark 3.5. There exists an example of a closed ind-subgroup $G \subset \mathrm{Aut}(\mathbb{A}^3)$ and a proper closed subgroup $H \subset G$ such that $\mathrm{Lie}(H) = \mathrm{Lie}(G)$, see [13, Theorem 17.3.1]. Therefore, the uniqueness in Theorem 3.4 is a special phenomenon.

We prove the existence and the uniqueness statements separately.

Existence proof. By Lemma 2.21, there exists a unique minimal solvable connected algebraic subgroup $G_i \subset \mathrm{Aut}(X)$ such that $\mathfrak{b}_i \subset \mathrm{Lie}(G_i)$. We have $G_i = T_i \ltimes U_i$, where T_i is a maximal torus and U_i is the unipotent radical of G_i.

By the uniqueness of the subgroups G_i we have $G_i \subset G_{i+1}$ and $U_i \subset U_{i+1}$. Therefore, $U := \varinjlim U_i$ is a nested unipotent ind-subgroup of $\mathrm{Aut}(X)$. The maximal tori T_i of G_i can be chosen such that $T_i \subset T_{i+1}$. Since the sequence $(T_i)_i$ stabilizes, we can assume that $T_i = T_1 =: T$ for all i. Thus, $G := \varinjlim G_i = T \ltimes U \subset \mathrm{Aut}(X)$ is a connected nested ind-subgroup. By Theorem 2.14, G is closed.

Since U is solvable, see [22, Theorem 5.3.1], $G = T \ltimes U$ is also solvable. We have

$$\mathrm{Lie}(G) = \varinjlim \mathrm{Lie}(G_i) = \mathrm{Lie}(T) \ltimes \varinjlim \mathrm{Lie}(U_i) = \mathrm{Lie}(T) \ltimes \mathrm{Lie}(U),$$

where the factors are solvable, see Theorem 2.13. Therefore, $\mathrm{Lie}(G)$ is a locally integrable solvable subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$ filtered by the locally finite subalgebras $\mathrm{Lie}(G_i)$. By assumption, $\mathfrak{b} = \varinjlim \mathfrak{b}_i \subset \mathrm{Lie}(G)$ is a locally integrable Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$. Therefore, we have $\mathfrak{b} = \mathrm{Lie}(G)$.

Let $\hat{G} \subset \mathrm{Aut}(X)$ be a solvable closed connected subgroup containing G. By Theorem 2.13, $\hat{G}$ is nested, and so $\mathrm{Lie}(\hat{G}) \subset \mathrm{Lie}(\mathrm{Aut}(X))$ is a solvable locally integrable Lie subalgebra containing $\mathrm{Lie}(G) = \mathfrak{b}$. Since $\mathfrak{b}$ is a locally integrable Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$, we have $\mathrm{Lie}(\hat{G}) = \mathfrak{b}$.

Let $\{\hat{G}_\alpha\}_{\alpha \in \mathcal{A}}$ be a totally ordered ascending chain of solvable closed connected subgroups containing G. By Theorem 2.13, they are all nested by connected algebraic subgroups and have derived lengths $\le \dim(X) + 1$. Consider the closure $\bar{S}$ of the solvable subgroup $S = \cup_{\alpha \in \mathcal{A}} \hat{G}_\alpha$, see Lemma 2.3. Then $\bar{S}$ is solvable (see [22, Lemma 5.1.3(2)]) closed connected subgroup of derived length $\le \dim(X) + 1$ with $\mathrm{Lie}(\bar{S}) = \mathfrak{b}$ that majorates our chain. By Zorn's Lemma, the set of all solvable closed connected subgroups of $\mathrm{Aut}(X)$ containing G has a maximal element, say $\mathcal{B}$ (cf. [14, Proposition 3.10]). Thus, $\mathcal{B} \supset G$ is a solvable closed connected subgroup of $\mathrm{Aut}(X)$ maximal among the subgroups of $\mathrm{Aut}(X)$ with the same properties. That is, $\mathcal{B}$ is a Borel subgroup of $\mathrm{Aut}(X)$ such that $\mathrm{Lie}(\mathcal{B}) = \mathfrak{b}$. □

Uniqueness proof. Let $\mathcal{B}$ be a Borel subgroup of $\mathrm{Aut}(X)$ with $\mathrm{Lie}(\mathcal{B}) = \mathfrak{b}$, and let $H \subset \mathrm{Aut}(X)$ be a solvable closed connected subgroup with $\mathrm{Lie}(H) = \mathfrak{b}$. By Lemma 3.1 resp. by Theorem 2.13, $\mathcal{B} = \varinjlim \mathcal{B}_i$ resp. $H = \varinjlim H_i$ is nested by solvable connected algebraic subgroups $\mathcal{B}_i$ resp. H_i. This produces the ascending filtrations $\mathfrak{b} = \varinjlim \mathfrak{b}_i = \varinjlim \mathfrak{h}_i$ by locally finite Lie subalgebras $\mathfrak{b}_i := \mathrm{Lie}(\mathcal{B}_i)$ resp. $\mathfrak{h}_i := \mathrm{Lie}(H_i)$, see Remark 2.6. For any i there exists j such that $\mathfrak{h}_i \subset \mathfrak{b}_j$ and $\mathfrak{b}_i \subset \mathfrak{h}_j$. Therefore, we have $H_i \subset \mathcal{B}_j$ and $\mathcal{B}_i \subset H_j$. It follows that $H = \mathcal{B}$. □

The following immediate corollary of Theorem 3.4 and Lemma 3.3 proves Theorem 1.3.

Corollary 3.6. *The conjugacy classes of the Borel subgroups of* $\mathrm{Aut}(X)$ *are in natural one to one correspondence with the* Ad*-conjugacy classes of locally integrable Borel subalgebras of* $\mathrm{Lie}(\mathrm{Aut}(X))$.

Remark 3.7. Let $\mathcal{B} \subset \mathrm{Aut}(X)$ be a Borel subgroup. It is not true, in general, that $\mathrm{Lie}(\mathcal{B})$ is a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$, that is, a maximal solvable subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X))$. See Proposition 4.13 below for the corresponding example.

4. Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^n))$ for small n

In this section we prove Theorem 1.1, see Propositions 4.9 ($n = 2$) and 4.13 ($n = 3$).

4.1. **Borel subgroups of $\mathrm{Aut}(\mathbb{A}^n)$.** Recall that the lower and upper triangular (de Jonquères) subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ are

$$\mathrm{JONQ}^+(\mathbb{A}^n) = \{\phi\colon (x_1, \dots, x_n) \longmapsto (\alpha_1 x_1 + p_1, \dots, \alpha_n x_n + p_n) \,|\, \alpha_i \in \Bbbk^*,\ p_i \in \Bbbk[x_{i+1}, \dots, x_n]\},$$

resp.

$$\mathrm{JONQ}^-(\mathbb{A}^n) = \{\phi\colon (x_1, \dots, x_n) \longmapsto (\alpha_1 x_1 + p_1, \dots, \alpha_n x_n + p_n) \,|\, \alpha_i \in \Bbbk^*,\ p_i \in \Bbbk[x_1, \dots, x_{i-1}]\}.$$

They are conjugate via the reversion $(x_1, \dots, x_n) \mapsto (x_n, \dots, x_1)$.

The following results are well known: see [14, Theorem 1.1 and Corollary 1.2] for (a), (d) and (e), [29, Corollary 1.5] for (b), and [8, Theorem 1] for (c).

Theorem 4.1.

(a) *For all $n \geq 2$, the triangular de Jonquières subgroups $\mathrm{JONQ}^{\pm}(\mathbb{A}^n)$ are Borel subgroups maximal among the solvable subgroups of $\mathrm{Aut}(\mathbb{A}^n)$.*
(b) *For all $n \geq 3$, there exist Borel subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ non-conjugate to $\mathrm{JONQ}^{\pm}(\mathbb{A}^n)$.*
(c) *Every Borel subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ is conjugate to both $\mathrm{JONQ}^{\pm}(\mathbb{A}^2)$.*
(d) *The subgroup $\mathrm{JONQ}^{\pm}(\mathbb{A}^2)$ is maximal among the closed subgroups of $\mathrm{Aut}(\mathbb{A}^2)$.*
(e) *The subgroup $\mathrm{JONQ}^{\pm}(\mathbb{A}^2)$ is not maximal among all proper subgroups of $\mathrm{Aut}(\mathbb{A}^2)$.*

Note that the locally integrable Borel subalgebra

$$\mathfrak{j}_n^{\pm} := \mathrm{Lie}(\mathrm{JONQ}^{\pm}(\mathbb{A}^n)) \subset \mathrm{Der}(\Bbbk[x_1, \dots, x_n])$$

consists of the lower resp. upper triangular derivations.

4.2. **The case of the affine line.** Here we classify the Borel subalgebras of the Lie algebra $\mathrm{Der}(\Bbbk[x])$. The polynomials in $\Bbbk[x]$ of the form

$$f(x) = \lambda(x - \alpha)^k + \mu(x - \alpha), \quad \text{where} \quad k \geq 0 \quad \text{and} \quad \alpha, \lambda, \mu \in \Bbbk,$$

will be called *special*. Non-special polynomials exist in any degree ≥ 3.

Proposition 4.2.

