Published for SISSA by Springer

Received: May 14, 2022
Revised: June 23, 2022
Accepted: July 5, 2022
Published: August 9, 2022

Reduction with degenerate Gram matrix for one-loop integrals

Bo Feng,[a,b,d,e] **Chang Hu,**[c,f] **Tingfei Li**[a] **and Yuekai Song**[a]

[a] *Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou, 310027, P.R. China*
[b] *Beijing Computational Science Research Center, Beijing 100084, P.R. China*
[c] *Hangzhou Institute of Advanced Study, UCAS, Hangzhou, 310027, P.R. China*
[d] *Center of Mathematical Science, Zhejiang University, Hangzhou, 310027, P.R. China*
[e] *Peng Huanwu Center for Fundamental Theory, Hefei, Anhui, 230026, P.R. China*
[f] *University of Chinese Academy of Sciences, 100190 Beijing, P.R. China*

E-mail: fengbo@zju.edu.cn, isiahalbert@126.com, tfli@zju.edu.cn, songyk@zju.edu.cn

Abstract: An improved PV-reduction (Passarino-Veltman) method for one-loop integrals with auxiliary vector R has been proposed in [1, 2]. It has also been shown that the new method is a self-completed method in [3]. Analytic reduction coefficients can be easily produced by recursion relations in this method, where the Gram determinant appears in denominators. The singularity caused by Gram determinant is a well-known fact and it is important to address these divergences in a given frame. In this paper, we propose a systematical algorithm to deal with this problem in our method. The key idea is that now the master integral of the highest topology will be decomposed into combinations of master integrals of lower topologies. By demanding the cancellation of divergence for obtained general reduction coefficients, we solve decomposition coefficients as a Taylor series of the Gram determinant. Moreover, the same idea can be applied to other kinds of divergences.

Keywords: Scattering Amplitudes, Renormalization and Regularization

ArXiv ePrint: 2205.03000

Open Access, © The Authors.
Article funded by SCOAP[3].

https://doi.org/10.1007/JHEP08(2022)110

JHEP08(2022)110

Contents

1 Introduction **1**

2 Background **2**
2.1 Review of improved PV-reduction 2

3 Reduction for degenerate Gram matrix **6**
3.1 The reduction of scalar bubble with degenerate Gram matrix 6
3.2 Algorithm for general case 7
3.3 The degenerate triangle 10
3.4 Two-loop example: sunset 14

4 Self-consistence **19**
4.1 Proof with recursion relation 19
4.2 Another proof 24

5 Discussion **29**

A Expansion for degenerate Gram determinant **29**

B Divergence of reduction coefficients of two-loop sunset topology **40**

JHEP08(2022)110

1 Introduction

The calculation of scattering amplitudes at loop level is an important issue in High Energy Physics. A widely used method is to reduce a loop amplitude into a linear combination of some scalar integrals under dimensional regularization [4–20]

$$I^{1-\text{loop}} = \sum_{i_{d_0+1}} C_{i_{d_0+1}} I_{d_0+1}^{i_{d_0+1}} + \sum_{i_{d_0}} C_{i_{d_0}} I_{d_0}^{i_{d_0}} + \cdots + \sum_{i_1} C_{i_1} I_1^{i_1} \,, \tag{1.1}$$

where i_s is the set of propagators appearing in the basis. The coefficient C_{i_s} ($s = 1, \cdots, d_0+1$) is simply a rational function of some Lorentz invariant contractions of external momenta as well as the masses and spacetime dimension, while the terms $I_s^{i_s}$ are the s-gon master integrals. With the general expansion (1.1), the computation of general one-loop amplitudes has been switched to determining those coefficients C_{i_s}. Many tools have been invented to shovel the brambles, such as integration-by-parts (IBP) [21, 22], PV (Passarino-Veltman) reduction [6], OPP (Ossola-Papadopoulo-Pittau) reduction [19, 23–25], unitarity cut method, etc [17, 20, 26–32].

All these methods can be divided into two categories, i.e., the reduction at the integrand level or the integral level. For reduction at the integrand level, [19] shows how to extract

the coefficients of the 4-, 3-, 2- and 1-point one-loop scalar integrals from the full one-loop integrand of arbitrary scattering processes in an algebraical way. For the reduction at the integral level, an efficient way is the unitarity cut method. The main idea is to compare the imaginary part of two sides of (1.1). However, since the loop-integral is well-defined only after using the dimensional regularization, the unitarity cut method in pure 4D needs to be generalized to $(4-2\epsilon)$-dimension, which has been done in [31, 33]. Based on this generalization, the analytic expressions for reduction coefficients (except the tadpole coefficients) have been derived in a series of papers [34–38].

Recently, we proposed a new framework for general one-loop tensor reduction by employing an auxiliary vector R and two kinds of differential operators [1, 2]. Similar to other reduction methods, our method also suffers from divergences for vanishing Gram determinant, which appears as the inverse of Gram matrix in the recursion constructions of reduction coefficients. It is well known that the vanishing Gram determinant indicates external momenta are not completely independent of each other, thus the master integral of the highest topology will not be in the basis anymore, and it can be decomposed into combinations of master integrals of lower topologies.[1] With this decomposition, the reduction coefficients will be reorganized so that the divergences between different terms will cancel with each other. To see this picture clearly, calculating analytic coefficients will be important, which is exactly the merit of our new method. In this paper, we will systematically study the degenerate case with vanishing Gram determinant using the results in [1, 2]. It turns out that the decomposition coefficients can easily be solved as a series in the Gram determinant.

This paper is organized as follows. In section 2, we will review the improved PV-reduction method with auxiliary vector R in our previous work and briefly discuss the general structure of reduction coefficients. In section 3, based on the explicit computation for bubbles in the limit $K^2 \to 0$, we establish an algorithm to deal with the vanishing Gram determinant for general topologies. We demonstrate the algorithm with the triangle in the main text and the box and pentagon in appendix A. In section 4, an important consistency has been proved, i.e., our algorithm should be independent of the auxiliary vector R in reduction coefficients and the choice of tensor ranks. Finally, we give a summary and some discussions in section 5.

2 Background

2.1 Review of improved PV-reduction

In this section, we briefly review how to reduce one-loop tensor integrals using our *improved PV-reduction method*. After that, we will explain how to deal with the degenerated Gram determinant in our framework.

