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CONSTRUCTIVE APPROXIMATION



Shape Preserving Approximation of Periodic Functions: Conclusion

D. Leviatan[1] · I. O. Shevchuk[2]

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Abstract
We give here the final results about the validity of Jackson-type estimates in comonotone approximation of 2π-periodic functions by trigonometric polynomials ($q = 1$). For coconvex ($q = 2$) and the so called co-q-monotone, $q > 2$, approximations, everything is known by now. Thus, this paper concludes the research on Jackson type estimates of Shape Preserving Approximation of periodic functions by trigonometric polynomials. It is interesting to point out that the results for comonotone approximation of a periodic function are substantially different than the analogous results for comonotone approximation of a continuous function on a finite interval, by algebraic polynomials.

Keywords Comonotone approximation by trigonometric polynomials · Degree of approximation · Jackson-type estimates

Mathematics Subject Classification 41A29 · 42A10 · 41A25

1 Introduction

We give here the final results about the validity of Jackson-type estimates in comonotone approximation of 2π-periodic functions by trigonometric polynomials ($q = 1$). For coconvex approximation ($q = 2$) everything is known by now (see, e.g., [8]), and for the so called co-q-monotone, $q > 2$, approximation, no Jackson-type estimates

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✉ D. Leviatan
leviatan@tauex.tau.ac.il

I. O. Shevchuk
shevchukio@knu.ua

[1] Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, 6139001 Tel Aviv, Israel

[2] Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine

are possible (see, e.g., [6]). Thus, this paper concludes the research on Jackson type estimates of Shape Preserving Approximation of periodic functions by trigonometric polynomials.

Let $C = C^0$ be the space of continuous 2π-periodic functions f on $\mathbb{R}$, endowed with the uniform norm $\|f\| := \max_{x\in\mathbb{R}} |f(x)|$, and for $r \in \mathbb{N}$, denote by C^r the subspace of r times continuously differentiable functions $f \in C$.

In this paper we discuss the approximation of $f \in C^r, r \geq 0$, that changes its monotonicity a few times in its period $(-\pi, \pi]$, by trigonometric polynomials that follow the monotonicity of f (i.e., comonotone approximation). Note that a nonconstant trigonometric polynomial may only have a finite, even number of changes of monotonicity in a period. Hence, one is limited to discussion of comonotone approximation of f that has $2s$, $s \geq 1$, changes of monotonicity.

Let $Y_s := \{y_i\}_{i\in\mathbb{Z}}$, $s \in \mathbb{N}$, $y_i \in \mathbb{R}$, be such that $y_i < y_{i+1}$ and $y_{i+2s} = y_i + 2\pi$, $i \in \mathbb{Z}$. Denote by $\Delta^{(1)}(Y_s)$, the collection of functions $f \in C$, such that $(-1)^{i-1} f$ is nondecreasing on the interval $[y_{i-1}, y_i]$, $i \in \mathbb{Z}$, and note that if we denote

$$\Pi(x) := \prod_{i=1}^{2s} \sin \frac{x - y_i}{2}, \tag{1.1}$$

then for $f \in C^1$, $f \in \Delta^{(1)}(Y_s)$, if and only if

$$f'(x)\Pi(x) \geq 0, \quad x \in \mathbb{R}.$$

For $f \in \Delta^{(1)}(Y_s)$, let

$$E_n^{(1)}(f, Y_s) := \inf_{T_n \in \Delta^{(1)}(Y_s)} \|f - T_n\|,$$

denote the error of best approximation of f by comonotone trigonometric polynomials T_n of degree $< n$.

We are interested in estimates of the Jackson type,

$$E_n^{(1)}(f, Y_s) \leq \frac{c(k, r, s)}{n^r} \omega_k \left(f^{(r)}, \frac{1}{n} \right), \quad n \geq 1, \tag{1.2}$$

where $\omega_k \left(f^{(r)}, \cdot \right)$ is the kth modulus of smoothness of $f^{(r)}$. Specifically, we are interested in the validity of (1.2), depending on the various parameters, $s \geq 1$, $k \geq 1$ and $r \geq 0$.

The first result in this direction was obtained by Lorentz and Zeller [9, Theorem 1], who proved such estimate (with $r = 0$, $s = 1$ and $k = 1$) for the approximation of a continuous bell-shaped function on $[-\pi, \pi]$, by bell-shaped trigonometric polynomials. Clearly, such a bell-shaped function may be looked upon as an even 2π-periodic function, increasing on $[-\pi, 0]$ and decreasing on $[0, \pi]$. (They applied this result to get a corresponding estimate for the approximation of a continuous nondecreasing function on $[-1, 1]$, by nondecreasing algebraic polynomials.)

If $f \in C^r \cap \Delta^{(1)}(Y_s)$, and either $k = 1$ or $r > 2s$, then the validity of (1.2) is proved in [7, Theorem 1.1]. (For $r = 0$ and $k = 1$, this was shown earlier by [10].)

In this paper we determine all triplets (r, k, s), for which (1.2) is valid, and prove that (1.2) is invalid for all other triplets.

Remark 1.1 If (1.2) is invalid and $r \geq 2$, then all is not lost, as it still follows by [2], that the estimate

$$E_n^{(1)}(f, Y_s) \leq \frac{c(k, r, s)}{n^r} \omega_k \left(f^{(r)}, \frac{1}{n} \right), \quad n \geq N(r, k, Y_s), \tag{1.3}$$

holds, where $N(r, k, Y_s)$ depends on the smallest distance between the points y_i, $i \in \mathbb{Z}$. In addition, for $n > N(Y_s)$ we may guarantee that the distance between any two consecutive y_i's is bigger than $\frac{5\pi}{n}$. Therefore, if we take $n > N(Y_s)$, then we may combine Lemma 4.4 and Lemma 4.6 with Lemma 5.1 below, and get (1.3) for $r = 1$ and $k = 2, 3$, as well. Hence, (1.3) is valid, but as we show in this paper, (1.2) is valid for these pairs only for $s = 1$, and is invalid otherwise. Finally, the validity of (1.3) when $r = 0$ and $k = 2$, is proved in [3]. However, if $r = 0$ and $k \geq 3$, and if $r = 1$ and $k \geq 4$, then even (1.3) is invalid, since (see [4]), for each Y_s there is a function $f \in C^r \cap \Delta^{(1)}(Y_s)$, such that for these pairs (r, k),

$$\limsup_{n \to \infty} \frac{n^r}{\omega_k(f^{(r)}, 1/n)} E_n^{(1)}(f, Y_s) = \infty.$$

Recall that for $g \in C$ or $g \in C[a, b]$, we denote the kth modulus of smoothness of g, $k \geq 1$, respectively, by

$$\omega_k(g, t) = \sup_{0<h\leq t} \left\| \Delta_h^k(g, \cdot) \right\| \quad \text{and} \quad \omega_k(g, t; [a, b]) = \sup_{0<h\leq t} \left\| \Delta_h^k(g, \cdot) \right\|_{[a, b-kh]},$$

where

$$\Delta_h^k(g, \cdot) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} g\big(\cdot + ih\big) \quad \text{and} \quad \|g\|_{[c,d]} = \max_{x \in [c,d]} |g(x)|.$$

2 Main Results

Our first positive result is

Theorem 2.1 *Let $r = 2s - 2$. If $f \in C^r \cap \Delta^{(1)}(Y_s)$, then*

$$E_n^{(1)}(f, Y_s) \leq \frac{c(r)}{n^r} \omega_2 \left(f^{(r)}, \frac{1}{n} \right), \quad n \geq 1.$$

However, Theorem 2.1 cannot be extended to higher order moduli. Namely, we prove

Theorem 2.2 *Let $r = 2s - 2$. There is a constant $c(r) > 0$, such that for each $n \geq 1$, there exist a collection Y_s and a function $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that*

$$E_n^{(1)}(F, Y_s) > \sqrt{n}\frac{c(r)}{n^r}\omega_3\left(F^{(r)}, \frac{1}{n}\right).$$

Hence, for every $A > 0$ and $N > 0$, there exist $n > N$, Y_s and an $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that

$$E_n^{(1)}(F, Y_s) > \frac{A}{n^r}\omega_3\left(F^{(r)}, \frac{1}{n}\right).$$

We also prove

Theorem 2.3 *Let $r = 2s - 1$. If $f \in C^r \cap \Delta^{(1)}(Y_s)$, then*

$$E_n^{(1)}(f, Y_s) \leq \frac{c(r)}{n^r}\omega_3\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq 1. \tag{2.1}$$

Remark 2.4 Theorem 2.3 clearly implies (2.1) with ω_2 and ω_1 replacing ω_3. The case of ω_1 was proved in [7, Theorem 1.1], while that of ω_2 is new.