(a) *Every Borel subalgebra of $\mathrm{Der}(\Bbbk[x])$ is of dimension 1 or 2.*
(b) *The one-dimensional Borel subalgebras of $\mathrm{Der}(\Bbbk[x])$ are precisely the subalgebras $\Bbbk p(x)\partial/\partial x$, where $p \in \Bbbk[x]$ is non-special.*
(c) *The two-dimensional Borel subalgebras of $\mathrm{Der}(\Bbbk[x])$ are precisely the metabelian Lie subalgebras*

$$\big(\Bbbk(x - \alpha)^k \oplus \Bbbk(x - \alpha)\big)\, \partial/\partial x, \tag{1}$$

where $\alpha \in \Bbbk$, $k \geq 0$ and $k \neq 1$.

See the proof at the end of this subsection.

Corollary 4.3. *Any derivation $f(x)\partial/\partial x \in \mathrm{Der}(\Bbbk[x])$ with a non-special polynomial $f \in \Bbbk[x]$ is contained in the unique Borel subalgebra $\Bbbk f(x)\partial/\partial x$ of $\mathrm{Der}(\Bbbk[x])$.*

Remark 4.4. Letting in (1) $\alpha = 0$ and $k = 0$ we see that

$$\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^1)) = \Bbbk x\partial/\partial x \ltimes \Bbbk\partial/\partial x$$

is a Borel subalgebra of $\mathrm{Der}(\Bbbk[x])$. In fact, it is a unique locally integrable Borel subalgebra of $\mathrm{Der}(\Bbbk[x])$.

Recall that

$$[f\partial/\partial x, g\partial/\partial x] = (fg' - f'g)\partial/\partial x.$$

The proof of Proposition 4.2 is preceded by the following notation and Lemma 4.6.

Notation 4.5. Given $f \in \Bbbk[x]$ we set $\nu(f)$ stand for the zero order at $x = 0$ of f. If $f, g \in \Bbbk[x]$ and $\nu(f) \neq \nu(g)$, then

$$\nu(fg' - f'g) = \nu(f) + \nu(g) - 1. \tag{2}$$

Especially, if $\nu(f), \nu(g) \geq 2$, then

$$\nu(fg' - f'g) > \max\{\nu(f), \nu(g)\}.$$

Lemma 4.6. *Consider a subalgebra $\mathfrak{g} \subset \mathrm{Der}(\Bbbk[x])$. Let $u_i = f_i(x)\partial/\partial x \in \mathfrak{g}$, $i = 1, 2$ be such that $\min\{\nu(f_1), \nu(f_2)\} \geq 2$ and $\nu(f_1) \neq \nu(f_2)$. Then the commutant $[\mathfrak{g}, \mathfrak{g}]$ contains two elements $v_i = g_i(x)\partial/\partial x$, $i = 1, 2$ such that $\min\{\nu(g_1), \nu(g_2)\} \geq 3$ and $\nu(g_1) \neq \nu(g_2)$.*

Proof. Consider the elements

$$v_1 = [u_1, [u_1, u_2]] = g_1(x)\partial/\partial x \quad \text{and} \quad v_2 = [u_1, u_2] = g_2(x)\partial/\partial x.$$

Using (2), we obtain:

$$\nu(g_1) = 2\nu(f_1) + \nu(f_2) - 2 > \nu(g_2) = \nu(f_1) + \nu(f_2) - 1 \geq 3.$$

□

Proof of Proposition 4.2. Let $\mathfrak{g} \subset \mathrm{Der}(\Bbbk[x])$ be a Lie subalgebra of dimension ≥ 4, and let $u_i = f_i(x)\partial/\partial x \in \mathfrak{g}$, $i = 1, 2, 3, 4$ be linearly independent. Suppose that

$$\nu(f_i) \leq \nu(f_{i+1}), \quad i = 1, 2, 3.$$

By replacing the u_i by their appropriate linear combinations, we may assume that

$$\nu(f_i) < \nu(f_{i+1}), \quad i = 1, 2, 3.$$

It follows that

$$2 \leq \nu(f_3) < \nu(f_4).$$

Applying Lemma 4.6, we see that all successive commutants of $\mathfrak{g}$ do not vanish, so $\mathfrak{g}$ is not solvable. Thus, every solvable subalgebra, and in particular every Borel subalgebra of $\mathrm{Der}(\Bbbk[x])$, is of dimension ≤ 3. Finite dimensional Lie subalgebras of $\mathrm{Der}(\Bbbk[x])$ are classified in [3, Proposition 3]. Every one-dimensional Lie subalgebra is abelian. Up to isomorphism, there is a unique two-dimensional Lie subalgebra; it is not abelian but metabelian, and therefore solvable. Up to isomorphism, there exists a unique subalgebra $\mathfrak{h}$ of dimension 3, isomorphic to $\mathfrak{sl}(2, \Bbbk)$, see Remark 2.5. This proves (a). Now, (b) and (c) follow from the explicit description of these subalgebras, see [3, Proposition 3]. □

4.3. **The case of the affine plane.**

4.3.1. *Borel subgroups.* We consider the following remarkable subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ (resp. of $\mathrm{Aut}_0(\mathbb{A}^2)$):

- the subgroup $\mathfrak{U}_2 \subset \mathrm{Aut}(\mathbb{A}^2)$ of automorphisms of $\mathbb{A}^2$ whose Jacobian is equal to 1;
- the triangular de Jonquières subgroups

$$\mathrm{JONQ}^+(\mathbb{A}^2) = \{\phi\colon (x,y) \longmapsto (\alpha x + p(y), \beta y + \gamma) \,|\, \alpha,\beta \in \Bbbk^*, \gamma \in \Bbbk,\, p \in \Bbbk[t]\}$$

and

$$\mathrm{JONQ}^-(\mathbb{A}^2) = \{\phi\colon (x,y) \longmapsto (\alpha x + \gamma, \beta y + q(x)) \,|\, \alpha,\beta \in \Bbbk^*, \gamma \in \Bbbk,\, q \in \Bbbk[t]\}$$

$$(\text{resp.}\quad \mathrm{Jonq}^\pm(\mathbb{A}^2) = \{\phi \in \mathrm{JONQ}^\pm(\mathbb{A}^2) \,|\, \phi(0) = 0\}),$$

- the standard maximal torus

$$\mathbb{T} = \{\psi\colon (x,y) \longmapsto (\alpha x, \beta y)\} \subset \mathrm{Jonq}^+(\mathbb{A}^2) \cap \mathrm{Jonq}^-(\mathbb{A}^2);$$

- the unipotent radical

$$U^\pm = \{\phi \in \mathrm{JONQ}^\pm(\mathbb{A}^2) \,|\, \alpha = \beta = 1\} \quad (\text{resp.}\quad U_0^\pm = \{\phi \in \mathrm{Jonq}^\pm(\mathbb{A}^2) \,|\, \alpha = \beta = 1\}).$$

We have

$$\mathrm{JONQ}^\pm(\mathbb{A}^2) = \mathbb{T} \ltimes U^\pm \quad (\text{resp.}\quad \mathrm{Jonq}^\pm(\mathbb{A}^2) = \mathbb{T} \ltimes U_0^\pm).$$

4.3.2. $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ *as a bigraded Lie algebra.* Recall that $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$ consists of polynomial vector fields on $\mathbb{A}^2$ with constant divergence, where the divergence of a vector field $\delta = f\partial/\partial x + g\partial/\partial y$ is $\mathrm{div}(\delta) = \partial f/\partial x + \partial g/\partial y$. The Lie subalgebra $\mathrm{Vec}^0(\mathbb{A}^2) \subset \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$ of vector fields over $\mathbb{A}^2$ with zero divergence coincides with $\mathrm{Lie}(\mathfrak{U}_2)$, see for example [13, Proposition 15.7.2]. There is a decomposition

$$\mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2) = \mathrm{Vec}^0(\mathbb{A}^2) \oplus \Bbbk(x\partial/\partial x + y\partial/\partial y). \tag{3}$$

Letting

$$\mathrm{Vec}_0^0(\mathbb{A}^2) := \mathrm{Vec}^0(\mathbb{A}^2) \cap \mathrm{Vec}_0^{\mathrm{c}}(\mathbb{A}^2), \quad \text{where} \quad \mathrm{Vec}_0^{\mathrm{c}}(\mathbb{A}^2) = \{\delta \in \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2) \,|\, \delta(0) = 0\},$$

we obtain

$$\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2)) = \mathrm{Vec}_0^0(\mathbb{A}^2) \oplus \Bbbk(x\partial/\partial x + y\partial/\partial y),$$

see [22, Section 6.2].

The standard $\mathbb{Z}^2$-grading on $\Bbbk[x,y]$ induces a $\mathbb{Z}^2$-grading on $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$ and on $\mathrm{Lie}(\mathfrak{U}_2) = \mathrm{Vec}^0(\mathbb{A}^2)$. For $i,j \geq 0$ we have

$$\mathrm{bideg}(x^i y^j \partial/\partial x) = (i-1, j) \quad \text{and} \quad \mathrm{bideg}(x^i y^j \partial/\partial y) = (i, j-1).$$

Thus, the bidegrees of homogeneous derivations are the pairs $(m,n) \in \mathbb{Z}^2$, where $m,n \geq -1$ and $(m,n) \neq (-1,-1)$. If $\partial \in \mathrm{Der}(\Bbbk[x,y])$ is a homogeneous derivation of bidegree (a,b) and $x^i y^j \notin \ker(\partial)$, then $\mathrm{bideg}(\partial(x^i y^j)) = (a+i, b+j)$. For non-commuting homogeneous derivations ∂ and δ we have $\mathrm{bideg}([\delta,\partial]) = \mathrm{bideg}(\delta) + \mathrm{bideg}(\partial)$.