[1] In this paper, the integrals of *the highest topology* refer to the integrals containing the maximum number of propagators while *lower topology* means the integrals containing fewer propagators, which is distinct from its definition in mathematics.

JHEP08(2022)110

For a general one-loop m-rank tensor integral with $(n+1)$ propagators

$$I_{n+1}^{\mu_1\mu_2\cdots\mu_m} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{\ell^{\mu_1}\ell^{\mu_2}\dots\ell^{\mu_m}}{D_0\prod_{j=1}^n D_j} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{\ell^{\mu_1}\ell^{\mu_2}\dots\ell^{\mu_m}}{(\ell^2-M_0^2)\prod_{j=1}^n((\ell-K_j)^2-M_j^2)} \tag{2.1}$$

the algorithm of traditional PV (Passarino-Veltman) reduction method [6] is following. First, we use the external momenta to write down the matched general tensor structure. For example, with rank 3 tensor integral of the form (2.1), we write down (where permutation symmetry of μ_1, μ_2, μ_3 is imposed)

$$I_{n+1}^{\mu_1\mu_2\mu_3} = a_i(g^{\mu_1\mu_2}K_i^{\mu_3} + g^{\mu_1\mu_3}K_i^{\mu_2} + g^{\mu_2\mu_3}K_i^{\mu_1}) + b_{ijl}K_i^{\mu_1}K_j^{\mu_2}K_l^{\mu_3} \tag{2.2}$$

with unknown expansion coefficients a_i, b_{ijk}. Secondly, we contract both sides of (2.2) with various $g^{\mu\nu}$ and K_i to establish enough algebraic relations to solve these coefficients. It is easy to see that with higher and higher tensor rank, there will be more and more different tensor structures to be written down in (2.2) and more and more algebraic relations to be established to fix them.

A nice observation made in [1, 2] is that the complicated tensor structure can be simply recovered from

$$I_{n+1}^{(m)} \equiv \int \frac{d^D\ell}{i\pi^{D/2}} \frac{(2R\cdot\ell)^m}{D_0\prod_{j=1}^n D_j} \tag{2.3}$$

by writing $R = \sum_{i=1}^m x_i R_i$ and extracting terms with coefficients $x_1 \dots x_m$ (or taking the proper differentiation action). For example,

$$(\ell\cdot P)\ell^2 \sim (\partial_R\cdot\partial_R)(P\cdot\partial_R)(2R\cdot\ell)^3\,. \tag{2.4}$$

The vector R in (2.3) is called *auxiliary vector*. Involving the auxiliary vector R not only greatly simplified the tensor structure but also provided a very simple way to establish algebraic relations to solve the reduction problem, as we will review shortly.

The reduction of (2.3) can be denoted as

$$I_{n+1}^{(m)} = C_{n+1\to n+1}^{(m)} I_{n+1} + C_{n+1\to n+1;\widehat{i}}^{(m)} I_{n+1;\widehat{i}} + C_{n+1\to n+1;\widehat{ij}}^{(m)} I_{n+1;\widehat{ij}} + \dots \tag{2.5}$$

where $I_{n+1;\widehat{i_1,\dots,i_a}}$ represents the scalar integral got by removing propagators $D_{i_1}, D_{i_2}, ..., D_{i_a}$ from the integral (2.3). For example,

$$I_{8;\widehat{0,2,5,6}} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{D_1\cdot D_3\cdot D_4\cdot D_7}; \quad I_{7;\widehat{1,3,5}} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{D_0\cdot D_2\cdot D_4\cdot D_6}. \tag{2.6}$$

For later convenience, in this paper we define

$$s_{ij} = K_i\cdot K_j, \quad s_{0i} = R\cdot K_i, \quad f_i = M_0^2 - M_i^2 + K_i^2\,, \tag{2.7}$$

and the Gram matrix is given by $\boldsymbol{G} = [G_{ij} = s_{ij}]$ and its determinant is denoted by $|G|$.

Now we review how to solve reduction coefficients in (2.5). Without loss of generality, we only review how to calculate $C_{n+1\to n+1;r+1,\widehat{r+2,\dots,n}}^{(m)}$. Other reduction coefficients can be

JHEP08(2022)110

got by a label permutation and a proper momentum shift.[2] Having introduced the R, we can construct the following two types of differential operators

$$\mathcal{D}_i \equiv K_i \cdot \frac{\partial}{\partial R}, \quad i = 1, \ldots, n; \qquad \mathcal{T} \equiv \eta^{\mu\nu} \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu}. \tag{2.8}$$

Acting with these two operators on both sides of (2.5) and comparing the coefficients of each Master Integral.[3] we get

$$\mathcal{T} C^{(m)}_{n+1\to n+1;\widehat{r+1,\ldots,n}} = 4m(m-1) M_0^2 C^{(m-2)}_{n+1\to n+1;\widehat{r+1,\ldots,n}}, \tag{2.9}$$

and

$$\mathcal{D}_i C^{(m)}_{n+1\to n+1;\widehat{r+1,\ldots,n}} = -m C^{(m-1)}_{n+1;\widehat{i}\to n+1;\widehat{r+1,\ldots,n}} + m f_i C^{(m-1)}_{n+1\to n+1;\widehat{r+1,\ldots,n}}. \tag{2.10}$$

where $C^{(m-1)}_{n+1;\widehat{i}\to n+1;\widehat{r+1,\ldots,n}-1}$ is the reduction coefficient of integral $I^{(m-1)}_{n+1;\widehat{i}}$ to the Master Integral $I_{n+1;\widehat{r+1,\ldots,n}}$. After expanding reduction coefficients according to its tensor structure

$$C^{(m)}_{n+1\to n+1;\widehat{r+1,r+2,\ldots,n}} = \sum_{2a_0+\sum_{k=1}^n a_k = m} \left\{ c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m) (M_0^2)^{a_0+r-n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\}, \tag{2.11}$$

equations (2.9) and (2.10) will lead to algebraic recursion relations for *expansion coefficients* $c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m)$ as a rational function of spacetime dimension D, rank m and kinematics $(K_i \cdot K_j)$ and M_i^2. A very nice feature is that these algebraic recursion relations can be solved to give the recursive construction for expansion coefficients as following,