Again,Theorem 2.3 cannot be extended to higher order moduli. Namely, we prove

Theorem 2.5 *Let $r = 2s - 1$. There is a constant $c(r) > 0$, such that for each $n \geq 1$, there exist a collection Y_s and a function $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that*

$$E_n^{(1)}(F, Y_s) > \sqrt[3]{n}\frac{c(r)}{n^r}\omega_4\left(F^{(r)}, \frac{1}{n}\right).$$

Hence, for every $A > 0$ and $N > 0$, there exist $n > N$, Y_s and an $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that

$$E_n^{(1)}(F, Y_s) > \frac{A}{n^r}\omega_4\left(F^{(r)}, \frac{1}{n}\right).$$

Our last positive result is

Theorem 2.6 *Let $r = 2s$ and let $k \geq 3$. If $f \in C^r \cap \Delta^{(1)}(Y_s)$, then*

$$E_n^{(1)}(f, Y_s) \leq \frac{c(r,k)}{n^r}\omega_k\left(f^{(r)}, \frac{1}{n}\right), \quad n \geq 1. \tag{2.2}$$

Remark 2.7 Theorem 2.6, evidently implies (2.2) also for $k = 1, 2$, which were proved in [7, Theorem 1.1].

Finally, if $0 \leq r < 2s - 2$ and $k \geq 2$, then (1.2) cannot be had. Namely,

Theorem 2.8 *Let $0 \le r < 2s-2$. There is a constant $c(r) > 0$, such that for each $n \ge 1$, there exist a collection Y_s and a function $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that*

$$E_n^{(1)}(F, Y_s) > n\frac{c(r)}{n^r}\omega_2\left(F^{(r)}, \frac{1}{n}\right).$$

Hence, for every $A > 0$ and $N > 0$, there exist $n > N$, Y_s and an $F \in C^r \cap \Delta^{(1)}(Y_s)$, such that

$$E_n^{(1)}(F, Y_s) > \frac{A}{n^r}\omega_2\left(F^{(r)}, \frac{1}{n}\right).$$

We can now summarize the results. Given $s \ge 1$, the pairs (r,k), for which (1.2) holds, are $k = 1$ and $r \ge 0$, $k = 2$ and $r \ge 2s-2$, $k = 3$ and $r \ge 2s-1$, and $k \ge 4$ and $r \ge 2s$. For all other pairs (1.2) is invalid. In the tables below we denote by "+" the pairs (r,k) for which (1.2) is valid. We denote by "$\oplus$" the pairs (r,k) for which (1.2) is invalid but (1.3) is valid, and finally, "$-$" denotes that even (1.3) is invalid.

r	⋮	⋮	⋮	⋮	⋮	⋰
$2s+1$	+	+	+	+	+	⋯
$2s$	+	+	+	+	+	⋯
$2s-1$	+	+	+	$\oplus$	$\oplus$	⋯
$2s-2$	+	+	$\oplus$	$\oplus$	$\oplus$	⋯
$2s-3$	+	$\oplus$	$\oplus$	$\oplus$	$\oplus$	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	+	$\oplus$	$\oplus$	$\oplus$	$\oplus$	⋯
1	+	$\oplus$	$\oplus$	$-$	$-$	⋯
0	+	$\oplus$	$-$	$-$	$-$	⋯
	1	2	3	4	5	k

$s > 2$

r	⋮	⋮	⋮	⋮	⋮	⋰
5	+	+	+	+	+	⋯
4	+	+	+	+	+	⋯
3	+	+	+	$\oplus$	$\oplus$	⋯
2	+	+	$\oplus$	$\oplus$	$\oplus$	⋯
1	+	$\oplus$	$\oplus$	$-$	$-$	⋯
0	+	$\oplus$	$-$	$-$	$-$	⋯
	1	2	3	4	5	k

$s = 2$

r	⋮	⋮	⋮	⋮	⋮	⋰
3	+	+	+	+	+	⋯
2	+	+	+	+	+	⋯
1	+	+	+	$-$	$-$	⋯
0	+	+	$-$	$-$	$-$	⋯
	1	2	3	4	5	k

$s = 1$

We emphasize that these tables differ substantially from the corresponding tables for comonotone approximation of continuous functions on a finite interval, by algebraic polynomials (see, e.g, [5]).

3 Divided Differences

Let $m \in \mathbb{N}$ and let the function g be defined at the distinct points $t_i \in \mathbb{R}$, $0 \le i \le m$. Denote $[t_0; g] := g(t_0)$ and by

$$[t_0, t_1, \dots, t_m; g] := \sum_{i=0}^{m} \frac{g(t_i)}{\prod_{j=0,\, j\neq i}^{m} (t_i - t_j)},$$

the divided difference of order m of g at the knots t_i.

Lemma 3.1 *Let $t_0 < t_1 < \cdots < t_m$. Assume that either*

$$(-1)^{m-i}\big(g(t_i) - g(t_{i-1})\big) \ge 0, \quad 1 \le i \le m, \tag{3.1}$$

or

$$(-1)^{m-i}\big(g(t_i) - g(t_{i-1})\big) \le 0, \quad 1 \le i \le m. \tag{3.2}$$

Then, respectively,

$$[t_0, \dots, t_m; g] \ge 0, \quad \text{or} \quad [t_0, \dots, t_m; g] \le 0, \tag{3.3}$$

and

$$(t_m - t_0)^m \big|[t_0, t_1, \dots, t_m; g]\big| \ge \max_{0\le i\le m} g(t_i) - \min_{0\le i\le m} g(t_i), \tag{3.4}$$

Therefore, for $m \ge 2$,

$$\big|[t_0, \dots, t_m; g]\big| = \frac{\big|[t_1, \dots, t_m; g]\big| + \big|[t_0, \dots, t_{m-1}; g]\big|}{t_m - t_0}, \tag{3.5}$$

and if $2 \le r \le m$, then

$$\big|[t_0, \dots, t_m; g]\big| \prod_{i=r}^{m} (t_i - t_0) \ge \big|[t_1, \dots, t_r; g]\big|. \tag{3.6}$$

Proof Since all statements depend only on $g(t_i)$, $0 \le i \le m$, and there is an algebraic polynomial of degree $\le m$, interpolating g at t_i, $0 \le i \le m$, without loss of generality we assume that g is that polynomial. Then (3.1), respectively, (3.2) implies that $g'(x) = a \prod_{i=1}^{m-1}(x - x_i)$, where $a \ge 0$, respectively, $a \le 0$, and $x_i \in (t_0, t_m)$, $1 \le i \le m-1$. Note that $g^{(m)}(x) \equiv (m-1)!a$, so that $[t_0, \dots, t_m; g] = \frac{a}{m}$. Thus, (3.3) holds.