The homogeneous elements of $\mathrm{Vec}^0(\mathbb{A}^2)$ with bidegree (a,b), where $a,b \geq 0$, are (up to a scaling) of the form

$$\partial_{a,b} = x^a y^b((b+1)x\partial/\partial x - (a+1)y\partial/\partial y).$$

For $\min\{a,b\} = -1$ they are of the form

$$\partial_{-1,n} = (n+1)y^n \partial/\partial x \quad \text{and} \quad \partial_{m,-1} = -(m+1)x^m \partial/\partial y.$$

Every homogeneous locally nilpotent derivation ∂ of $\Bbbk[x,y]$ belongs to $\mathrm{Vec}^0(\mathbb{A}^2)$. Up to a scalar factor, ∂ coincides with $\partial_{-1,n}$ or $\partial_{m,-1}$. These homogeneous derivations verify the following commutation relations:

(4)
$$[\partial_{a,b},\partial_{a',b'}] = \det\begin{pmatrix} a'+1 & a+1 \\ b'+1 & b+1 \end{pmatrix}\partial_{a+a',b+b'} \quad \text{for all } a,b,a',b' \geq -1,\ (a,b),\ (a',b') \neq (-1,-1),$$

where we let $\partial_{-1,-1}=0$. Every semisimple homogeneous derivation of $\mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$ has bidegree $(0,0)$. They are of the form $\delta_{\alpha,\beta} = \alpha x\partial/\partial x + \beta y\partial/\partial y$ and satisfy the commutation relations

$$[\delta_{\alpha,\beta},\delta_{\gamma,\eta}] = 0 \quad \text{and} \quad [\delta_{\alpha,\beta},\partial_{a,b}] = (\alpha a+\beta b)\partial_{a,b}. \tag{5}$$

Furthermore, we have

$$\partial_{a,b}(x^k y^l) = \det\begin{pmatrix} k & a+1 \\ l & b+1 \end{pmatrix} x^{k+a}y^{l+b}. \tag{6}$$

4.3.3. *Borel subalgebras of* $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$. Let us begin with the following remark.

Remark 4.7. Let us show that $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ contains non-conjugate Borel subalgebras. Let $\mathfrak{g} \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ be a Borel subalgebra Ad-conjugate to the triangular subalgebra $\mathfrak{j}_2^+ := \mathrm{Lie}(\mathrm{JONQ}^+(\mathbb{A}^2))$. Then $\mathfrak{g}$ is locally integrable, and therefore consists of locally finite derivations. However, $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$ contains elements that are not locally finite; for example, $\partial = \partial_{1,0} = x^2\partial/\partial x - 2xy\partial/\partial y \in \mathrm{Vec}^0(\mathbb{A}^2)$ is such an element. Indeed, the degrees of the iterates $\partial^n(x)$ are unbounded. By Corollary 2.4, there exists a Borel subalgebra $\mathfrak{f} \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ containing ∂. Therefore, $\mathfrak{f}$ is not conjugate to $\mathfrak{j}_2^+$ and does not correspond to a Borel subgroup of $\mathrm{Aut}(\mathbb{A}^2)$.

Consider the following Lie subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ (resp. $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$):

- the metabelian (resp. abelian) Lie subalgebras of triangular locally nilpotent derivations

$$\mathfrak{u}_2^+ = \mathrm{Lie}(U^+) = \Bbbk[y]\partial/\partial x \oplus \Bbbk\partial/\partial y \quad \text{and} \quad \mathfrak{u}_2^- = \mathrm{Lie}(U^-) = \Bbbk\partial/\partial x \oplus \Bbbk[x]\partial/\partial y$$

$$(\text{resp.} \quad \mathfrak{u}_{2,0}^+ = \mathrm{Lie}(U_0^+) = y\Bbbk[y]\partial/\partial x \quad \text{and} \quad \mathfrak{u}_{2,0}^- = \mathrm{Lie}(U_0^-) = x\Bbbk[x]\partial/\partial y);$$

- the abelian toral Lie subalgebra

$$\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T}) = \Bbbk x\partial/\partial x \oplus \Bbbk y\partial/\partial y;$$

- the triangular Lie subalgebras of derived length 3 (resp. 2)

$$\mathfrak{j}_2^\pm = \mathrm{Lie}(\mathrm{JONQ}^\pm(\mathbb{A}^2)) = \mathrm{Lie}(\mathbb{T} \ltimes U^\pm) = \mathfrak{t}_2 \ltimes \mathfrak{u}_2^\pm$$

$$(\text{resp.} \quad \mathfrak{j}_{2,0}^\pm = \mathrm{Lie}(\mathrm{Jonq}^\pm(\mathbb{A}^2)) = \mathrm{Lie}(\mathbb{T} \ltimes U_0^\pm) = \mathfrak{t}_2 \ltimes \mathfrak{u}_{2,0}^\pm),$$

where

$$\mathfrak{j}_2^+ = \Bbbk[y]\partial/\partial x \oplus \Bbbk x\partial/\partial x \oplus \Bbbk y\partial/\partial y \oplus \Bbbk\partial/\partial y \tag{7}$$

$$(\text{resp.} \quad \mathfrak{j}_{2,0}^+ = y\Bbbk[y]\partial/\partial x \oplus \Bbbk x\partial/\partial x \oplus \Bbbk y\partial/\partial y). \tag{8}$$

Using (4) we see that

$$\mathrm{Lie}(\mathfrak{U}_2) = \mathrm{Vec}^0(\mathbb{A}^2) = \langle \mathfrak{u}_2^+, \mathfrak{u}_2^- \rangle_{\mathrm{Lie}}.$$

Therefore, by (3) we have

$$\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2) = \langle \mathfrak{u}_2^+, \mathfrak{u}_2^- \rangle_{\mathrm{Lie}} \oplus \Bbbk(x\partial/\partial x + y\partial/\partial y).$$

Similar equalities hold for all $\mathbb{A}^n$ with $n \geq 2$; see, for example, the proof of [30, Lemma 2].

In the proof of Proposition 4.9 below we use the following simple lemma.

Lemma 4.8. *Let A be a diagonalizable endomorphism of a vector space V. Consider the decomposition $V = \oplus_i V_i$, where the V_i are eigenspaces of A corresponding to different eigenvalues λ_i, $i = 1, \ldots, m$. Let a subspace $U \subset V$ be invariant under A, and let $u = u_1 + \cdots + u_m \in U$, where $u_i \in V_i$. Then $u_i \in U$ for all i.*

Proof. By symmetry, it suffices to show that $u_1 \in U$. The latter is obviously true if $u = u_1$. Arguing by recursion, we can assume that our claim is true for $u = u_1 + \cdots + u_{k-1}$. Now suppose that $u = u_1 + \cdots + u_k$, where $1 < k \leq m$. We have

$$U \ni Au - \lambda_k u = \sum_{j=1}^{k-1} (\lambda_j - \lambda_k) u_j.$$

By the inductive hypothesis, $(\lambda_j - \lambda_k)u_j \in U$ for $j = 1, \ldots, k-1$, where $\lambda_j - \lambda_k \neq 0$. In particular, $u_1 \in U$. □

Proposition 4.9. *The subalgebra $\mathfrak{j}_2^{\pm} = \mathrm{Lie}(\mathrm{JONQ}^{\pm}(\mathbb{A}^2))$ is a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$.*

Proof. To prove that $\mathfrak{j}_2^+$ is a Borel subalgebra, it suffices to show that any Lie subalgebra $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ properly containing $\mathfrak{j}_2^+$ also contains a subalgebra isomorphic to $\mathfrak{sl}(2, \Bbbk)$. Therefore, $\mathfrak{h}$ cannot be solvable.

Choose $v \in \mathfrak{h} \setminus \mathfrak{j}_2^+$. As a vector space, $\mathfrak{j}_2^+$ is generated by its homogeneous elements. Hence, v contains homogeneous components that do not belong to $\mathfrak{j}_2^+$. We can assume that none of the homogeneous components of v belong to $\mathfrak{j}_2^+$.

Since $\mathfrak{t}_2 \subset \mathfrak{j}_2^+$, the semisimple derivations $A = \mathrm{ad}(x\partial/\partial x)$ and $B = \mathrm{ad}(y\partial/\partial y)$ from $\mathrm{End}(\mathrm{Der}(\Bbbk[x, y]))$ leave $\mathfrak{h}$ invariant. Let $v_{a,b}$ be the homogeneous component of v of bidegree (a, b). Then $v_{a,b}$ is an eigenvector of A (resp. of B) with eigenvalue a (resp. b). Let us show that $v_{a,b} \in \mathfrak{h} \setminus \mathfrak{j}_2^+$.

Note that $A|_{\mathfrak{h}}$ is locally finite. Therefore, v belongs to a finite-dimensional A-invariant vector subspace $V \subset \mathfrak{h}$. Being semisimple, $A|_V$ is diagonalizable and defines an eigenspace decomposition of V. If $v = \sum_{(a,b)} v_{a,b}$ is the usual weight decomposition, then the A-homogeneous components of v are of the form $v_a = \sum_b v_{a,b}$. They belong to V due to Lemma 4.8. The same argument applied to $B = \mathrm{ad}(y\partial/\partial y) \in \mathrm{End}(\mathrm{Der}(\Bbbk[x, y]))$ shows that the B-homogeneous components $v_{a,b}$ of v_a also belong to V. Thus, $v_{a,b} \in \mathfrak{h} \setminus \mathfrak{j}_2^+$ for all (a, b).