$$\boldsymbol{c}^{(0,1,\cdots,r)}(a_1,\cdots,a_n;m) = \boldsymbol{T}^{-1}\widetilde{\boldsymbol{G}}^{-1}\boldsymbol{O}^{(0,1,\cdots,r)}(a_1,\cdots,a_n;m), \tag{2.12}$$

$$c^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n times}}}(2k) = \frac{2k-1}{D+2k-n-2} \times \tag{2.13}$$

$$\left[(4 - \boldsymbol{\alpha}^T \widetilde{\boldsymbol{G}}^{-1} \boldsymbol{\alpha}) c^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n times}}}(2k-2) + \boldsymbol{\alpha}^T \widetilde{\boldsymbol{G}}^{-1} \boldsymbol{c}^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n-1 times}}}(2k-2) \right],$$

where $\widetilde{\boldsymbol{G}} = [\beta_{ij} = \frac{s_{ij}}{M_0^2}]$ is the $n \times n$ dimensionless Gram matrix and $\boldsymbol{T} = \mathrm{diag}(a_1+1, a_2+1, \cdots, a_n+1)$ is a diagonal matrix. Other vectors are defined as

$$\boldsymbol{\alpha}^T = \left(\frac{f_1}{M_0^2}, \frac{f_2}{M_0^2}, \cdots, \frac{f_n}{M_0^2} \right), \tag{2.14}$$

$$[\boldsymbol{c}^{(0,1,\cdots,r)}(a_1,\cdots,a_n;m)]_i = c^{(0,1,\cdots,r)}_{a_1,a_2,\cdots,a_i+1,\cdots,a_n}(m), \tag{2.15}$$

[2] More details can be found in [1, 2].

[3] Note that Master Integral contains no R, the differential operators will directly act on reduction coefficients.

JHEP08(2022)110

and

$$
\begin{aligned}
[\boldsymbol{O}^{(0,1,\cdots,r)}(a_1,\cdots,a_n;m)]_i = & m\alpha_i c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m-1) - m\delta_{0a_i} c^{(0,1,\cdots,r)}_{a_1,\cdots,\widehat{a_i},\cdots,a_n}(m-1;\widehat{i}) \\
& - (m+1-\sum_{l=1}^{n} a_l) c^{(0,1,\cdots,r)}_{a_1,\cdots,a_i-1,\cdots,a_n}(m) \\
\boldsymbol{c}^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n-1 times}}}(m) = & \left(0,0,\cdots,0, c^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n-1 times}}}(m;\widehat{r+1}),\cdots, c^{(0,1,\cdots,r)}_{\underbrace{0,\cdots,0}_{\text{n-1 times}}}(m;\widehat{n-1})\right).
\end{aligned}
\tag{2.16}
$$

With the known boundary conditions, one can get all reduction coefficients by applying the recursions (2.12), (2.13) iteratively. Above formulas are easy to be implemented into Mathematica and one can generate analytic expressions for reduction coefficients for any tensor rank (Exactly because of these benefits involving the auxiliary vector R, we call the method as "improved PV-reduction method"). Knowing the analytic expressions for reduction coefficients, we can discuss many topics. For example, we can find the reduction for integrals with propagators having power higher than one as shown in [3]. Another important application is for the case where some reduction coefficients become divergent, as will be discussed in this paper.

Now we want to point out an important observation from (2.12), (2.13): with $\boldsymbol{G}^{-1}$ we will have $|G|$ appearing in denominators of all reduction coefficients. When applying above PV-reduction method, we have made a hidden assumption that $D \geq n$ and all K_i's in propagators are generic (i.e., linear independent). With this hidden assumption, $|G| \neq 0$ and above formulas are well-defined. However, in the practical applications, we do meet the degeneration of the Gram determinant, i.e., $|G| = 0$, which means that these K_i's are not linear independent anymore and the kinematics lives in a smaller space. For this case, the formulas (2.12) and (2.13) are ill-defined. Thus, for the completion of the improved PV-reduction method, we must provide an algorithm to solve this issue. Briefly, $|G| = 0$ means that the scalar integrals of the highest topology are not in the basis anymore, and they can be reduced to a combination of lower topologies.

Later we need to use the divergence behavior when $|G| \to 0$. Noticing that $G^{-1} = G^*/|G|$, where G^* is the adjugate matrix of Gram matrix, there is a Gram determinant appearing in the denominator for each iteration, thus one can conclude that the reduction coefficients have the highest divergent part about the determinant of Gram matrix[4]

$$
\begin{aligned}
C^{(m)}_{n+1\to n+1;\widehat{r+1,\ldots,n}} &\sim \begin{cases} \frac{1}{|G|^m}, & r = n, n-1 \\ \frac{1}{|G|^{m-n+r+1}}, & r < n-1. \end{cases} \\
c^{(0,\ldots,n)}_{a_1,\ldots,a_n} &\sim |G|^{-\frac{m+\sum_i^n a_i}{2}}
\end{aligned}
\tag{2.17}
$$

[4]In fact, only those reduction coefficients with $m + r \geq n$ are nonzero. Because there are not enough $\ell \cdot R$ in the numerator to cancel $(n - r)$ propagators for $m + r < n$.

JHEP08(2022)110

3 Reduction for degenerate Gram matrix

In the previous section, we find that the reduction coefficients diverge when $|G| \to 0$, while $I^{(m)}_{n+1}$ is still well-defined. The naive conflict in (2.5) indicates that the expansion on the r.h.s. is not proper. In other words, the scalar integrals are not independent of each other anymore. As we will show in this section, these divergences tell us a lot of information about the degenerating behavior of the basis. We will use various examples, i.e., integrals of bubble, triangle, box, and pentagon, to demonstrate the strategy to deal with the divergences in our framework.