By the inequality between the arithmetic and geometric means, we have for $m > 1$,

$$\begin{aligned}\int_{t_0}^{t_m} |g'(x)|\, dx &= |a| \int_{t_0}^{t_m} \prod_{i=1}^{m-1} |x - x_i|\, dx \le \frac{|a|}{m-1} \sum_{i=1}^{m-1} \int_{t_0}^{t_m} |x - x_i|^{m-1}\, dx \\ &= \frac{|a|}{m(m-1)} \sum_{i=1}^{m-1} \big((t_m - x_i)^m + (x_i - t_0)^m\big) \le \frac{|a|}{m}(t_m - t_0)^m.\end{aligned}$$

This inequality is trivial (in fact, an equality) for $m = 1$.

Hence,

$$\max_{0\le i\le m} g(t_i) - \min_{0\le i\le m} g(t_i) \le \int_{t_0}^{t_m} |g'(x)|\,dx \le \frac{|a|}{m}(t_m - t_0)^m$$
$$= \big|[t_0, t_1, \ldots, t_m; g]\big|(t_m - t_0)^m,$$

thus, (3.4) is proved.

Finally, (3.3) readily implies (3.5), and (3.6) follows from (3.5) by induction on $m - r$. □

Lemma 3.2 *Let $0 \le l < m$. If $f \in C^l[a, b]$ and $x_i \in [a, b]$, $0 \le i \le m$, are distinct, then*

$$|[x_0, \ldots, x_m; f]| \le \frac{c}{l!}\omega_{m-l}(f^{(l)}, b - a; [a, b]) \sum_{j=l}^{m} \frac{1}{\prod_{i=l, i\ne j}^{m} |x_j - x_i|}, \quad (3.7)$$

where c is an absolute constant.

Proof For $l \ge 1$, let $U_l := \{u_i\}_{i=1}^{l}$ and denote

$$f_l(x; U_l) := f^{(l)}\big(x_0 + u_1(x_1 - x_0) + \cdots + u_l(x - x_{l-1})\big).$$

By virtue of [1, Chapter 4 (7.12)] we have,

$$[x_0, \ldots, x_l; f] = \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{l-1}} f_l(x_l; U_l)\,du_l \cdots du_1$$
$$= \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{l-1}} [x_l; f_l(\cdot; U_l)]\,du_l \cdots du_1.$$

Assuming by induction that for some $m > l$,

$$[x_0, \ldots, x_{m-1}; f] = \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{l-1}} [x_l, \ldots, x_{m-1}; f_l(\cdot; U_l)]\,du_l \cdots du_1, \quad (3.8)$$

we will prove (3.8) for $m + 1$.

Indeed,

$$[x_0, \ldots, x_m; f] = \frac{[x_0, \ldots, x_{m-1}; f] - [x_0, \ldots, x_{m-2}, x_m; f]}{x_{m-1} - x_m}$$
$$= \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{l-1}} \frac{[x_l, \ldots, x_{m-1}; f_l(\cdot; U_l)] - [x_l, \ldots, x_{m-2}, x_m; f_l(\cdot; U_l)]}{x_{m-1} - x_m}\,du_l \cdots du_1$$
$$= \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{l-1}} [x_l, \ldots, x_m; f_l(\cdot; U_l)]\,du_l \cdots du_1.$$

Hence,

$$\left|[x_0, \dots, x_m; f]\right| \le \frac{\|f^{(l)}\|_{[a,b]}}{l!} \sum_{j=l}^{m} \frac{1}{\prod_{i=l, i\neq j}^{m} |x_j - x_i|}. \tag{3.9}$$

Evidently, (3.9) holds also for $l = 0$.

Finally, by Whitney's inequality there is an algebraic polynomial P of degree $< m - l$, such that $\|f^{(l)} - P\|_{[a,b]} \le c\omega_{m-l}(f^{(l)}, b - a; [a, b])$. Since for any polynomial Q such that $Q^{(l)} = P$, $[x_0, \dots, x_m; f] = [x_0, \dots, x_m; f - Q]$, we conclude that (3.9) yields (3.7). □

4 Auxiliary Lemmas

In the sequel the various positive constants c and c_i may depend only on $r \ge 0$, $s \ge 1$ and $k \ge 2$. Also, we assume that $f \neq \text{const}$.

Lemma 4.1 *Let $h > 0$ and $y_{2s} - y_1 \le (6s - 2)h$. If $f \in C^{2s-1} \cap \Delta^{(1)}(Y_s)$, then*

$$\|f - f(y_{2s})\| \le ch^{2s-1}\omega_{k+1}(f^{(2s-1)}, h). \tag{4.1}$$

Hence, if $f \in C^{2s} \cap \Delta^{(1)}(Y_s)$, then

$$\|f - f(y_{2s})\| \le ch^{2s}\omega_k(f^{(2s)}, h).$$

Proof Let $g(t) := f(t) - f(y_{2s})$ and note that $f \in \Delta^{(1)}(Y_s)$ implies $\|g\| = \|g\|_{[y_1, y_{2s}]}$. Set $H := 2^k 6sh$, and we may assume that $kH < 2\pi$, for otherwise, by Whitney's inequality, the periodicity of f yields (4.1). Indeed, in view of the periodicity, $f(y_{2ms}) = f(y_{2s})$, $m \in \mathbb{Z}$, hence the constant polynomial $P_0(x) \equiv f(y_{2s})$ interpolates f as many times as one wishes provided one takes a big enough interval. Specifically, it interpolates f at least $(2s + k)$ times in the interval $[0, (2s + k)2\pi]$. Whitney's inequality, thus implies, for $kH \ge 2\pi$,

$$\begin{aligned}\|f - f(y_{2s})\| &= \|f - f(y_{2s})\|_{[0,(2s+k)2\pi]} \le c\omega_{2s+k}(f, (2s + k)2\pi)\\ &\le c\omega_{2s+k}(f, (2s + k)kH) \le cH^{2s-1}\omega_{k+1}(f^{(2s-1)}, H)\\ &\le ch^{2s-1}\omega_{k+1}(f^{(2s-1)}, h),\end{aligned}$$

which yields (4.1).

Therefore, we assume that $kH < 2\pi$, we let

$$t_0 := y_1 - h, \quad t_i := y_i, \quad 1 \le i \le 2s, \quad \text{and} \quad t_{2s+i} = t_0 + iH, \quad 1 \le i \le k,$$

and we set $x_{2s} := t_0$, $x_i := t_{i+1}$, $0 \le i \le 2s - 1$, and $x_{2s+i} = t_{2s+i}$, $1 \le i \le k$. Then $|x_j - x_i| \ge h, i \neq j, 2s - 1 \le i, j \le 2s + k$, so that by Lemma 3.2 with

$l = 2s - 1$ and $m = 2s + k$, we have

$$h^{k+1}\big|[t_0, \dots, t_{2s+k}; g]\big| \le c\omega_{k+1}(f^{(2s-1)}, h). \tag{4.2}$$