Let us return to the proof of the proposition. First, note that $v_{a,b} \in \Bbbk^* \partial_{a,b}$. Consequently, $\partial_{a,b} \in \mathfrak{h} \setminus \mathfrak{j}_2^+$ for some pair $(a, b) \in \mathbb{Z}^2$ such that $a \geq 0$, $b \geq -1$, and $(a, b) \neq (0, 0), (0, -1)$. Thus, it suffices to examine the following possibilities for the bidegree (a, b) of $\partial_{a,b} \in \mathfrak{h}$:

- $(a, b) = (a, -1)$, with $a \geq 1$. Since $\partial_{-1,0} = \partial/\partial x \in \mathfrak{j}_2^+$, formula (4) implies

$$\partial_{a-1,-1} \in \Bbbk^*[\partial_{-1,0}, \partial_{a,-1}] \subset \mathfrak{h}.$$

By iteration, we deduce that $x\partial/\partial y \in \mathfrak{h}$. Since $y\partial/\partial x \in \mathfrak{j}_2^+ \subset \mathfrak{h}$, we have

$$\mathfrak{sl}(2,\Bbbk) \simeq \langle x\partial/\partial y,\ y\partial/\partial x\rangle_{\mathrm{Lie}} \subset \mathfrak{h}.$$

- $a \geq 1$, $b \geq 0$. Since $\partial_{0,-1} = \partial/\partial y \in \mathfrak{j}_2^+$, we obtain by (4),
$$\partial_{a,b-1} \in \Bbbk^*[\partial_{0,-1}, \partial_{a,b}] \subset \mathfrak{h}.$$
Iterating, we arrive at the case where $b = -1$. So, $\partial_{a,-1} \in \mathfrak{h}$ for some $a \geq 1$. This brings us back to the previous case.
- $(a,b) = (0,b)$, where $b \geq 1$. Then, by (4),
$$\partial_{0,b-1} \in \Bbbk^*[\partial_{0,-1}, \partial_{0,b}] \subset \mathfrak{h}.$$
Iterating, we arrive at the case where $b = 1$, that is, $\partial_{0,1} \in \mathfrak{h}$. We then have
$$\mathfrak{sl}(2,\Bbbk) \simeq \Bbbk\partial_{0,-1} \oplus \Bbbk\partial_{0,0} \oplus \Bbbk\partial_{0,1} \subset \mathfrak{h}.$$

This proves the assertion for $\mathrm{Lie}(\mathrm{JONQ}^+(\mathbb{A}^2))$; the proof is similar for $\mathfrak{j}_2^-$. □

Remark 4.10. Let us show that the triangular subalgebra $\mathfrak{j}_2^+ \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ is not a Borel subalgebra of $\mathrm{Der}(\Bbbk[x,y])$. It suffices to verify that the subalgebra

$$\mathfrak{g} := \langle \mathfrak{j}_2^+, \delta\rangle_{\mathrm{Lie}} \subset \mathrm{Der}(\Bbbk[x,y]), \quad \text{where} \quad \delta = xy\partial/\partial x,$$

strictly containing $\mathfrak{j}_2^+$ is solvable. It is easy to see that

$$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{u}_2^+ \oplus \Bbbk\delta \oplus \Bbbk x\partial/\partial x \quad \text{and} \quad \mathfrak{g}^{(2)} \subset \mathfrak{j}_2^+.$$

Now the claim follows.

As per the following proposition, an analogue of Proposition 4.9 does not hold for the Lie algebra $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$.

Proposition 4.11. *The de Jonquières Lie subalgebra $\mathfrak{j}_{2,0}^+ \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ is not a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$. More precisely, the distinct Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ that strictly contain $\mathfrak{j}_{2,0}^+$ form an infinite sequence*

$$\mathfrak{h}_k := \langle \mathfrak{j}_{2,0}^+,\ \partial_{0,k}\rangle_{\mathrm{Lie}}, \quad k = 1,2,\dots$$

None of the $\mathfrak{h}_k$ are locally integrable.

Proof. We have $\partial_{0,k} \in \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$, and consequently $\mathfrak{h}_k \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$, see (8). For a fixed $k \geq 1$, from formula (6) we obtain

$$\partial_{0,k}^{(d)}(x) = c_d x y^{kd} \quad \text{where} \quad c_d \neq 0, \quad d > 0.$$

It follows that $\partial_{0,k}$ is not locally finite. This proves the second assertion.

To show that $\mathfrak{h}_k$ is solvable, note that $\mathfrak{i}_k := \mathfrak{u}_{2,0}^+ \oplus \Bbbk\,\partial_{0,k}$ is an ideal of $\mathfrak{h}_k$ and that $[\mathfrak{h}_k,\mathfrak{h}_k] = \mathfrak{i}_k$. Furthermore, $\mathfrak{u}_{2,0}^+$ is an abelian ideal in $\mathfrak{i}_k$, whence $\mathfrak{h}_k^{(2)} \subset \mathfrak{u}_{2,0}^+$, and $\mathfrak{h}_k^{(3)} = 0$.

Let us show that each $\mathfrak{h}_k$ is a maximal solvable subalgebra of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$. Suppose that a Lie subalgebra $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ strictly contains $\mathfrak{h}_k$. Proceeding as in the proof of Proposition 4.9, we see that $\mathfrak{h}$ must contain a homogeneous derivation $\partial_{a,b}$ with $(a,b) \neq (0,k)$, where $a \geq 0$. If $a = 0$, then we let $\partial_{a,b} = \partial_{0,l}$, where $l = b \neq k$. If $a > 0$, then we can replace $\partial_{a,b}$ by

$$\partial_{0,l} \in \Bbbk^* \operatorname{ad}(\partial_{-1,m})^{(a)}(\partial_{a,b}), \quad \text{where} \quad l = l(m) = am + b \neq k. \tag{9}$$

We claim that the Lie subalgebra

$$\mathfrak{h}_{k,l} := \langle \partial_{0,k},\, \partial_{0,l} \rangle_{\mathrm{Lie}} \subset \mathfrak{h}$$

is not solvable. Let $P_{k,l} \subset \mathbb{N}$ be the sub-semigroup generated by k and l. By (4), we have $\partial_{0,n} \in \mathfrak{h}_{k,l}$ for all $n \in P_{k,l}$. It is easy to see by induction, using (4), that, for any fixed $d > 0$, the derived ideal $\mathfrak{h}_{k,l}^{(d)}$ contains $\partial_{0,n}$ for all sufficiently large $n \in P_{k,l}$, hence it is nonzero.

This proves that $\mathfrak{h}_k$ is a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ strictly containing $\mathfrak{u}^+_{2,0}$. By the preceding, $\mathfrak{h}_k$ cannot contain $\partial_{0,l}$ for $l \neq k$, hence $\mathfrak{h}_k \neq \mathfrak{h}_l$.

Finally, let us show that the $\mathfrak{h}_k$ are the only Borel subalgebras of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ containing $\mathfrak{j}^+_{2,0}$. Indeed, consider a Lie subalgebra $\mathfrak{h} \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))$ strictly containing $\mathfrak{u}^+_{2,0}$. Suppose, for the sake of contradiction, that $\mathfrak{h}$ is solvable and that $\mathfrak{h} \neq \mathfrak{h}_k$ for all $k \geq 1$. As before, $\mathfrak{h}$ admits a vector basis consisting of homogeneous derivations $\partial_{a,b}$, and there exists $\partial_{a,b} \in \mathfrak{h}$ such that $a \geq 1$. Then, by (9), it follows that also $\partial_{0,am+b} \in \mathfrak{h}$ for all $m \geq 1$, which, as we have already noted, contradicts the assumption that $\mathfrak{h}$ is solvable. □

4.4. **The case of $\mathbb{A}^3$.** Unlike the affine plane case, the triangular subalgebra of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3))$ is not maximal among the solvable subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^3)$. In other words, it is a locally integrable Borel subalgebra (see Theorem 4.1(a)) and not a Borel subalgebra.

We use the following notation.