3.1 The reduction of scalar bubble with degenerate Gram matrix

Let us start with the simplest example, i.e., the reduction of the bubble with tensor rank 1. Our method gives

$$I^{(1)}_2 = C^{(1)}_{2\to 2}\, I_2 + C^{(1)}_{2\to 2;\widehat{0}}\, I_{2;\widehat{0}} + C^{(1)}_{2\to 2;\widehat{1}}\, I_{2;\widehat{1}}\,, \tag{3.1}$$

where

$$\begin{aligned} C^{(1)}_{2\to 2} &= \frac{(M_0^2 - M_1^2 + s_{11})\, s_{01}}{s_{11}}\,, \\ C^{(1)}_{2\to 2;\widehat{1}} &= \frac{s_{01}}{s_{11}}\,, \quad C^{(1)}_{2\to 2;\widehat{2}} = -\frac{s_{01}}{s_{11}}\,. \end{aligned} \tag{3.2}$$

It is easy to see that with $|G| = k_1 \cdot k_1 = s_{11} \to 0$, all coefficients in (3.2) become divergent. But $I^{(1)}_2$ is a definitely well-defined integral even with $s_{11} = 0$, so the divergence on the r.h.s. in (3.1) implies that the expansion is not proper. More explicitly, the three scalar integrals $I_2, I_{2;\widehat{0}}, I_{2;\widehat{1}}$ are not linearly independent anymore, i.e., we should have

$$I_2 = \sum_{i=0}^{1} \mathcal{B}_{2\to 2;\widehat{i}} I_{2;\widehat{i}} = \sum_{i=0}^{1}\sum_{a=0}^{\infty} \mathcal{B}^{(a)}_{2\to 2;\widehat{i}} s_{11}^a I_{2;\widehat{i}}\,, \tag{3.3}$$

where the decomposition coefficient $\mathcal{B}_{2\to 2;\widehat{i}}$ is expanded as a Taylor series of $|G| = s_{11}$.

Now we show how to use the (3.3) to make the r.h.s. of (3.1) well-defined under the limit $|G| \to 0$. Putting (3.2) and (3.3) to (3.1), the r.h.s. becomes

$$\begin{aligned} &\left(\frac{(M_0^2 - M_1^2 + s_{11})\, s_{01}}{s_{11}} \sum_{a=0}^{\infty} \mathcal{B}^{(a)}_{2\to 2;\widehat{0}} s_{11}^a + \frac{s_{01}}{s_{11}}\right) I_{2;\widehat{0}} \\ &+ \left(\frac{(M_0^2 - M_1^2 + s_{11})\, s_{01}}{s_{11}} \sum_{a=0}^{\infty} \mathcal{B}^{(a)}_{2\to 2;\widehat{1}} s_{11}^a - \frac{s_{01}}{s_{11}}\right) I_{2;\widehat{1}}. \end{aligned} \tag{3.4}$$

To cancel the divegence, we just need

$$\mathcal{B}^{(0)}_{2\to 2;\widehat{0}} = \frac{-1}{(M_0^2 - M_1^2)}\,, \quad \mathcal{B}^{(0)}_{2\to 2;\widehat{1}} = \frac{1}{(M_0^2 - M_1^2)}\,. \tag{3.5}$$

Although we have only fixed the first coefficient in the above computation, the smoothness of r.h.s. in (3.1) does tell us the information of later coefficients. To get them, it is

JHEP08(2022)110

natural to consider the reduction of higher tensor rank. For rank 2, we have

$$
\begin{aligned}
C^{(2)}_{2\to 2} &= \frac{D\left(M_0^2-M_1^2\right)^2 s_{01}^2}{(D-1)s_{11}^2}+\frac{Ds_{01}^2+2M_0^2 s_{00}+2M_1^2 s_{00}}{D-1}\\
&\quad+\frac{2M_0^2\left((D-2)s_{01}^2+M_1^2 s_{00}\right)-2DM_1^2 s_{01}^2-M_0^4 s_{00}-M_1^4 s_{00}}{(D-1)s_{11}}-\frac{s_{11}s_{00}}{D-1}\,,\\
C^{(2)}_{2\to 2;\widehat{1}} &= \frac{D\left(M_0^2-M_1^2\right)s_{01}^2}{(D-1)s_{11}^2}+\frac{(3D-4)s_{01}^2+M_0^2\left(-s_{00}\right)+M_1^2 s_{00}}{(D-1)s_{11}}+\frac{s_{00}}{D-1}\,,\\
C^{(2)}_{2\to 2;\widehat{2}} &= \frac{D\left(M_1^2-M_0^2\right)s_{01}^2}{(D-1)s_{11}^2}+\frac{-Ds_{01}^2+M_0^2 s_{00}-M_1^2 s_{00}}{(D-1)s_{11}}+\frac{s_{00}}{D-1}\,.
\end{aligned}
\tag{3.6}
$$

Now the denominators have s_{11}^2 in (3.6). When combining (3.3) and (3.6), the smoothness under the limit requires the cancellation of poles $\frac{1}{s_{11}^2}$ and $\frac{1}{s_{11}}$. One can easily check that the result (3.5) will remove the pole $\frac{1}{s_{11}^2}$ automatically, while the cancellation of the pole $\frac{1}{s_{11}}$ gives

$$
\begin{aligned}
\mathcal{B}^{(1)}_{2\to 2;\widehat{0}} &= -\frac{DM_0^2-(D-4)M_1^2}{D\left(M_0-M_1\right)^3\left(M_0+M_1\right)^3}\,,\\
\mathcal{B}^{(1)}_{2\to 2;\widehat{1}} &= -\frac{(D-4)M_0^2-DM_1^2}{D\left(M_0-M_1\right)^3\left(M_0+M_1\right)^3}\,.
\end{aligned}
\tag{3.7}
$$

Now the idea is clear. To get the higher-order decomposition coefficients $\mathcal{B}^{(a)}_{2\to 2;\widehat{i}}$, we just need to impose the smoothness limit of tensor reduction with higher ranks. In fact, we can get expansion coefficients $\mathcal{B}^{(a)}_{2\to 2;\widehat{i}}$ with $a\le m$ for bubble through calculation with tensor rank $(m+1)$.