Set

$$\tilde{g}(t) := \begin{cases} g(t), & t \in [t_0, t_{2s}], \\ 0, & t \in [t_{2s}, t_0 + 2\pi), \end{cases}$$

and note that $\tilde{g}$ satisfies either (3.1) or (3.2), with $m = 2s + k$. By (3.4) and (3.6)

$$\begin{aligned}(t_{2s} - t_1)^{2s-1}\big|[t_0, \dots, t_{2s+k}; \tilde{g}]\big| \prod_{i=2s}^{2s+k}(t_i - t_0) &\ge (t_{2s} - t_1)^{2s-1}\big|[t_1, \dots, t_{2s}; \tilde{g}]\big| \\ &\ge \max_{1\le i\le 2s} \tilde{g}(t_i) - \min_{1\le i\le 2s} \tilde{g}(t_i) \\ &\ge \|\tilde{g}\|_{[t_1,t_{2s}]} = \|g\|_{[t_1,t_{2s}]} = \|g\|,\end{aligned}$$

where we used the fact that $\tilde{g}(t_{2s}) = 0$. Hence,

$$\begin{aligned}\big|[t_0, \dots, t_{2s+k}; \tilde{g}]\big| &\ge \frac{\|g\|}{(t_{2s} - t_1)^{2s-1}(t_{2s} - t_0)\prod_{j=2s+1}^{2s+k}(t_i - t_0)} \\ &= \frac{\|g\|}{(t_{2s} - t_1)^{2s-1}(t_{2s} - t_0)k!H^k}.\end{aligned} \tag{4.3}$$

On the other hand,

$$\big|[t_0, \dots, t_{2s+k}; \tilde{g} - g]\big| = \left|\sum_{i=1}^{k} \frac{g(t_{2s+i})}{\prod_{j=0, j\ne 2s+i}^{2s+k}(t_{2s+i} - t_j)}\right|,$$

and we have

$$\prod_{j=0, j\ne 2s+i}^{2s+k} |t_{2s+i} - t_j| = H^k i!(k-i)! \prod_{j=1}^{2s}(t_{2s+i} - t_j) \ge H^k i!(k-i)!(t_{2s+1} - t_{2s})^{2s}.$$

Therefore,

$$\big|[t_0, \dots, t_{2s+k}; \tilde{g} - g]\big| \le \frac{\|g\|H^{-k}}{(t_{2s+1} - t_{2s})^{2s}k!}\sum_{i=1}^{k}\binom{k}{i} = \frac{\|g\|H^{-k}}{(t_{2s+1} - t_{2s})^{2s}k!}(2^k - 1).$$

Thus, combining with (4.3), we obtain

$$\frac{\left|[t_0,\dots,t_{2s+k};\tilde{g}-g]\right|}{\left|[t_0,\dots,t_{2s+k};\tilde{g}]\right|} \le \frac{(t_{2s}-t_1)^{2s-1}(t_{2s}-t_0)}{(t_{2s+1}-t_{2s})^{2s}}(2^k-1)$$
$$\le \frac{(2^k-1)(6sh)^{2s}}{(H-6sh))^{2s}} = \frac{1}{(2^k-1)^{2s-1}} < \frac{1}{2}.$$

Finally, by virtue of (4.3),

$$\left|[t_0,\dots,t_{2s+k};g]\right| \ge \left|[t_0,\dots,t_{2s+k};\tilde{g}]\right| - \left|[t_0,\dots,t_{2s+k};\tilde{g}-g]\right| \tag{4.4}$$
$$\ge \frac{1}{2}\left|[t_0,\dots,t_{2s+k};\tilde{g}]\right| \ge \frac{c\|g\|}{h^{2s+k}} = \frac{c\|f-f(t_{2s})\|}{h^{2s+k}}.$$

Now, we combine (4.4) with (4.2) to obtain (4.1). □

Lemma 4.2 *Let $h>0$ and $y_{2s}-y_1 \le (6s-2)h$. If $f \in C^{2s-2}\cap \Delta^{(1)}(Y_s)$, then*

$$\|f-f(y_1)\| \le ch^{2s-2}\omega_2(f^{(2s-2)},h). \tag{4.5}$$

Proof Note that $f\in\Delta^{(1)}(Y_s)$ implies $\|f-f(y_1)\| = \|f-f(y_1)\|_{[y_1,y_{2s}]}$.

We may assume that $h \le y_1-y_0$, for otherwise, $h > y_1-y_0$, so that

$$2\pi = y_{2s}-y_0 = y_{2s}-y_1+y_1-y_0 < (6s-2)h+h,$$

whence $h > 2\pi/(6s-1)$.

Hence, Whitney's inequality and the periodicity of f yield (4.5), as in the proof of Lemma 4.1.

Thus, we assume that $h \le y_1-y_0$ and let $a := y_1-h$. Set $x_i := y_{i+2}$, $0\le i\le 2s-3$, $x_{2s-2}:=a$, $x_{2s-1}:=y_1$ and $x_{2s}:=y_{2s}$. By Lemma 3.2 with $l=2s-2$ and $m=2s$, we obtain

$$h(y_{2s}-y_1)\left|[a,y_1\dots,y_{2s};f]\right| \le c\omega_2(f^{(2s-2)},h).$$

On the other hand, Lemma 3.1 yields

$$\left|[a,y_1\dots,y_{2s};f]\right| = \frac{\left|[y_1\dots,y_{2s};f]\right|+\left|[a,y_1\dots,y_{2s-1};f]\right|}{y_{2s}-a} \ge \frac{\left|[y_1\dots,y_{2s};f]\right|}{(6s-1)h}$$
$$\ge \frac{\|f-f(y_1)\|_{[y_1,y_{2s}]}}{(y_{2s}-y_1)^{2s-1}(6s-1)h} = c\frac{\|f-f(y_1)\|}{(y_{2s}-y_1)^{2s-1}h}.$$

Hence,

$$\|f-f(y_1)\| \le c\frac{(y_{2s}-y_1)^{2s-1}h}{h(y_{2s}-y_1)}\omega_2(f^{(2s-2)},h) \le ch^{2s-2}\omega_2(f^{(2s-2)},h).$$

This completes the proof. □

Given $t_1 < \cdots < t_\nu$, $1 \le \nu \le r+1$, denote

$$\pi_\nu(x) = \prod_{i=1}^{\nu}(x - t_i).$$

Lemma 4.3 *Let $h > 0$, $3h \le b-a \le 3(r+1)h$ and $a+h \le t_1 < \cdots < t_{r+1} \le b-h$. If a function $f \in C^r[a.b]$, $r \ge 1$, satisfies*

$$f'(x)\pi_{r+1}(x) \ge 0, \quad x \in [a, b],$$

or if $f \in C[a, b]$ (that is, $r = 0$) changes monotonicity once, at t_1, then

$$\|f - f(a)\|_{[a,b]} \le ch^r \omega_2(f^{(r)}, h; [a, b]). \tag{4.6}$$

Proof By virtue of Lemma 3.2 with $l = r$ and $m = r+2$, setting

$$x_i := t_{i+1},\ 0 \le i \le r, \quad x_{r+1} := a \quad \text{and} \quad x_{r+2} := b,$$

we readily obtain,

$$h^2\big|[a, t_1, \ldots, t_{r+1}, b; f]\big| = h^2\big|[t_1, \ldots, t_{r+1}, a, b; f]\big| \le c\omega_2(f^{(r)}, h; [a, b]).$$

On the other hand Lemma 3.1 implies

$$(b-a)^{r+2}\big|[a, t_1, \ldots, t_{r+1}, b; f]\big| \ge \|f - f(a)\|_{[a,b]}.$$