Notation 4.12. The ind-subgroup $\mathrm{JONQ}^+(\mathbb{A}^3)$ and its Lie algebra $\mathfrak{j}^+_3$ of triangular derivations of $\mathbb{k}[x,y,z]$ are nested and admit the semidirect product decompositions

$$\mathrm{JONQ}^+(\mathbb{A}^3) = \mathbb{T}_3 \ltimes U_3^+ \quad \text{resp.} \quad \mathfrak{j}^+_3 = \mathfrak{t}_3 \ltimes \mathfrak{u}^+_3,$$

where $\mathbb{T}_3 \simeq \mathbb{G}_{\mathrm{m}}^3$ is the diagonal 3-torus, U_3^+ is the unipotent radical of $\mathrm{JONQ}^+(\mathbb{A}^3)$,

$$\mathfrak{t}_3 = \mathrm{Lie}(\mathbb{T}_3) = \mathbb{k}x\partial/\partial x \oplus \mathbb{k}y\partial/\partial y \oplus \mathbb{k}z\partial/\partial z$$

and

$$\mathfrak{u}^+_3 = \mathrm{Lie}(U_3^+) = \mathbb{k}[y,z]\partial/\partial x \oplus \mathbb{k}[z]\partial/\partial y \oplus \mathbb{k}\partial/\partial z.$$

Proposition 4.13. *The Lie algebra $\mathfrak{j}^+_3$ of triangular derivations of $\mathbb{k}[x,y,z]$ is not maximal among the solvable subalgebras of $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^3)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^3)$.*

Proof. Let

$$\delta = z(x\partial/\partial x - y\partial/\partial y) \in \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^3) \setminus \mathfrak{j}^+_3\,.$$

Consider the Lie subalgebra $\mathfrak{h} = \langle \mathfrak{j}^+_3, \delta \rangle_{\mathrm{Lie}} \subset \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^3)$ which strictly contains $\mathfrak{j}^+_3$. We claim that $\mathfrak{h}$ is also solvable. Indeed, an easy calculation shows that

$$\mathrm{ad}(\delta)(\mathbb{k}[y,z]\partial/\partial x) \subset \mathbb{k}[y,z]\partial/\partial x, \quad \mathrm{ad}(\delta)(\mathbb{k}[z]\partial/\partial y) \subset \mathbb{k}[z]\partial/\partial y,$$

and

$$\mathrm{ad}(\delta)(\mathbb{k}\partial/\partial z) = \mathbb{k}(x\partial/\partial x - y\partial/\partial y).$$

So,

$$\mathrm{ad}(\delta)(\mathfrak{u}^+_3) \subset \mathfrak{u}^+_3 \oplus \mathbb{k}(x\partial/\partial x - y\partial/\partial y). \tag{10}$$

Moreover,

$$\mathrm{ad}(\delta)(x\partial/\partial x) = \mathrm{ad}(\delta)(y\partial/\partial y) = 0 \quad \text{and} \quad \mathrm{ad}(\delta)(z\partial/\partial z) = -\delta.$$

Therefore, $\mathrm{ad}(\delta)(\mathfrak{t}_3) = \Bbbk\delta$. This shows that $\mathfrak{h} = \mathfrak{j}_3^+ \oplus \Bbbk\delta$ (as vector spaces). It follows that

$$[\mathfrak{h},\mathfrak{h}] = [\mathfrak{j}_3^+,\mathfrak{j}_3^+] \oplus [\mathfrak{j}_3^+,\delta] \subset \mathfrak{u}_3^+ \oplus \Bbbk(x\partial/\partial x - y\partial/\partial y) \oplus \Bbbk\delta.$$

Since $\mathrm{ad}(\delta)(x\partial/\partial x - y\partial/\partial y) = 0$, using (10) we deduce $\mathfrak{h}^{(2)} \subset \mathfrak{j}_3^+$. Therefore, the Lie algebra $\mathfrak{h}$ is solvable. □

Remark 4.14. Note that δ is not locally finite. Indeed, $\delta : xz^n \mapsto xz^{n+1}$.

5. Subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$

This section is devoted to the proof of Theorem 1.5.

5.1. **Generalities.** Let us recall the notation and some results from [4, Section 4] and [5, Section 7.1].

Notation 5.1. Let ζ be a primitive root of unity of order $d > 1$ and e be an integer number with $1 \le e < d$ and $\gcd(d,e) = 1$. We denote by $G_{d,e}$ the cyclic subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ generated by $(x,y) \mapsto (\zeta^e x, \zeta y)$. We consider the normal affine toric surface

$$X_{d,e} = \mathrm{Spec}(k[x,y]^{G_{d,e}})$$

with the acting torus

$$\mathbb{T}' := \mathbb{T}/G_{d,e} \simeq \mathbb{G}_{\mathrm{m}}^2.$$

The unique singular point of $X_{d,e}$ is the image of the origin of $\mathbb{A}^2$, see e.g. [12, Section 2.6]. Consider the normalizers

$$\mathcal{N}_{d,e} = \mathrm{Norm}_{\mathrm{Aut}(\mathbb{A}^2)}(G_{d,e}) \quad \text{and} \quad N_{d,e} = \mathrm{Norm}_{\mathrm{GL}(2,\Bbbk)}(G_{d,e}) = \mathcal{N}_{d,e} \cap \mathrm{GL}(2,\Bbbk).$$

Let $\tau\colon (x,y) \mapsto (y,x)$ be the twist. We have

$$N_{d,e} = \begin{cases} \mathrm{GL}(2,\Bbbk) & \text{if } e = 1, \\ \mathrm{Norm}_{\mathrm{GL}(2,\Bbbk)}(\mathbb{T}) = \langle \mathbb{T}, \tau \rangle & \text{if } e \ne 1 \text{ and } e^2 \equiv 1 \mod d, \\ \mathbb{T} & \text{if } e^2 \not\equiv 1 \mod d. \end{cases}$$

We consider the lower and upper triangular subgroups $\mathrm{Jonq}^{\pm}(\mathbb{A}^2)$, where $\mathrm{Jonq}^-(\mathbb{A}^2) = \tau\mathrm{Jonq}^+(\mathbb{A}^2)\tau$, see subsection 4.3.1. Let also

$$\mathcal{N}_{d,e}^{\pm} = \mathcal{N}_{d,e} \cap \mathrm{Jonq}^{\pm}(\mathbb{A}^2) \quad \text{and} \quad N_{d,e}^{\pm} = N_{d,e} \cap \mathrm{Jonq}^{\pm}(\mathbb{A}^2).$$

We have

$$\mathcal{N}_{d,e}^+ \cap \mathcal{N}_{d,e}^- = \mathrm{Jonq}^+(\mathbb{A}^2) \cap \mathrm{Jonq}^-(\mathbb{A}^2) = \mathbb{T}\,.$$

Letting $\mathcal{B}_{d,e}^{\pm} = \mathcal{N}_{d,e}^{\pm}/G_{d,e}$ we get $\mathcal{B}_{d,e}^+ \cap \mathcal{B}_{d,e}^- = \mathbb{T}'$.

Proposition 5.2 (see [4, Lemmas 4.3 and 4.5])**.** *We have*

$$\mathrm{Aut}(X_{d,e}) \simeq \mathcal{N}_{d,e}/G_{d,e}, \tag{11}$$

and $\mathbb{T}'$ is a maximal torus of $\mathrm{Aut}(X_{d,e})$. Furthermore,

$$\mathcal{N}_{d,e}^+ = \mathrm{Norm}_{\mathrm{Jonq}^+(\mathbb{A}^2)}(G_{d,e}) = \{\phi\colon (x,y) \mapsto (\alpha x + y^e p(y^d), \beta y) \,|\, p \in \Bbbk[t], \alpha,\beta \in \Bbbk^*\}$$

and

$$\mathcal{N}_{d,e}^- = \mathrm{Norm}_{\mathrm{Jonq}^-(\mathbb{A}^2)}(G_{d,e}) = \{\phi\colon (x,y) \mapsto (\alpha x, \beta y + x^{e'} q(x^d)) \,|\, q \in \Bbbk[t], \alpha,\beta \in \Bbbk^*\}.$$

The group

$$\mathcal{N}^{\pm}_{d,e} = \mathbb{T} \ltimes U^{\pm}_{d,e} \tag{12}$$

is metabelian. The commutant

$$[\mathcal{N}^{\pm}_{d,e}, \mathcal{N}^{\pm}_{d,e}] = \{\phi \in \mathcal{N}^{\pm}_{d,e} \,|\, \alpha = \beta = 1\}$$

is abelian and coincides with the unipotent radical $U^{\pm}_{d,e}$ of $\mathcal{N}^{\pm}_{d,e}$.

The group $\mathrm{Aut}(X_{d,e})$ admits a free amalgamated product structure.

Theorem 5.3 ([4, Theorem 4.2 and Proposition 4.4])**.**

(*a*) *In case $e^2 \not\equiv 1 \mod d$ we have*

$$\mathcal{N}_{d,e} \simeq \mathcal{N}^{+}_{d,e} *_{\mathbb{T}} \mathcal{N}^{-}_{d,e}$$

and

$$\mathrm{Aut}(X_{d,e}) \simeq \mathcal{B}^{+}_{d,e} *_{\mathbb{T}'} \mathcal{B}^{-}_{d,e}\,.$$

(*b*) *In case $e^2 \equiv 1 \mod d$ we have*

$$\mathcal{N}_{d,e} \simeq \mathcal{N}^{+}_{d,e} *_{N^{+}_{d,e}} N_{d,e}$$

and

$$\mathrm{Aut}(X_{d,e}) \simeq \mathcal{B}^{+}_{d,e} *_{N^{+}_{d,e}/G_{d,e}} N_{d,e}/G_{d,e}.$$

Theorem 5.4 ([5, Theorem 1.1])**.**

(*a*) *Let $e^2 \equiv 1 \mod d$. Then the $\mathcal{N}^{\pm}_{d,e}$ (resp. the $\mathcal{B}^{\pm}_{d,e}$) are Borel subgroups of $\mathcal{N}_{d,e}$ (resp. of $\mathrm{Aut}(X_{d,e})$), and they are conjugate. Any Borel subgroup of $\mathcal{N}_{d,e}$ (resp. of $\mathrm{Aut}(X_{d,e})$) is conjugate to $\mathcal{N}^{+}_{d,e}$ (resp. to $\mathcal{B}^{+}_{d,e}$).*