Before ending this subsection, let us point out that the decomposition coefficients given in (3.5) and (3.7) have a nice symmetry between $M_0\leftrightarrow M_1$ and they become singular under the limit $M_1\to M_0$. In that limit, $I_{2;\widehat{0}}$ and $I_{2;\widehat{1}}$ are not linearly independent of each other, and we should expand

$$
I_{2;\widehat{1}}=\sum_{a=0}^{\infty}\mathcal{B}^{(a)}_{(2;\widehat{1})\to(2;\widehat{0})}\left(M_1^2-M_0^2\right)^a I_{2;\widehat{0}}\,.
\tag{3.8}
$$

Putting (3.8) into (3.3) with known $\mathcal{B}^{(a)}_{2\to 2;\widehat{i}}$ and requiring the smoothness under the limit $M_1\to M_0$, we can calculate all the $\mathcal{B}^{(a)}_{(2;\widehat{1})\to(2;\widehat{0})}$. For example, with results (3.5) and (3.7), we get

$$
\mathcal{B}^{(0)}_{(2;\widehat{1})\to(2;\widehat{0})}=1\,,\quad \mathcal{B}^{(1)}_{(2;\widehat{1})\to(2;\widehat{0})}=\frac{D-2}{2M_0^2}\,,\quad \mathcal{B}^{(2)}_{(2;\widehat{1})\to(2;\widehat{0})}=\frac{D^2-6D+8}{8M_0^4}\,.
\tag{3.9}
$$

3.2 Algorithm for general case

Now we are ready to discuss the degeneration of scalar $(n+1)$-gon integral when the Gram determinant $|G|\to 0$. The Gram matrices corresponding to lower topologies don't

JHEP08(2022)110

degenerate generally with $|G| \to 0$, thus the scalar $(n+1)$-gon integral will become a linear combination of scalar m-gon integrals with $m \leq n$, i.e.,

$$I_{n+1} = \sum_{|\mathbf{b}_j|=1}^{n} \mathcal{B}_{n+1;\widehat{\mathbf{b}_j}} I_{n+1;\widehat{\mathbf{b}_j}} = \sum_{|\mathbf{b}_j|=1}^{n} \sum_{a=0}^{\infty} \mathcal{B}^{(a)}_{n+1;\widehat{\mathbf{b}_j}} |G|^a I_{n+1;\widehat{\mathbf{b}_j}} \,, \tag{3.10}$$

where at the second equation, the Taylor expansion of $|G|$ is given explicitly. To determine these decomposition coefficients $\mathcal{B}^{(a)}_{n+1;\widehat{\mathbf{b}_j}}$, we put (3.10) to (2.5) and get

$$\begin{aligned} I^{(m)}_{n+1} &= \sum_{|\mathbf{b}_j|=0}^{n} C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} I_{n+1;\widehat{\mathbf{b}_j}} = C^{(m)}_{n+1\to n+1} I_{n+1} + \sum_{|\mathbf{b}_j|=1}^{n} C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} I_{n+1;\widehat{\mathbf{b}_j}} \\ &= \sum_{|\mathbf{b}_j|=1}^{n} \left(C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} + C^{(m)}_{n+1\to n+1} \mathcal{B}_{n+1;\widehat{\mathbf{b}_j}} \right) I_{n+1;\widehat{\mathbf{b}_j}} \,. \end{aligned} \tag{3.11}$$

For later convenience, let us write

$$\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} = \left(C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} + C^{(m)}_{n+1\to n+1} \mathcal{B}_{n+1;\widehat{\mathbf{b}_j}} \right) . \tag{3.12}$$

The smoothness of $I^{(m)}_{n+1}$ under the limit $|G| \to 0$ means that $\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}$ is a Taylor series of $|G|$. Using this condition, we can get all the decomposition coefficients.

Now we show how to do it more directly instead of solving linear equations. Since $C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}$ is a Laurent series about the Gram determinant $|G|$, for example,

$$C^{(m)}_{n+1\to n+1} = \sum_{i=-m}^{\infty} C^{(m;i)}_{n+1\to n+1} |G|^i \,, \tag{3.13}$$

using (3.12), we can write

$$\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}} = \frac{\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} - C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n+1\to n+1}} = -\frac{C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n+1\to n+1}} + \frac{\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n+1\to n+1}} \,. \tag{3.14}$$

In (3.14), $\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}$ is regular and $C^{(m)}_{n+1\to n+1} \sim |G|^{-m}$, so we just need to keep the first term in the numerator of (3.14) to get the coefficients $\mathcal{B}^{(k<m)}_{n+1;\widehat{\mathbf{b}_j}}$, i.e.,

$$\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}} = -\frac{C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n+1\to n+1}} + \mathcal{O}(|G|^m) \,. \tag{3.15}$$

Using (3.15), we can get

$$\begin{aligned} \mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} &= \left(C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} + \left(-\frac{C^{(m')}_{n+1\to n+1;\widehat{\mathbf{b}_j}}}{C^{(m')}_{n+1\to n+1}} + \mathcal{O}(|G|^{m'}) \right) C^{(m)}_{n+1\to n+1} \right) \\ &= \frac{C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m')}_{n+1\to n+1} - C^{(m')}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m)}_{n+1\to n+1}}{C^{(m')}_{n+1\to n+1}} + \mathcal{O}(|G|^{m'-m}) \,. \end{aligned} \tag{3.16}$$

JHEP08(2022)110

Here we use the fact $C^{(m)}_{n+1\to n+1} \sim |G|^{-m}$. To get the accurate expansion of $\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}$ up to k-th order of $|G|$, we set $m'-m=k+1$, then we can write

$$\mathcal{F}^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} = \frac{C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m+k+1)}_{n+1\to n+1} - C^{(m+k+1)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m)}_{n+1\to n+1}}{C^{(m+k+1)}_{n+1\to n+1}} + \mathcal{O}(|G|^{k+1})\,. \quad (3.17)$$

If we take $k=0$ in (3.17), it gives

$$F^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} = \lim_{|G|\to 0}\left(\frac{C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m+1)}_{n+1\to n+1} - C^{(m+1)}_{n+1\to n+1;\widehat{\mathbf{b}_j}} C^{(m)}_{n+1\to n+1}}{C^{(m+1)}_{n+1\to n+1}}\right), \quad (3.18)$$

which gives the reduction coefficients for $I^{(m)}_{n+1}$ when $|G|=0$. The same result (3.18) has been obtained by methods in projective space in [39]. For the expansion (3.10), if we care about only the zeroth order, the result is also well known, which is given by [5, 19, 40] (see eq. (5.8) in [19])

$$\begin{vmatrix} I_{n+1} & -I_{n+1,\widehat{0}} & -I_{n+1,\widehat{1}} & \dots & -I_{n+1,\widehat{n}} \\ 1 & Y_{0\,0} & Y_{0\,1} & \dots & Y_{0\,n} \\ 1 & Y_{1\,0} & Y_{1\,1} & \dots & Y_{1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & Y_{n\,0} & Y_{n\,1} & \dots & Y_{n\,n} \end{vmatrix} = 0\,, \quad (3.19)$$

with

$$Y_{ij} = M_i^2 + N_j^2 - (K_i - K_j)^2,\ \ i,j=0,\dots,n. \quad (3.20)$$

A good feature of the result (3.15) is that we can get the higher-order expansion, which may be helpful for a better numerical evaluation around the point $|G|=0$.