Combining the last two inequalities completes the proof of (4.6). □

Lemma 4.4 *Let $r \ge 1$, $h > 0$, $3h \le b-a \le 3rh$, and $a+h \le t_1 < \cdots < t_r \le b-h$. If a function $f \in C^r[a, b]$ satisfies*

$$f'(x)\pi_r(x) \ge 0, \quad x \in [a, b],$$

then, there is an algebraic polynomial $p = p(\cdot, f; (a, b))$ of degree $\le r+2$, such that $f(a) = p(a)$,

$$\|f - p\|_{[a,b]} \le ch^r \omega_3(f^{(r)}, h; [a, b]), \tag{4.7}$$

and

$$p'(x)\pi_r(x) \ge 0, \quad x \in [a, b], \tag{4.8}$$

Proof Let $t_0 := a$, $t_{r+1} := b$, and denote by L the Lagrange polynomial of degree $\le r+1$ interpolating f' at t_i, $0 \le i \le r+1$. Set

$$x_i := t_{i+1},\ 0 \le i \le r-2, \quad x_{r-1} := x, \quad x_r := t_r, \quad x_{r+1} := t_0, \quad \text{and} \quad x_{r+2} := t_{r+1}.$$

We apply Lemma 3.2 with $l = r-1$, $m = r+2$, and get for $x \in (a,b)$, $x \ne t_i$, $1 \le i \le r$,

$$h^2 \big|[x, a, t_1 \dots, t_r, b; f']\big| \min\{|x-a|, |x-t_r|, |x-b|\} \le c\omega_3(f^{(r)}, h; [a,b]).$$

Hence,

$$|f'(x) - L(x)| = \big|[x, t_0, \dots, t_{r+1}; f']\big| \left|\prod_{i=0}^{r+1}(x-t_i)\right| \le ch^{r-1}\omega_3(f^{(r)}, h; [a,b]).$$

Let

$$p(x) := f(a) + \int_a^x L(t)\,dt.$$

Then (4.7) is evident, and since $L(t_i) = f'(t_i) = 0, \quad 1 \le i \le r$,

$$p'(x)\pi_r(x) = L(x)\pi_r(x) = \frac{f'(a)}{\pi(a)}\pi_r^2(x)\frac{x-b}{a-b} + \frac{f'(b)}{\pi(b)}\pi_r^2(x)\frac{x-a}{b-a} \ge 0, \quad x \in [a,b],$$

which proves (4.8). □

Lemma 4.5 *Let $r \ge 2$, $1 \le \nu \le r-1$, $h > 0$, $3h \le b-a \le 3\nu h$ and $a+h \le t_1 < \cdots < t_\nu \le b-h$.*

If a function $f \in C^r[a,b]$ satisfies

$$f'(x)\pi_\nu(x) \ge 0, \quad x \in [a,b], \tag{4.9}$$

then there is an algebraic polynomial $p = p(\cdot, f; (a,b))$ of degree $< r+k-1$, such that $p(a) = f(a)$,

$$\|f - p\|_{[a,b]} \le ch^r\omega_k(f^{(r)}, h; [a,b]), \tag{4.10}$$

and

$$p'(x)\pi_\nu(x) \ge 0, \quad x \in [a,b]. \tag{4.11}$$

Proof Denote $\omega := \omega_k(f^{(r)}, h, [a,b])$. Let $t_0 := a$ and $t_{\nu+i} := b-h+\frac{ih}{r+k-\nu-2}$, $1 \le i \le r+k-\nu-2$. We apply Lemma 3.2 with $l = r-1$, $m = r+k-1$, setting

$$x_0 := x, \quad x_m = a, \quad \text{and} \quad x_i = t_i, 1 \le i \le m-1,$$

and get

$$h^k \big|[x, t_0, \dots, t_{r+k-2}; f']\big| = h^k \big|[x, t_1, \dots, t_{r+k-2}, a; f']\big| \le c\omega.$$

Denote by L the Lagrange polynomial of degree $< r+k-2$, that interpolates f' at t_i, $0 \le i \le r+k-2$. Then, for $x \in [a,b]$, $x \ne t_i$, $0 \le i \le r+k-2$,

$$f'(x) - L(x) = [x, t_0, \dots, t_{r+k-2}; f'] \prod_{i=0}^{r+k-2} (x - t_i),$$

whence

$$|f'(x) - L(x)| \le c_0 h^{r-\nu-1} |\pi_\nu(x)| \omega, \quad x \in [a,b].$$

Now, the desired polynomial may be taken in the form

$$p(x) := f(a) + \int_a^x \Big(L(t) + c_0 h^{r-1-\nu} \omega \pi_\nu(t) \Big)\, dt.$$

Evidently, (4.10) is valid, and by (4.9)

$$\begin{aligned} p'(x)\pi_\nu(x) &= f'(x)\pi_\nu(x) + (L(x) - f'(x))\pi_\nu(x) + c_0 h^{r-1-\nu} \omega \pi_\nu^2(x) \\ &\ge -|(L(x) - f'(x))\pi_\nu(x)| + c_0 h^{r-\nu-1} \omega \pi_\nu^2(x) \\ &\ge -c_0 h^{r-\nu-1} \omega \pi_\nu^2(x) + c_0 h^{r-\nu-1} \omega \pi_\nu^2(x) = 0, \end{aligned}$$

which proves (4.11). □

Finally, we need the next result.

Lemma 4.6 *Let $r \ge 1$, and assume that $f \in C^r[a,b]$ is such that $f'(x) \ge 0$, $x \in [a,b]$. Then, there exists a polynomial $P = P(\cdot, f; (a,b))$ of degree $< r+k$ such that*

$$\|f - P\| \le c(b-a)^r \omega_k(f^{(r)}, b-a; [a,b]), \tag{4.12}$$

$$P(a) = f(a), \quad P(b) = f(b), \tag{4.13}$$

and

$$P'(x) \ge 0, \quad x \in [a,b]. \tag{4.14}$$

Proof Set $\omega := \omega_k(f^{(r)}, b-a; [a,b])$ and let $h := b-a$. Whitney's inequality implies that the best algebraic polynomial p of degree $< r+k-1$, approximating f', satisfies

$$\|f' - p\|_{[a,b]} \le c_1 h^{r-1} \omega.$$

Hence, the polynomial

$$\tilde{P}(x) := f(a) + \int_a^x (p(t) + c_1 h^{r-1}\omega)\, dt,$$

of degree $< r + k$, satisfies (4.12), (4.14) and $\tilde{P}(a) = f(a)$.

Now, let

$$P(x) := f(a) + (f(b) - f(a)) \frac{\tilde{P}(x) - \tilde{P}(a)}{\tilde{P}(b) - \tilde{P}(a)}.$$

Evidently, (4.13) and (4.14) are satisfied, and since

$$P(x) - f(x) \equiv \left(\tilde{P}(x) - f(x)\right) + \left(f(b) - \tilde{P}(b)\right) \frac{\tilde{P}(x) - \tilde{P}(a)}{\tilde{P}(b) - \tilde{P}(a)},$$

we get

$$\|P - f\|_{[a,b]} \le \|\tilde{P} - f\|_{[a,b]} + |f(b) - \tilde{P}(b)| \le 2\|\tilde{P} - f\|_{[a,b]} \le ch^r\omega,$$

which is (4.12). □

5 Proofs of the Positive Results

Since f is 2π-periodic, Whitney's inequality implies for any $k \ge 1$,

$$\|f - f(0)\| \le c(k)\omega_k(f, 2\pi),$$

that yields, for any N and all $n \le N$, and $f \in \Delta^{(1)}(Y_s)$,

$$E_n^{(1)}(f, Y_s) \le E_1^{(1)}(f, Y_s) \le \|f - f(0)\| \le c(N, k)\omega_k(f, 1/n). \tag{5.1}$$

So, in the sequel, we assume that $n > 6s$.