(*b*) *Let $e^2 \not\equiv 1 \mod d$. Then $\mathcal{N}_{d,e}$ (resp. $\mathrm{Aut}(X_{d,e})$) has exactly two conjugacy classes of Borel subgroups represented by the $\mathcal{N}^{\pm}_{d,e}$ (resp. by the $\mathcal{B}^{\pm}_{d,e}$).*

5.2. **On the Lie algebra** $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$**.**

Lemma 5.5. *We have*

$$\mathrm{Lie}(\mathcal{N}^{\pm}_{d,e}) = \mathrm{Lie}(\mathbb{T}) \ltimes \mathrm{Lie}(U^{\pm}_{d,e}) = \mathfrak{t}_2 \ltimes \mathfrak{u}^{\pm}_{d,e}, \tag{13}$$

where the abelian Lie subalgebras

$$\mathfrak{u}^{+}_{d,e} = \mathrm{Lie}(U^{+}_{d,e}) = \{\delta_p := y^e p(y^d)\partial/\partial x \,|\, p \in \Bbbk[y]\} = \bigoplus_{k\geq 0} \Bbbk y^{e+kd}\partial/\partial x \tag{14}$$

and

$$\mathfrak{u}^{-}_{d,e} = \mathrm{Lie}(U^{-}_{d,e}) = \{\delta'_q := x^{e'} q(x^d)\partial/\partial y \,|\, q \in \Bbbk[x]\} = \bigoplus_{l\geq 0} \Bbbk x^{e'+ld}\partial/\partial y \tag{15}$$

are made up of triangular locally nilpotent derivations from $\mathrm{Lie}(\mathcal{N}_{d,e})$*, while the toral subalgebra*

$$\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T}) = \langle x\partial/\partial x, y\partial/\partial y\rangle_{\mathrm{Lie}} = \Bbbk x\partial/\partial x \oplus \Bbbk y\partial/\partial y \tag{16}$$

is made up of semisimple derivations from $\mathrm{Lie}(\mathcal{N}_{d,e})$*.*

Proof. Let $\mathfrak{G}$ be an ind-group and $G \subset \mathfrak{G}$ be an algebraic subgroup. Then we have $\mathrm{Lie}(G) \subset \mathrm{Lie}(\mathfrak{G})$. If $\mathfrak{G} = \varinjlim G_i$ is nested by algebraic subgroups G_i, then $\mathrm{Lie}(\mathfrak{G}) = \varinjlim \mathrm{Lie}(G_i)$.

The equality (13) is a direct consequence of (12). Note that the replica δ_p (resp. δ'_q) of $\partial/\partial x$ (resp. of $\partial/\partial y$) is a locally nilpotent derivations of $\Bbbk[x,y]$.

Every $u \in U^+_{d,e}$ (resp. $v \in U^-_{d,e}$) is contained in a $\mathbb{G}_{\mathrm{a}}$-subgroup

$$U^+_p := \exp(\Bbbk\delta_p) = \{(x + ty^e p(y^d), y) \,|\, t \in \Bbbk\} \quad (\text{resp.} \quad U^-_q := \exp(\Bbbk\delta'_q)).$$

Now (14) and (15) follow, and (16) is immediate. □

Corollary 5.6. *The homogeneous locally nilpotent derivations*

$$\partial^+_{e+kd} = y^{e+kd}\partial/\partial x \quad \textit{resp.} \quad \partial^-_{e'+ld} = x^{e'+ld}\partial/\partial y, \quad k,l = 0,1,2,\dots$$

generate the abelian Lie algebra $\mathfrak{u}^+_{d,e}$ *resp.* $\mathfrak{u}^-_{d,e}$ *as a vector space.*

Proof. Using (14) resp. (15), for $p(y) = y^k$ resp. $q(x) = x^l$ we obtain

$$\mathrm{Lie}(U^+_p) = \Bbbk\partial^+_{e+kd} \subset \mathfrak{u}^+_{d,e} \quad \text{resp.} \quad \mathrm{Lie}(U^-_q) = \Bbbk\partial^-_{e'+ld} \subset \mathfrak{u}^-_{d,e}, \quad k,l = 0,1,2,\dots,$$

and therefore $\mathfrak{u}^+_{d,e} = \bigoplus_{k\ge 0} \Bbbk\partial^+_{e+kd}$ resp. $\mathfrak{u}^-_{d,e} = \bigoplus_{l\ge 0} \Bbbk\partial^-_{e'+ld}$. □

Remark 5.7. The subgroup $\mathcal{N}_{d,e}$ is defined as the normalizer of $G_{d,e}$ in $\mathrm{Aut}(\mathbb{A}^2)$, see Notation 5.1. However, since the origin of $\mathbb{A}^2$ is the unique fixed point of $G_{d,e}$, we have $\mathcal{N}_{d,e} \subset \mathrm{Aut}_0(\mathbb{A}^2)$.

Let G be a connected algebraic group and $F \subset G$ be a finite subgroup. It is well known that the Lie algebra of the centralizer $\mathrm{Cent}_G(F)$ coincides with the fixed-point subalgebra $\mathrm{Lie}(G)^{\mathrm{Ad}(F)}$, see for example [26, Chapter 1, Section 2.11, Exercise 4]. In our setup, there is the following partial analog, cf. [22, Section 6.2, formula (2)].

Lemma 5.8. *We have* $\mathrm{Lie}(\mathcal{N}_{d,e}) \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$. *In other words, the* Ad*-action of* $G_{d,e}$ *on* $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ *is identical on the Lie subalgebra* $\mathrm{Lie}(\mathcal{N}_{d,e})$*, and then also on the Lie subalgebras* $\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T})$, $\mathrm{Lie}(U^\pm_{d,e})$*, and* $\mathrm{Lie}(\mathcal{N}^\pm_{d,e})$ *of* $\mathrm{Lie}(\mathcal{N}_{d,e})$.

Proof. We have

$$\mathrm{Lie}(\mathcal{N}_{d,e}) = \mathrm{Lie}(\mathcal{N}^0_{d,e}) \quad \text{and} \quad \mathcal{N}^0_{d,e} \subset \mathrm{Cent}_{\mathrm{Aut}_0(\mathbb{A}^2)}(G_{d,e}).$$

The second inclusion follows from the fact that the conjugation map sends $\mathcal{N}_{d,e}$ into the finite group $\mathrm{Aut}(G_{d,e})$; it is therefore constant on the connected component $\mathcal{N}^0_{d,e}$ and coincides with the identity there. Now, the inclusion $\mathrm{Lie}(\mathcal{N}_{d,e}) = \mathrm{Lie}(\mathcal{N}^0_{d,e}) \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$ follows. □

Remark 5.9. From the decomposition

$$\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2)) \oplus \Bbbk\partial/\partial x \oplus \Bbbk\partial/\partial y$$

it follows that $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})} = \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$.

The following corollary is immediate.

Corollary 5.10. *The surjection* $\mathcal{N}_{d,e} \to \mathrm{Aut}(X_{d,e}) = \mathcal{N}_{d,e}/G_{d,e}$*, see* (11)*, induces isomorphisms*

$$\mathrm{Lie}(\mathrm{Aut}(X_{d,e})) \simeq \mathrm{Lie}(\mathcal{N}_{d,e}/G_{d,e}) \simeq \mathrm{Lie}(\mathcal{N}_{d,e}).$$

Remarks 5.11.

1. The $\mathbb{T}$-action on $\mathrm{Vec}(\mathbb{A}^2)$ leaves invariant the subalgebra $\mathrm{Lie}(\mathcal{N}_{d,e})$. By Lemma 4.8, for any $\partial \in \mathrm{Lie}(\mathcal{N}_{d,e})$ every homogeneous component of ∂ also belongs to $\mathrm{Lie}(\mathcal{N}_{d,e})$.
2. Since $\mathfrak{t}_2 = \mathrm{Lie}(\mathbb{T}) \subset \mathrm{Lie}(\mathcal{N}_{d,e})$, the homogeneous derivations $\alpha x\partial/\partial x + \beta y\partial/\partial y \in \mathfrak{t}_2$ and $\partial^+_{e+kd} \in \mathfrak{u}^+_{d,e}$, $\partial^-_{e'+ld} \in \mathfrak{u}^-_{d,e}$ belong to $\mathrm{Lie}(\mathcal{N}_{d,e})$.

Lemma 5.12. *A homogeneous derivation $\partial \in \mathrm{Vec}^c(\mathbb{A}^2)$ of bidegree (a,b) is fixed by the $\mathrm{Ad}(G_{d,e})$-action if and only if $ae + b \equiv 0 \mod d$. Therefore, we have*

$$\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})} = \Bbbk x\partial/\partial x \oplus \Bbbk y\partial/\partial y \oplus \bigoplus_{k\geq 0} \Bbbk \partial^+_{e+kd} \oplus \bigoplus_{l\geq 0} \Bbbk \partial^-_{e'+ld} \oplus \bigoplus_{(a,b)\in\Lambda_{d,e}} \Bbbk \partial_{a,b}, \tag{17}$$

where the submonoid $\Lambda_{d,e} \subset \mathbb{Z}^2$ is defined as follows:

$$\Lambda_{d,e} := \{(a,b) \,|\, ae + b \equiv 0 \mod d,\ \ a \geq 0,\ \ b \geq 0,\ \ a + b > 0\}. \tag{18}$$

Proof. Any derivation $\alpha x\partial/\partial x + \beta y\partial/\partial y \in \mathfrak{t}$ of bidegree $(0,0)$ is fixed by the $\mathrm{Ad}(\mathbb{T})$-action, hence also by the $G_{d,e}$-action. Since $\mathrm{Vec}^c(\mathbb{A}^2) = \mathrm{Vec}^0(\mathbb{A}^2) \oplus \Bbbk\delta$, where $\delta = x\partial/\partial x + y\partial/\partial y$ has bidegree $(0,0)$, it suffices to prove the assertion for $\mathrm{Vec}^0(\mathbb{A}^2)$.