Having established the general formula in (3.15) and (3.18), let us redo the example of the bubble in the previous subsection:

- Up to the order s_{11}^0: we have

$$\mathcal{B}_{2;\widehat{1}} = -\frac{C^{(1)}_{2\to 2;\widehat{1}}}{C^{(1)}_{2\to 2}} = \frac{1}{M_0^2-M_1^2} + \mathcal{O}\left(|G|\right),$$
$$\mathcal{B}_{2;\widehat{0}} = -\frac{C^{(1)}_{2\to 2;\widehat{0}}}{C^{(1)}_{2\to 2}} = \frac{1}{M_1^2-M_0^2} + \mathcal{O}\left(|G|\right). \quad (3.21)$$

  after using the result (3.2).

- Up to the order s_{11}^1: we have

$$\mathcal{B}_{2;\widehat{1}} = -\frac{C^{(2)}_{2\to 2;\widehat{1}}}{C^{(2)}_{2\to 2}} = \frac{1}{M_0^2-M_1^2} + \frac{\left(DM_1^2-(D-4)M_0^2\right)|G|}{D\left(M_0^2-M_1^2\right)^3} + \mathcal{O}\left(|G|^2\right),$$
$$\mathcal{B}_{2;\widehat{0}} = -\frac{C^{(2)}_{2\to 2;\widehat{0}}}{C^{(2)}_{2\to 2}} = \frac{1}{M_1^2-M_0^2} + \frac{\left((D-4)M_1^2-DM_0^2\right)|G|}{D\left(M_0^2-M_1^2\right)^3} + \mathcal{O}\left(|G|^2\right). \quad (3.22)$$

  after using the result (3.6).

JHEP08(2022)110

- Up to the order s_{11}^2: we have

$$\begin{aligned}
\mathcal{B}_{2;\widehat{1}} = & -\frac{C^{(3)}_{2\to 2;\widehat{1}}}{C^{(3)}_{2\to 2}} = \frac{1}{M_0^2 - M_1^2} + \frac{\left(DM_1^2 - (D-4)M_0^2\right)|G|}{D\left(M_0^2 - M_1^2\right)^3} \\
& + \frac{\left(\left(D^2 - 10D + 24\right)M_0^4 - 2\left(D^2 - 4D - 12\right)M_1^2 M_0^2 + D(D+2)M_1^4\right)|G|^2}{D(D+2)\left(M_0^2 - M_1^2\right)^5} \\
& + \mathcal{O}\left(|G|^3\right) , \\
\mathcal{B}_{2;\widehat{0}} = & -\frac{C^{(3)}_{2\to 2;\widehat{0}}}{C^{(3)}_{2\to 2}} = \frac{1}{M_1^2 - M_0^2} + \frac{\left((D-4)M_1^2 - DM_0^2\right)|G|}{D\left(M_0^2 - M_1^2\right)^3} \\
& - \frac{\left(\left(D^2 - 10D + 24\right)M_1^4 - 2\left(D^2 - 4D - 12\right)M_1^2 M_0^2 + D(D+2)M_0^4\right)|G|^2}{D(D+2)\left(M_0^2 - M_1^2\right)^5} \\
& + \mathcal{O}\left(|G|^3\right) .
\end{aligned} \tag{3.23}$$

  after using the result in [2] (see eq. (4.11) to eq. (4.16)).

3.3 The degenerate triangle

In this subsection, we will present some results for the degenerate triangle. First, we parameterize the Gram determinant as[5]

$$G_{\text{tri}} = s_{11}s_{22} - s_{12}^2 = |G| , \tag{3.24}$$

where $s_{ij} \equiv K_i \cdot K_j$ and the parameter $|G|$ controls the speed of degeneration. We expand the Master Integral I_3 as

$$I_3 = \sum_{i_1=0}^{2} \mathcal{B}_{3;\widehat{i_1}} I_{3;\widehat{i_1}} + \sum_{0\le i_1<i_2}^{2} \mathcal{B}_{3;\widehat{i_1 i_2}} I_{3;\widehat{i_1 i_2}} . \tag{3.25}$$

We will calculate several typical $\mathcal{B}_{3;\widehat{\mathbf{b}_j}}$ by (3.15), while others can be obtained by proper permutation.

- $\mathcal{B}_{3;\widehat{2}}$: using the tensor rank 1 we get

$$\mathcal{B}_{3;\widehat{2}} = \frac{s_{11}}{M_0^2\left(s_{11} - s_{12}\right) - M_2^2 s_{11} + s_{12}\left(M_1^2 - s_{11} + s_{12}\right)} + \mathcal{O}\left(|G|\right) , \tag{3.26}$$

[5] One should be very careful about the kinematics. For example, in the calculation of tensor triangle, we cannot treat s_{11}, s_{12}, s_{22} and $|G|$ as four independent Lorentz invariant combinations. Instead, we should replace, for example, every s_{22} with $(|G| + s_{12}^2)/s_{11}$.

JHEP08(2022)110

while using the tensor rank 2 we get

$$\begin{aligned}\mathcal{B}_{3;\widehat{2}} = & \frac{s_{11}}{M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)} \\ & -\frac{s_{11}|G|}{(D-1)\left(M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^3} \times \\ & \Big(s_{11}\left(DM_0^2-DM_2^2-3M_0^2-2M_1^2+M_2^2\right)+(D-1)s_{12}^2+M_1^4+s_{11}^2+ \\ & s_{12}\left(-DM_0^2+DM_1^2+M_0^2-M_1^2\right)+M_0^4-2M_1^2M_0^2+(1-D)s_{11}s_{12}\Big) \\ & +\mathcal{O}\left(|G|^2\right) . \end{aligned} \tag{3.27}$$

One can see that for zeroth order, results in (3.26) and (3.27) are same. Similar results hold for other coefficients.