Let $x_j := \frac{j\pi}{n}$, $I_j := [x_j, x_{j+1}]$, $j \in \mathbb{Z}$, $h := \frac{\pi}{n}$, and for each $i \in \mathbb{Z}$, let j_i be the index, such that $x_{j_i} \le y_i < x_{j_i+1}$. Denote by O the interior of the union

$$\cup_{i\in\mathbb{Z}}[x_{j_i-1}, x_{j_i+2}],$$

and let $O_\mu = (x_{\mu^-}, x_{\mu^+})$, $\mu \in \mathbb{Z}$, be the connected components of O, enumerated from right to left. Note that $x_{\mu^+} - x_{\mu^-} \le 6sh$ for all $\mu \in \mathbb{Z}$.

Denote $I := [-\pi, \pi]$. Since f is 2π-periodic and $n > 6s$, we may without loss of generality, assume that $\pi \notin O$, so that there is no O_μ such that $\pm\pi \in O_\mu$. We may also assume, by shifting the indices if needed, that $y_i \in I$, $1 \le i \le 2s$.

There are two possibilities for the location of the extremum points. Either there is only one component of O in the period I, or there are at least two. In the former

case, each O_μ contains $2s$ points y_i, so that Lemma 4.2 and Lemma 4.1 imply that the polynomial $T_n(x) \equiv f(0)$ provides the validity of the estimates for all appropriate n's, in Theorem 2.1 and in Theorems 2.3 and 2.6, respectively.

Thus, we are left with the case where there are at least two components of O in I.

Under our assumption that $\pi \notin O$, denote by $\Sigma_{m,n}(Y_s)$ the set of continuous piecewise polynomials S, on I, such that

$$S\big|_{I_j} = P_j, \quad \text{if} \quad I_j \subset I \setminus O,$$

and

$$S\big|_{O_\mu} = p_\mu, \quad O_\mu \subset I,$$

where P_j and p_μ are algebraic polynomials of degree $< m$.

In the sequel, by writing $S'(x)$, we implicitly assume that $x \neq x_j$, $x_j \notin O$.

We need the following result [7, Corollary 4.4].

Lemma 5.1 *Let $s \geq 1$ and $m \geq 2$. There is a constant $\tilde{c} = c(s,m)$, such that for $n > \tilde{c}$, if $f \in C \cap \Delta^{(1)}(Y_s)$, $S \in \Sigma_{m,n}(Y_s)$ and $S'(x)\Pi(x) \geq 0$, $x \in I$, where $\Pi(x)$ was defined in (1.1), then*

$$E^{(1)}_{\tilde{c}n}(f, Y_s) \leq \tilde{c}(\omega_m(f, 1/n) + \|f - S\|_I).$$

Remark 5.2 In Lemma 5.1, $S \in \Sigma_{m,n}(Y_s)$ is defined on I, while in [7, Corollary 4.4], $S \in \overline{\Sigma}_{m,n}(Y_s)$ is defined on $\mathbb{R}$. But, there is no difficulty in extending $S \in \Sigma_{m,n}(Y_s)$ to an element $S \in \overline{\Sigma}_{m,n}(Y_s)$.

We point out that in view of (5.1), we will only have to prove the results for $n > (\tilde{c}(s,m) + 6s)^2 =: c_2$. To this end, for all $n > 6s$, we construct an appropriate S as follows.

Recall that we assume that there are at least two components in a period, so that in each component O_μ there are $1 \leq \nu \leq 2s - 1$ extremal points.

In each interval $I_j \subset I \setminus O$, either f or $-f$ is monotone nondecreasing, so for $r \geq 1$, we apply Lemma 4.6 to obtain, for any appropriate $k \geq 2$, a polynomial $\tilde{P}_j$, of degree $< r + k$, of the same monotonicity, interpolating f at both endpoints of the interval, and such that,

$$\|f - \tilde{P}_j\|_{I_j} \leq \frac{c}{n^r}\omega_k(f^{(r)}, 1/n; I_j). \tag{5.2}$$

For $r = 0$ and $k = 2$, (5.2) follows by taking $\tilde{P}_j$ to be the linear, interpolating f at both endpoints of I_j.

In the proof below we apply the various lemmas of Section 4, where we always take a generic component $O_\mu \subset I$ and $y_{l+i} \in O_\mu$, $1 \leq i \leq \nu < 2s$, and we set $t_i = y_{l+i}$, $1 \leq i \leq \nu$. Note that $t_1 = y_{l+1}$ may be either a maximum of f or of $-f$, and the lemmas are applied accordingly.

When $r = 2s$ (Theorem 2.6), we may have components $O_\mu \subset I$ with ν extremum points, for various $1 \le \nu \le r-1$. Therefore, we apply Lemma 4.5 and obtain, for any $k \ge 3$, a polynomial $\tilde{p}_\mu$ of degree $< r+k$, comonotone with f in that component, such that $\tilde{p}(x_{\mu^-}) = f(x_{\mu^-})$, and

$$\|f - \tilde{p}_\mu\|_{\overline{O}_\mu} \le \frac{c}{n^r}\omega_k(f^{(r)}, 1/n). \tag{5.3}$$

When $r = 2s-1$ (Theorem 2.3), we may have components $O_\mu \subset I$, with ν extremum points, for various $1 \le \nu \le r$. Therefore, we have to apply Lemma 4.4 for $\nu = r$ and Lemma 4.5 for $1 \le \nu \le r-1$, as the case maybe, and thus obtain a polynomial $\tilde{p}_\mu$ of degree $\le r+2$, comonotone with f in that component, such that $\tilde{p}(x_{\mu^-}) = f(x_{\mu^-})$, and

$$\|f - \tilde{p}_\mu\|_{\overline{O}_\mu} \le \frac{c}{n^r}\omega_3(f^{(r)}, 1/n). \tag{5.4}$$

Finally, when $r = 2s-2$ (Theorem 2.1), we may have components $O_\mu \subset I$ with ν extremum points, for various $1 \le \nu \le r+1$. If there is no component with $\nu = r$ extreme points, then we apply Lemma 4.3 for $\nu = r+1$ and Lemma 4.5 with $k = 2$ for $1 \le \nu \le r-1$, as the case maybe, and obtain a polynomial $\tilde{p}_\mu$ of degree $< r+2$, comonotone with f in that component, such that $\tilde{p}(x_{\mu^-}) = f(x_{\mu^-})$, and

$$\|f - \tilde{p}_\mu\|_{\overline{O}_\mu} \le \frac{c}{n^r}\omega_2(f^{(r)}, 1/n). \tag{5.5}$$

If it so happens that there is a component with $\nu = r$ extreme points, then we apply Lemma 4.4 for that component and Lemma 4.5 with $k = 3$ for any other, and thus obtain a polynomial $\tilde{p}_\mu$ of degree $< r+3$, comonotone with f in that component, such that $\tilde{p}(x_{\mu^-}) = f(x_{\mu^-})$, and

$$\|f - \tilde{p}_\mu\|_{\overline{O}_\mu} \le \frac{c}{n^r}\omega_3(f^{(r)}, 1/n). \tag{5.6}$$

Hence, in all cases, the piecewise polynomial

$$\tilde{S}(x) := \begin{cases} \tilde{p}_\mu(x), & x \in O_\mu \subset I, \\ \tilde{P}_j(x), & x \in I_j : I_j \subset I \setminus O, \end{cases}$$

of the respective degrees, is continuous in each interval $(x_{\mu^+}, x_{(\mu-1)^-})$, $\mu : O_\mu \subset I$, and by (5.2) through (5.6), it satisfies, for the respective r and k,

$$\tilde{S}'(x)\Pi(x) \ge 0, \quad x \in I, \text{ and } \sup_{x\in I}|f(x) - \tilde{S}(x)| \le \frac{c_3}{n^r}\omega_k(f^{(r)}, 1/n).$$