We can easily see that $\partial_{-1,n}$, resp. $\partial_{m,-1}$, is fixed under the $\mathrm{Ad}(G_{d,e})$-action if and only if $n = e + kd$, resp. $m = e' + ld$, for some $k, l \geq 0$, if and only if the bidegree $(a,b) = (-1,n)$ resp. $(a,b) = (m,-1)$ satisfies the relation $ae + b \equiv 0 \mod d$.

Recall that the weighted component of $\mathrm{Vec}^0(\mathbb{A}^2)$ of bidegree (a,b), where $a, b \geq 0$, is one-dimensional and is generated by

$$\partial_{a,b} = x^a y^b((b+1)x\partial/\partial x - (a+1)y\partial/\partial y).$$

Since $x\partial/\partial x$ and $y\partial/\partial y$ are fixed under the $\mathrm{Ad}(G_{d,e})$-action, $\partial_{a,b}$ is also fixed if and only if $G_{d,e}$ fixes $x^a y^b$, if and only if $ae + b \equiv 0 \mod d$. □

Question 2. *For which pairs (d,e) the equality*

$$\mathrm{Lie}(\mathcal{N}_{d,e}) = \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})} \tag{19}$$

holds? In particular, is it true that

$$xy(x\partial/\partial x - y\partial/\partial y) \in \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{4,3})} \setminus \mathrm{Lie}(\mathcal{N}_{4,3}),$$

and, more generally, that (19) holds if and only if $ee' \in \{1, d+1\}$?

See Example 5.13 and Proposition 5.16 below for some partial results.

Example 5.13. Suppose that $ee' = 1$, that is, $e = e' = 1$. In this case, the derivations ∂^+_{1+kd} and ∂^-_{1+ld} belong to $\mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$, see e.g. (14) and (15). Thus, $\mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$ contains $\partial^+_1 = y\partial/\partial x$ and $\partial^-_1 = x\partial/\partial y$. Then also $[\partial^-_1, \partial^+_1] = x\partial/\partial x - y\partial/\partial y \in \mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$, and therefore

$$\mathrm{Lie}(\mathrm{Aut}(X_{d,1})) \supset \Bbbk y\partial/\partial x \oplus \Bbbk x\partial/\partial y \oplus \Bbbk(x\partial/\partial x - y\partial/\partial y) \simeq \mathfrak{sl}(2,\Bbbk). \tag{20}$$

From (4) we deduce

$$\partial_{ld,kd} = [\partial^-_{1+ld},\, \partial^+_{1+kd}] \in \mathrm{Lie}(\mathrm{Aut}(X_{d,1})) \quad \forall k, l \geq 0.$$

In particular, $\partial_{ld,0} \in \mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$ for $l \geq 0$. Then also

$$\mathrm{ad}(\partial^+_1)^s(\partial_{ld,0}) = \alpha_s \partial_{ld-s,s} \in \mathrm{Lie}(\mathrm{Aut}(X_{d,1})) \quad \forall s = 1, \ldots, ld, \ \text{ where } \ \alpha_s > 0,$$

see (4). Thus, $\partial_{a,b} \in \mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$ for all (a,b) with $a,b \geq 0$ and $a+b \equiv 0 \mod d$. Since $\mathfrak{t}_2 \subset \mathrm{Lie}(\mathrm{Aut}(X_{d,1}))$ it follows from (17) that

$$\mathrm{Lie}(\mathrm{Aut}(X_{d,1})) = \mathrm{Lie}(\mathcal{N}_{d,1}) = \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,1})}.$$

Remark 5.14. The surface $X_{d,1}$ is isomorphic to the d-th Veronese surface $V_d \subset \mathbb{A}^{d+1}$, where $d \geq 2$, that is, to the affine cone over the rational normal curve

$$C_d = \{(u^d : u^{d-1}v : \ldots : v^d) \,|\, (u:v) \in \mathbb{P}^1\} \subset \mathbb{P}^d.$$

Note that V_d is the closure of an $\mathrm{SL}(2,\Bbbk)$-orbit in the irreducible $\mathrm{SL}(2,\Bbbk)$-module $\mathbb{A}^{d+1}$ of degree d binary forms, see e.g. [28]. The $\mathfrak{sl}(2,\Bbbk)$-subalgebra (20) of $\mathrm{Lie}(\mathrm{Aut}(V_d))$ is the tangent Lie algebra of the $\mathrm{SL}(2,\Bbbk)$-action on V_d. The $\mathrm{SL}(2,\Bbbk)$-module $\mathbb{A}^{d+1}$ can be considered as the Zariski tangent space of the cone V_d at its vertex, equipped with the induced $\mathrm{SL}(2,\Bbbk)$-action. In turn, the $\mathrm{SL}(2,\Bbbk)$-action on the pair $(\mathbb{A}^{d+1}, V_d)$ induces the natural $\mathrm{PSL}(2,\Bbbk)$-action on the pair $(\mathbb{P}^d, C_d)$.

The interested reader can find further information on automorphisms and derivations of Veronese subalgebras over a field of characteristic zero in [1] and [2].

Example 5.15. We have $\mathrm{Lie}(\mathcal{N}_{3,2}) = \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{3,2})}$, see [22, Lemma 6.2.1]. In this example, $e' = e = 2$ and $d = ee' - 1 = 3$.

The last example can be generalized as follows.

Proposition 5.16. *The equality* (19) *holds provided that* $ee' = d+1$[3].

Proof. Recall that $\partial_e^+ \in \mathfrak{u}_{d,e}^+$ and $\partial_{e'}^- \in \mathfrak{u}_{d,e}^-$. Fix an arbitrary $(a,b) \in \Lambda_{d,e}$, see (18). We will show that under our assumptions,

$$\partial_{a,b} \in \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}} \subset \mathrm{Lie}(\mathrm{Aut}(X_{d,e})).$$

Since $\mathfrak{u}_{d,e}^{\pm}$ and $\mathfrak{t}_2$ are also subalgebras of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ (see Remark 5.11.2), the above inclusion implies (19).

We have an equivalence

$$ae + b \equiv 0 \mod d \Leftrightarrow a + be' \equiv 0 \mod d.$$

Let $u = (-1, e)$ and $v = (e', -1)$. There exists a pair (n,m) of strictly positive integers such that

$$(a,b) = nu + mv = (me' - n, ne - m). \tag{21}$$

Indeed, since by assumption $ee' = d+1$, the system (21) of diophantine equations admits an integer solution

$$(n,m) = \left(\frac{a+be'}{d}, \frac{ae+b}{d}\right), \quad \text{where} \quad n, m > 0.$$

By (4) we have

$$u + v = (e'-1, e-1) \quad \text{and} \quad \partial_{e'-1,e-1} = -\mathrm{ad}(\partial_e^+)(\partial_{e'}^-) \in \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}.$$

[3]For $d > 2$ the latter equality is equivalent to the conditions $e > 1$ and $e|(d+1)$ (cf. Theorem 1.5), and also to the conditions $e' > 1$ and $e'|(d+1)$.

Since $ee' = d+1$, the vector $(a,b) = (e'-1, e-1)$ is a unique lattice vector in $\Lambda_{d,e}$ satisfying the inequalities

$$a < e', \quad b < e. \tag{22}$$

Consider the strictly positive, integer valued function

$$f\colon \Lambda_{d,e} \to d\mathbb{Z}_{>0}, \quad f(a,b) = (ae+b) + (a+be') = (e+1)a + (e'+1)b.$$

Suppose that (22) fails for $(a,b) \in \Lambda_{d,e}$. Let, for example, $a \geq e' > 1$, and therefore $\rho := \lfloor \frac{a}{e'} \rfloor \geq 1$. Consider the lattice vector

$$(a',b') = (a,b) - \rho v = (a - \rho e', b + \rho) \in \Lambda_{d,e}.$$

Using (4) we deduce

$$\partial_{a,b} = \beta(\mathrm{ad}(\partial_{e'}^-))^\rho(\partial_{a',b'}), \quad \text{where} \quad \beta = [(b+2)(b+3)\cdots(b+\rho+1)]^{-1} \neq 0,$$

and

$$f(a',b') = f(a,b) - \rho d < f(a,b).$$

By symmetry, if $b \geq e > 1$, then letting $\eta = \lfloor \frac{b}{e} \rfloor \geq 1$ and $(a',b') = (a,b) - \eta u \in \Lambda_{d,e}$ we obtain

$$\partial_{a,b} = \alpha(\mathrm{ad}(\partial_e^+))^\eta(\partial_{a',b'}), \quad \text{where} \quad \alpha \neq 0,$$

and

$$f(a',b') = f(a,b) - \eta d < f(a,b).$$

Thus, any $(a,b) \in \Lambda_{d,e}$ distinct from $(e'-1, e-1) = u+v$ can be written in one of the forms

$$(a,b) = (a',b') + \rho v \quad \text{or} \quad (a,b) = (a',b') + \eta u,$$

where $\rho, \eta \geq 1$ and $f(a',b') < f(a,b)$. The minimum of f on $\Lambda_{d,e}$ is equal to $2d$ and is achieved at a single point $u+v = (e'-1, e-1)$. Continuing in this way, we finally achieve the point $(a',b') = (e'-1, e-1)$, where the function f attains its minimum.