- $\mathcal{B}_{3;\widehat{0}}$: using the tensor rank 1 we get

$$\mathcal{B}_{3;\widehat{0}} = \frac{s_{12}-s_{11}}{M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)}+\mathcal{O}\left(|G|\right) , \tag{3.28}$$

while using the tensor rank 2 we get

$$\begin{aligned}\mathcal{B}_{3;\widehat{0}} = & \frac{s_{12}-s_{11}}{M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)} \\ & +\frac{|G|}{(D-1)\left(s_{11}-s_{12}\right)\left(M_0^2\left(s_{12}-s_{11}\right)+M_2^2 s_{11}-s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^3} \times \\ & \Big(s_{11}^2(DM_0^2M_1^2-DM_2^2M_1^2+DM_2^4-DM_0^2M_2^2-M_1^4-M_0^2M_1^2+3M_2^2M_1^2- \\ & 2M_2^4+M_0^2M_2^2)+s_{12}s_{11}^2\left(-DM_1^2+DM_2^2-3M_1^2-5M_2^2\right)+s_{12}^2s_{11}(DM_1^2- \\ & DM_2^2+M_1^2+3M_2^2)+s_{12}s_{11}(DM_1^4-DM_0^2M_1^2-DM_2^2M_1^2+DM_0^2M_2^2- \\ & M_1^4+M_0^2M_1^2+M_2^2M_1^2-M_0^2M_2^2)+\left(2M_1^2+2M_2^2\right)s_{11}^3-s_{11}^4+4s_{12}s_{11}^3- \\ & 6s_{12}^2s_{11}^2+4s_{12}^3s_{11}-s_{12}^4\Big)+\mathcal{O}\left(|G|^2\right) . \end{aligned} \tag{3.29}$$

- $\mathcal{B}_{3;\widehat{12}}$: using the tensor rank 1 we get

$$\mathcal{B}_{3;\widehat{12}} = 0+\mathcal{O}\left(|G|\right) , \tag{3.30}$$

while using the tensor rank 2 we get

$$\mathcal{B}_{3;\widehat{12}} = \frac{(D-2)s_{11}|G|}{(D-1)s_{12}\left(M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^2}+\mathcal{O}\left(|G|^2\right) . \tag{3.31}$$

JHEP08(2022)110

Finally using the tensor rank 3 we get

$$\begin{aligned}\mathcal{B}_{3;\widehat{12}} =& \frac{(D-2)s_{11}|G|}{(D-1)s_{12}\left(M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^2} \\ &-\frac{(D-2)s_{11}|G|^2}{\left(D^2-1\right)s_{12}^3\left(M_0^2\left(s_{11}-s_{12}\right)-M_2^2 s_{11}+s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^4}\times \\ &\Big(s_{12}^3\left(-3DM_0^2+3DM_1^2-2M_0^2+2M_1^2\right)+s_{12}^2(DM_0^4-2DM_1^2M_0^2+ \\ &DM_1^4+2M_0^4-4M_1^2M_0^2+2M_1^4)+s_{11}s_{12}^2(4DM_0^2-2DM_1^2-2DM_2^2- \\ &4M_0^2-4M_1^2-4M_2^2)+s_{11}^2s_{12}\left(DM_2^2-DM_0^2\right)+s_{11}s_{12}(-DM_0^4+ \\ &DM_1^2M_0^2+DM_2^2M_0^2-DM_1^2M_2^2)+(2D+3)s_{12}^4+(-3D-2)s_{11}s_{12}^3 \\ &+(D+2)s_{11}^2s_{12}^2+\left(M_0^4-2M_2^2M_0^2+M_2^4\right)s_{11}^2\Big)+\mathcal{O}\left(|G|^3\right). \end{aligned} \tag{3.32}$$

- $\mathcal{B}_{3;\widehat{02}}$: using the tensor rank 1 we get

$$\mathcal{B}_{3;\widehat{02}} = 0+\mathcal{O}\left(|G|\right), \tag{3.33}$$

while using the tensor rank 2 we get

$$\begin{aligned}\mathcal{B}_{3;\widehat{02}} =& \frac{(D-2)s_{11}|G|}{(D-1)\left(s_{11}-s_{12}\right)\left(M_0^2\left(s_{12}-s_{11}\right)+M_2^2 s_{11}-s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^2} \\ &+\mathcal{O}\left(|G|^2\right). \end{aligned} \tag{3.34}$$

Finally using the tensor rank 3 we get

$$\begin{aligned}\mathcal{B}_{3;\widehat{02}} =& \frac{(D-2)s_{11}|G|}{(D-1)\left(s_{11}-s_{12}\right)\left(M_0^2\left(s_{12}-s_{11}\right)+M_2^2 s_{11}-s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^2} \\ &-\frac{(D-2)s_{11}|G|^2}{(D^2-1)\left(s_{11}-s_{12}\right)^3\left(M_0^2\left(s_{12}-s_{11}\right)+M_2^2 s_{11}-s_{12}\left(M_1^2-s_{11}+s_{12}\right)\right)^4}\times \\ &\Big(s_{12}s_{11}(-2DM_0^4+3DM_1^2M_0^2+DM_2^2M_0^2-DM_1^4-DM_1^2M_2^2-4M_0^4+ \\ &8M_1^2M_0^2-4M_1^4)+s_{11}^3\left(DM_0^2-DM_2^2-2M_0^2-6M_1^2-4M_2^2\right)+s_{11}^2(DM_0^4- \\ &DM_1^2M_0^2-DM_2^2M_0^2+DM_1^2M_2^2+2M_0^4-4M_1^2M_0^2+3M_1^4+M_2^4-2M_1^2M_2^2) \\ &+s_{12}s_{11}^2\left(-5DM_0^2+2DM_1^2+3DM_2^2+2M_0^2+14M_1^2+8M_2^2\right)+s_{12}^3(2M_1^2- \\ &3DM_0^2+3DM_1^2-2M_0^2)+s_{12}^2s_{11}(7DM_0^2-5DM_1^2-2DM_2^2+2M_0^2-4M_2^2- \\ &10M_1^2)-(D+10)s_{12}s_{11}^3+(4D+14)s_{12}^2s_{11}^2-(5D+10)s_{12}^3s_{11}+3s_{11}^4+ \\ &(2D+3)s_{12}^4+s_{12}^2\left(DM_0^4-2DM_1^2M_0^2+DM_1^4+2M_0^4-4M_1^2M_0^2+2M_1^4\right)\Big) \\ &+\mathcal{O}\left(|G|^3\right). \end{aligned} \tag{3.35}$$

Note that the expansion of $\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}}$ is independent of R although the numerator and the denominator depend on R on the r.h.s. of (3.15). And it's also independent of the tensor rank m up to the order $|G|^k$ as long as $m > k$. The above results confirm these nontrivial points, and we will give a proof in the later section.