The continuous piecewise polynomial,

$$S(x) := f(-\pi) + \int_{-\pi}^{x} \tilde{S}'(t)\,dt, \quad x \in I,$$

is in $\Sigma_{m,n}(Y_s)$, where $m = r + k$ for Theorem 2.6, $m = r + 3$ for Theorem 2.3, and $m = r + 2$ or $m = r + 3$, as the case may be, for Theorem 2.1. Further, it satisfies $S'(x)\Pi(x) \geq 0$, $x \in I$, and since there are at most $2s$ components O_μ, that is, at most $2s$ discontinuities of $\tilde{S}$ in I, we conclude that for the respective r and k,

$$\|f - S\|_I \leq 2s\frac{c_3}{n^r}\omega_k(f^{(r)}, 1/n).$$

Hence, by Lemma 5.1, we obtain

$$E_{n_1}^{(1)}(f, Y_s) \leq \frac{c}{n_1^r}\omega_k(f^{(r)}, 1/n_1), \quad n_1 > c_2.$$

This completes the proof, when we note, regarding Theorem 2.1, that $\omega_3 \leq 2\omega_2$. □

6 Counterexamples

In the proofs below, G is an even function, infinitely differentiable in $\mathbb{R}$, nondecreasing in $(0, \infty)$ and

$$G(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \leq 1. \end{cases}$$

In addition, we assume that $n > c_*$ for some $c_* > 0$.

We observe that for $1 \leq n \leq c_*$, $E_n^{(1)}(f, Y_s) \geq E_{[c_*+1]}^{(1)}(f, Y_s)$, and we apply the case $n = [c_* + 1]$, to obtain the results for $1 \leq n \leq c_*$.

We begin with the proof of the negative result involving all r, $0 \leq r < 2s - 2$, and then we deal with the negative results for the additional, specific r's.

Proof of Theorem 2.8 Let $0 \leq r < 2s - 2$ and set

$$\tau(x) := \begin{cases} 2^{r+1}\sin^{r+1}\frac{x}{2}, & \text{if } r \text{ is odd}, \\ -\sin^{r+1}x, & \text{if } r \text{ is even}. \end{cases}$$

Note that $|\tau^{(r+1)}(0)| = (r + 1)!$. Hence, there is a $0 < c_4 < 1$, such that

$$|\tau^{(r+1)}(x)| \geq \frac{1}{2}, \quad |x| \leq c_4. \tag{6.1}$$

Let $0 < b \leq c_4$, and note that $\|\tau^{(r)}\|_{[-2b,2b]} \leq cb$.

Thus, applying Taylor's expansion, we obtain

$$\|\tau^{(j)}\|_{[-2b,2b]} \leq \frac{(2b)^{r-j}}{(r-j)!}\|\tau^{(r)}\|_{[-2b,2b]} \leq cb^{r+1-j}, \quad 0 \leq j \leq r. \tag{6.2}$$

Denote by F the 2π-periodic function, such that

$$F(x) := G\left(\frac{x}{b}\right)\tau(x), \quad x \in [-\pi, \pi],$$

and note that

$$\|F - \tau\| \le (2b)^{r+1} \quad \text{and} \quad \|F^{(r)} - \tau^{(r)}\| \le cb. \tag{6.3}$$

Indeed, by (6.2),

$$\begin{aligned}\|F^{(r)} - \tau^{(r)}\| &= \|F^{(r)} - \tau^{(r)}\|_{[-2b,2b]} \le \|\tau^{(r)}\|_{[-2b,2b]} + \|F^{(r)}\|_{[-2b,2b]} \\ &\le cb + 2^r \max_{0\le j<r} \frac{1}{b^{r-j}} \|\tau^{(j)}\|_{[-2b,2b]} \|G^{(r-j)}\| < cb.\end{aligned}$$

Hence,

$$\begin{aligned}\omega_2(F^{(r)}, t) &\le \omega_2(\tau^{(r)}, t) + \omega_2(F^{(r)} - \tau^{(r)}, t) \\ &\le t^2\|\tau^{(r+2)}\| + 2^2\|F^{(r)} - \tau^{(r)}\| \le c_5 t^2 + c_6 b.\end{aligned} \tag{6.4}$$

Define the collection Y_s as follows.

If r is even, then we let $y_1 = \frac{-\pi}{2}$, $y_{2s} = \frac{\pi}{2}$ and $y_i \in [-b, b]$, $2 \le i \le 2s-1$, and we observe that $F(y_1) = 1$, is the single maximum in $[-\pi, -b]$, and $F(y_{2s}) = -1$, is the single minimum in $[b, \pi]$.

If r is odd, then we let $y_1 = -\pi$ and $y_i \in [-b, b]$, $2 \le i \le 2s$, and we observe that $F(y_1) = 2^{r+1}$, the single maximum in $[-\pi, \pi)$, outside $[-b, b]$.

Since $F(x) = 0$, $x \in [-b, b]$, in both cases we may regard F as $F \in \Delta^{(1)}(Y_s)$.

In both cases there are at least $r+1$ points $y_i \in [-b, b]$, and for every trigonometric polynomial $T_n \in \Delta^{(1)}(Y_s)$, of degree $< n$, we have $T_n'(y_i) = 0$. It follows that there is a point $\theta \in (-b, b)$, such that $T_n^{(r+1)}(\theta) = 0$. Hence, if $n > r+1$, then (6.1) and (6.3), imply

$$\begin{aligned}\frac{1}{2} &\le \left|\tau^{(r+1)}(\theta)\right| = \left|\tau^{(r+1)}(\theta) - T_n^{(r+1)}(\theta)\right| \\ &\le n^{r+1}\|\tau - T_n\| \le n^{r+1}\|F - T_n\| + n^{r+1}\|\tau - F\| \\ &\le n^{r+1}\|F - T_n\| + (2b)^{r+1}n^{r+1} < n^{r+1}\|F - T_n\| + \frac{1}{4},\end{aligned}$$

provided $b < 1/(8n)$, and we actually take $b = n^{-2}$ for $n > c_* := \max\{c_4^{-1/2}, r+1, 8\}$. Then, for $n > c_*$, we obtain by (6.4),

$$n^r\|F - T_n\| > \frac{1}{4n} = \frac{n}{4(c_5 + c_6)}\left(\frac{c_5}{n^2} + c_6 b\right) \ge cn\omega_2\left(F^{(r)}, \frac{1}{n}\right).$$

This completes the proof. □

Proof of Theorem 2.5 Let $r = 2s - 1$ and $0 < b < \pi/2$, and denote

$$\tau(x) := (\cos b - \cos x) \sin^r x.$$

Note that

$$\begin{aligned} &\omega_4(\tau^{(r-1)}, t) \le t^4 \|\tau^{(r+3)}\| \le c_7 t^4, \\ &\tau^{(j)}(0) = 0, \quad 0 \le j < r \quad \text{and} \quad \tau^{(r)}(0) = r!(\cos b - 1). \end{aligned} \tag{6.5}$$

Since $1 - \cos b > b^2/4$, there is a $0 < \delta < b$, such that

$$-\tau^{(r)}(x) > \frac{b^2}{4}, \quad x \in [-\delta, \delta]. \tag{6.6}$$

In addition,

$$\tau^{(r)}(x) = \sin^2 x\, \alpha(x) + (\sin^r x)^{(r)}(\cos b - \cos x),$$

where α is a trigonometric polynomial such that $\|\alpha\| < c$, and $\|(\sin^r \cdot)^{(r)}\| < c$.

Thus, we conclude that $\|\tau^{(r)}\|_{[-2b,2b]} \le cb^2$.