Summarizing, any $(a,b) \in \Lambda_{d,e}$ can be written as

$$(a,b) = u + v + \rho_1 v + \eta_1 u + \cdots + \eta_{k-1} u + \rho_k v, \quad k \geq 0,$$

where

$$1 + \rho_1 + \cdots + \rho_k = m \quad \text{and} \quad 1 + \eta_1 + \cdots + \eta_{k-1} = n,$$

one of the ρ_1, ρ_k or both can be zero, while the other coefficients are positive, and all partial sums

$$u+v, \quad u+v+\rho_1 v, \quad u+v+\rho_1 v + +\eta_1 u, \quad \ldots, \quad (a,b)$$

are lattice vectors from $\Lambda_{d,e}$. This allows to represent $\partial_{a,b}$ as follows:

$$\partial_{a,b} = c\,(\mathrm{ad}(\partial_{e'}^-))^{\rho_k}(\mathrm{ad}(\partial_e^+))^{\eta_{k-1}} \cdots (\mathrm{ad}(\partial_e^+))^{\eta_1}(\mathrm{ad}(\partial_{e'}^-))^{\rho_1}\mathrm{ad}(\partial_e^+)(\partial_{e'}^-) \in \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}, \tag{23}$$

where $c \in \Bbbk$ is nonzero, see (4). □

Corollary 5.17. *Under the assumption $ee' = d+1$ we have*

$$\mathrm{Lie}(\mathrm{Aut}(X_{d,e})) = \mathfrak{t}_2 \oplus \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}.$$

Proof. According to (23), $\partial_{a,b} \in \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}$ for all $(a,b) \in \Lambda_{d,e}$. In particular, for all $k, l > 0$ we have

$$\partial_{0,kd},\ \partial_{ld,0} \in \langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}. \tag{24}$$

Moreover, by (4),

$$\partial^+_{e+kd} = (e+kd+1)^{-1} \operatorname{ad}(\partial_e^+)(\partial_{0,kd}) \quad \text{and} \quad \partial^-_{e'+ld} = -(e'+ld+1)^{-1} \operatorname{ad}(\partial_{e'}^-)(\partial_{ld,0})$$

also belong to $\langle \partial_e^+, \partial_{e'}^- \rangle_{\mathrm{Lie}}$. Now the claim follows. □

By combining Theorem 1.3, Theorem 5.4, Lemma 5.5, and Corollary 5.10, we arrive at the following conclusion.

Corollary 5.18. *The locally integrable Borel subalgebras of* $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ *are precisely the* Ad*-conjugates of* $\mathfrak{t}_2 \ltimes \mathfrak{u}^{\pm}_{d,e}$.

Remark 5.19. However, $\mathfrak{t}_2 \ltimes \mathfrak{u}^{\pm}_{d,e}$ is not always a Borel subalgebra of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e})) \simeq \mathrm{Lie}(\mathcal{N}_{d,e})$. Indeed, suppose that $ee' = d+1$. Then we have $\partial_{0,d} \in \mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$, see (24). In this case, the solvable Lie subalgebra

$$\mathfrak{h} := \langle \mathfrak{t}_2 \ltimes \mathfrak{u}^{\pm}_{d,e}, \partial_{0,d} \rangle_{\mathrm{Lie}} \subset \mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$$

strictly contains $\mathfrak{t}_2 \ltimes \mathfrak{u}^{\pm}_{d,e}$; see the proof of Proposition 4.11.

5.3. **The Lie algebra of zero divergence vector fields on $X_{d,e}$ is not simple.** Recall that the subalgebra $\mathrm{Vec}^0(\mathbb{A}^2)$ of divergence-free vector fields is the unique proper ideal of the Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2)) = \mathrm{Vec}^{\mathrm{c}}(\mathbb{A}^2)$. Moreover, the codimension one subalgebra $\mathrm{Vec}^0(\mathbb{A}^2) \subset \mathrm{Lie}(\mathrm{Aut}(\mathbb{A}^2))$ is simple, see [30, Lemma 3].

In the Lie algebra $\mathrm{Lie}(\mathcal{N}_{d,e})$, we consider the codimension one ideal

$$\mathfrak{g}_{d,e} = \mathrm{Lie}(\mathcal{N}_{d,e}) \cap \mathrm{Lie}(\mathfrak{U}_2) = \mathrm{Lie}(\mathcal{N}_{d,e}) \cap \mathrm{Vec}^0(\mathbb{A}^2).$$

Since $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ can be naturally identified with $\mathrm{Lie}(\mathcal{N}_{d,e})$, see Corollary 5.10, we can consider $\mathfrak{g}_{d,e}$ as a maximal ideal of $\mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$.

Proposition 5.20. *The Lie subalgebra* $\mathfrak{g}_{d,e} \subset \mathrm{Lie}(\mathrm{Aut}(X_{d,e}))$ *is not simple.*

Proof. In the lattice

$$L_{d,e} = \{(a,b) \in \mathbb{Z}^2 \,|\, ae + b \equiv 0 \mod d\}$$

we consider the submonoid

$$\hat{\Lambda}_{d,e} = \{(a,b) \in L_{d,e} \,|\, a, b \geq -1, \quad (a,b) \neq (-1,-1)\}.$$

It properly contains the submonoid $\Lambda_{d,e}$ of (18). More precisely,

$$\hat{\Lambda}_{d,e} = \Lambda_{d,e} \cup \{(-1, e+kd) \,|\, k \geq 0\} \cup \{(e'+ld, -1) \,|\, l \geq 0\} \cup \{(0,0)\}.$$

We also consider the submonoid

$$\lambda_{d,e} := \hat{\Lambda}_{d,e} \setminus \{(-1,e), (e',-1), (0,0)\}$$

and the homogeneous subalgebra $I_{d,e} \subset \mathrm{Lie}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$ of codimension 4 supported by $\lambda_{d,e}$, where

$$I_{d,e} = \bigoplus_{k \geq 1} \Bbbk \partial^+_{e+kd} \oplus \bigoplus_{l \geq 1} \Bbbk \partial^-_{e'+ld} \oplus \bigoplus_{(a,b) \in \Lambda_{d,e}} \Bbbk \partial_{a,b},$$

cf. (17). We claim that $I_{d,e}$ is an ideal of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$. Using (4), it suffices to check the inclusion

$$(\hat{\Lambda}_{d,e} + \lambda_{d,e}) \cap \hat{\Lambda}_{d,e} \subset \lambda_{d,e}.$$

Suppose this inclusion is not true. Then the left-hand side contains one of the lattice vectors $(-1, e)$, $(e', -1)$ and $(0, 0)$. Assuming that $(-1, e) = \hat{v} + v$, where $\hat{v} \in \hat{\Lambda}_{d,e}$ and $v \in \lambda_{d,e}$, we obtain

$$(-1, e) = (-1, e + kd) + (0, ld) \quad \text{for some} \quad k, l > 0,$$

which is impossible. The same argument applies to $(e', -1)$. Similarly, if $(0, 0) = \hat{v} + v$, then either $\hat{v} = v = (0, 0)$, or $\{\hat{v}, v\} = \{(-1, 1), (1, -1)\}$. But this is impossible because $(0, 0)$, $(-1, 1)$, $(1, -1) \notin \lambda_{d,e}$.

Since $I_{d,e}$ is a homogeneous ideal of codimension 4 of $\mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$, then also $I^0_{d,e} := I_{d,e} \cap \mathfrak{g}_{d,e}$ is a homogeneous ideal of codimension 3 of the subalgebra $\mathfrak{g}_{d,e} \subset \mathrm{Lie}(\mathrm{Aut}_0(\mathbb{A}^2))^{\mathrm{Ad}(G_{d,e})}$. Indeed, it is transversal to the vector subspace

$$\Bbbk\partial^+_e \oplus \Bbbk\partial^-_{e'} \oplus \Bbbk\partial_{0,0} \subset \mathfrak{g}_{d,e}.$$

Therefore, the Lie algebra $\mathfrak{g}_{d,e}$ is not simple. □

Remarks 5.21.

1. This proof does not work for $\mathrm{Lie}(\mathfrak{U}_2)$, because this Lie algebra contains the derivations $\partial/\partial x$ and $\partial/\partial y$ whose adjoint action corresponds to the left or down shift by one unit on the lattice $\mathbb{Z}^2$, see (4). And, indeed, this algebra is simple, see [30, Lemma 3].

2. Any normal affine toric surface X carrying no non-constant invertible regular function is isomorphic to either $\mathbb{A}^2$ or to some $X_{d,e} = \mathbb{A}^2/\mathbb{G}_{d,e}$. It follows from Proposition 5.20 that $X \simeq \mathbb{A}^2$ (that is, X is smooth) if and only if the maximal ideal of divergence zero vector fields from $\mathrm{Lie}(\mathrm{Aut}(X))$ is simple (cf. [31]).

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