Again, by Taylor's expansion, we obtain

$$\|\tau^{(j)}\|_{[-2b,2b]} \le \frac{(2b)^{r-j}}{(r-j)!} \|\tau^{(r)}\|_{[-2b,2b]} \le cb^{r+2-j}, \quad 0 \le j \le r. \tag{6.7}$$

Denote by f the 2π-periodic function, such that

$$f(x) := G\left(\frac{x}{b}\right) \tau(x), \quad x \in [-\pi, \pi],$$

and note that

$$\|f - \tau\| = \|f - \tau\|_{[-2b,2b]} \le \|\tau\|_{[-2b,2b]} \le (2b)^{r+2}. \tag{6.8}$$

Also, by (6.7),

$$\begin{aligned} \|f^{(r-1)} - \tau^{(r-1)}\| &= \|f^{(r-1)} - \tau^{(r-1)}\|_{[-2b,2b]} \le \|\tau^{(r-1)}\|_{[-2b,2b]} + \|f^{(r-1)}\|_{[-2b,2b]} \\ &\le cb^3 + 2^r \max_{0 \le j < r} \frac{1}{b^{r-1-j}} \|\tau^{(j)}\|_{[-2b,2b]} \|G^{(r-1-j)}\| < cb^3. \end{aligned} \tag{6.9}$$

Define the collection Y_s so that $y_1 = -\pi$ and, for δ of (6.6), $y_i \in (-\delta, \delta)$, $2 \le i \le 2s$.

Recalling that r is odd, we conclude that

$$F(x) := \int_0^x f(t)\, dt,$$

is 2π-periodic and is in $\Delta^{(1)}(Y_s)$.

By (6.8) and (6.9),

$$\|F - \mathcal{T}\| \le (2b)^{r+3} \quad \text{and} \quad \|F^{(r)} - \mathcal{T}^{(r)}\| \le cb^3,$$

where

$$\mathcal{T}(x) := \int_0^x \tau(t)\,dt.$$

Hence,

$$\begin{aligned}\omega_4(F^{(r)}, t) &\le \omega_4(\mathcal{T}^{(r)}, t) + \omega_4(F^{(r)} - \mathcal{T}^{(r)}, t)\\ &\le t^4\|\mathcal{T}^{(r+4)}\| + 2^4\|F^{(r)} - \mathcal{T}^{(r)}\|\\ &\le c_7 t^4 + c_8 b^3. \end{aligned} \tag{6.10}$$

Let $T_n \in \Delta^{(1)}(Y_s)$. Then $T_n'(\delta) \ge 0$ and $T_n'(y_i) = 0$, $2 \le i \le 2s$.

Thus, there is a $\theta \in (-\delta, \delta)$, such that

$$T_n^{(r+1)}(\theta) = r![y_2, \dots, y_{2s}, \delta; T_n'] \ge 0.$$

Therefore, by (6.6), for $n > r + 1$, we have

$$\begin{aligned}\frac{b^2}{4} &\le |\tau^{(r)}(\theta)| \le |\tau^{(r)}(\theta) - T_n^{(r+1)}(\theta)| = |\mathcal{T}^{(r+1)}(\theta) - T_n^{(r+1)}(\theta)|\\ &\le n^{r+1}\|\mathcal{T} - T_n\| \le n^{r+1}\|\mathcal{T} - F\| + n^{r+1}\|F - T_n\|\\ &\le n^{r+1}(2b)^{r+3} + n^{r+1}\|F - T_n\| \le \frac{b^2}{8} + n^{r+1}\|F - T_n\|,\end{aligned}$$

provided $b < \frac{1}{2^4 n}$, and we actually take $b = n^{-4/3}$ for $n > c_* := \max\{r + 1, 2^{12}\}$. Then for $n > c_*$, we obtain by (6.10),

$$\|F - T_n\| \ge \frac{\sqrt[3]{n}}{8n^{r+4}} = \frac{\sqrt[3]{n}}{8(c_7 + c_8)n^r}\left(\frac{c_7}{n^4} + c_8 b^3\right) \ge \frac{c\sqrt[3]{n}}{n^r}\omega_4(F^{(r)}, 1/n).$$

This completes the proof. □

Proof of Theorem 2.2 Let $r = 2s - 2$, and set

$$\tau(x) = \sin^{r+1} x.$$

Note that $\tau^{(r+1)}(0) = (r + 1)!$, therefore, there is a constant $0 < c_9 < 1$, such that $\tau^{(r+1)}(x) \ge \frac{1}{2}$ for $|x| \le c_9$. Observing that $\tau^{(r)}(0) = 0$, we conclude that

$$|\tau^{(r)}(x)| \ge \frac{|x|}{2}, \quad |x| \le c_9. \tag{6.11}$$

For $0 < b < c_9$, denote by f the 2π-periodic function, such that

$$f(x) := G\left(\frac{x}{b}\right)\tau(x), \quad x \in [-\pi, \pi],$$

and we obtain, just as in (6.8) and (6.9),

$$\|f - \tau\| \le (2b)^{r+1}, \tag{6.12}$$

and if $r \ge 1$, then

$$\|f^{(r-1)} - \tau^{(r-1)}\| \le cb^2. \tag{6.13}$$

Define the collection Y_s so that $y_1 = -\pi$ and $y_i \in [b/2, b]$, $2 \le i \le 2s$.
Recalling that r is even, we conclude that

$$F(x) := \int_0^x f(t)\,dt,$$

is 2π-periodic and is in $\Delta^{(1)}(Y_s)$, and by (6.12),

$$\|F - \mathcal{T}\| \le (2b)^{r+2},$$

where

$$\mathcal{T}(x) := \int_0^x \tau(t)\,dt.$$

Thus, by (6.13),

$$\begin{aligned}\omega_3(F^{(r)}, t) &\le \omega_3(\mathcal{T}^{(r)}, t) + \omega_3(F^{(r)} - \mathcal{T}^{(r)}, t)\\ &\le t^3\|\mathcal{T}^{(r+3)}\| + 2^3\|F^{(r)} - \mathcal{T}^{(r)}\|\\ &\le c_{10}t^3 + c_{11}b^2.\end{aligned} \tag{6.14}$$

Since $T_n'(y_i) = 0$, $i = 2, \ldots, 2s$, for each polynomial $T_n \in \Delta^{(1)}(Y_s)$, there is a point $\theta \in [b/2, b]$, such that $T_n^{(r+1)}(\theta) = 0$.

Hence, by (6.11), for $n > r + 1$, we have

$$\begin{aligned}\frac{b}{4} &\le |\tau^{(r)}(\theta)| = |\tau^{(r)}(\theta) - T_n^{(r+1)}(\theta)| = |\mathcal{T}^{(r+1)}(\theta) - T_n^{(r+1)}(\theta)|\\ &\le n^{r+1}\|\mathcal{T} - T_n\| \le n^{r+1}\|\mathcal{T} - F\| + n^{r+1}\|F - T_n\|\\ &\le n^{r+1}(2b)^{r+2} + n^{r+1}\|F - T_n\| \le \frac{b}{8} + n^{r+1}\|F - T_n\|,\end{aligned}$$

provided $b < 1/(2^4 n)$, and we actually take $b = n^{-3/2}$ for $n > c_* := \max\{c_9^{-2/3}, r+1, 2^8\}$. Then for $n > c_*$, we obtain by (6.14),

$$\|F - T_n\| \geq \frac{b}{8n^{r+1}} = \frac{\sqrt{n}}{8n^{r+3}} = \frac{\sqrt{n}}{8(c_{10}+c_{11})n^r}\left(\frac{c_{10}}{n^3} + c_{11}b^2\right) \geq \frac{c\sqrt{n}}{n^r}\omega_3(F^{(r)}, 1/n).$$

This completes the proof. □

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