JHEP08(2022)110

The reduction coefficients $F^{(m)}_{3\to 3;\widehat{\mathbf{b}}_j}$ for $|G|=0$ (see (3.18)) are given in following:

- $m=0$:

$$F^{(0)}_{3\to 3;\widehat{0}} = \mathcal{B}^{(0)}_{3\to 3;\widehat{0}} = -\left(\frac{C^{(1)}_{3\to 3;\widehat{0}}}{C^{(1)}_{3\to 3}}\right)\Bigg|_{|G|=0}$$
$$= \frac{s_{12}-s_{11}}{M_1^2\left(s_{11}-s_{12}\right)-M_3^2 s_{11}+s_{12}\left(M_2^2-s_{11}+s_{12}\right)}\,, \tag{3.36a}$$
$$F^{(0)}_{3\to 3;\widehat{1}} = -\frac{s_{12}}{M_1^2\left(s_{11}-s_{12}\right)-M_3^2 s_{11}+s_{12}\left(M_2^2-s_{11}+s_{12}\right)}\,, \tag{3.36b}$$
$$F^{(0)}_{3\to 3;\widehat{2}} = \frac{s_{11}}{M_1^2\left(s_{11}-s_{12}\right)-M_3^2 s_{11}+s_{12}\left(M_2^2-s_{11}+s_{12}\right)}\,, \tag{3.36c}$$
$$F^{(0)}_{3\to 3;\widehat{12}} = F^{(0)}_{3\to 3;\widehat{01}} = F^{(0)}_{3\to 3;\widehat{02}} = 0\,. \tag{3.36d}$$

- $m=1$:

$$F^{(1)}_{3\to 3;\widehat{0}} = \frac{1}{(D-1)s_{11}\left(s_{11}-s_{12}\right)\left(\left(s_{12}-s_{11}\right)M_1^2+M_3^2 s_{11}-s_{12}\left(M_2^2-s_{11}+s_{12}\right)\right)^2}\times$$
$$\Big(s_{02}s_{11}^5+\left(-DM_1^2+M_1^2+DM_3^2-M_3^2\right)s_{01}s_{11}^4+(D-2)s_{12}s_{01}s_{11}^4+$$
$$\left(-DM_1^2+M_1^2-2M_2^2+DM_3^2-3M_3^2\right)s_{02}s_{11}^4+(D-5)s_{12}s_{02}s_{11}^4+$$
$$(7-3D)s_{12}^2s_{01}s_{11}^3+(DM_3^4-M_3^4-DM_1^2M_3^2+M_1^2M_3^2-DM_2^2M_3^2+$$
$$M_2^2M_3^2+DM_1^2M_2^2-M_1^2M_2^2)s_{01}s_{11}^3+(3DM_1^2-3M_1^2-2DM_2^2+$$
$$4M_2^2-DM_3^2+3M_3^2)s_{12}s_{01}s_{11}^3+(9-3D)s_{12}^2s_{02}s_{11}^3+(5-D)s_{12}^4s_{01}s_{11}$$
$$+\left(3DM_1^2-3M_1^2+4M_2^2-3DM_3^2+7M_3^2\right)s_{12}s_{02}s_{11}^3+(-3DM_1^2+$$
$$3M_1^2+3DM_2^2-7M_2^2-4M_3^2)s_{12}^2s_{01}s_{11}^2+(DM_2^4-2M_2^4-DM_1^2M_2^2+$$
$$M_1^2M_2^2-DM_3^2M_2^2+3M_3^2M_2^2-M_3^4+DM_1^2M_3^2-M_1^2M_3^2)s_{12}s_{01}s_{11}^2$$
$$+(3D-7)s_{12}^3s_{02}s_{11}^2+(-3DM_1^2+3M_1^2+DM_2^2-3M_2^2+2DM_3^2$$
$$-4M_3^2)s_{12}^2s_{02}s_{11}^2+(-DM_2^4+M_2^4+DM_1^2M_2^2-M_1^2M_2^2+DM_3^2M_2^2$$
$$-M_3^2M_2^2-DM_1^2M_3^2+M_1^2M_3^2)s_{12}s_{02}s_{11}^2+(M_2^4-DM_1^2M_2^2+M_1^2M_2^2+$$
$$DM_3^2M_2^2-3M_3^2M_2^2-DM_3^4+2M_3^4+DM_1^2M_3^2-M_1^2M_3^2)s_{02}s_{11}^3$$
$$+(DM_1^2-M_1^2-DM_2^2+3M_2^2+2M_3^2)s_{12}^3s_{01}s_{11}+(2-D)s_{12}^4s_{02}s_{11}$$
$$+(DM_1^2-M_1^2-DM_2^2+M_2^2)s_{12}^3s_{02}s_{11}-s_{12}^5s_{01}+(3D-9)s_{12}^3s_{01}s_{11}^2\Big)\,, \tag{3.37a}$$

$$F^{(1)}_{3\to 3;\widehat{1}} = \frac{1}{(D-1)s_{11}s_{12}\left(M_1^2\left(s_{12}-s_{11}\right)+M_3^2 s_{11}-s_{12}\left(M_2^2-s_{11}+s_{12}\right)\right)^2}\times$$
$$\Big(s_{11}s_{12}^3\left(DM_1^2-DM_2^2-M_1^2+M_2^2\right)s_{02}+s_{11}^2s_{12}^2(-2DM_1^2+2DM_3^2-$$
$$4M_3^2)s_{02}+s_{11}^3s_{12}\left(DM_1^2-DM_3^2-M_1^2+M_3^2\right)s_{02}+(2-D)s_{11}s_{12}^4s_{02}$$

JHEP08(2022)110

$$
\begin{aligned}
&+s_{11}^2 s_{12}(DM_1^4 - DM_2^2M_1^2 - DM_3^2M_1^2 + DM_2^2M_3^2 - M_1^4 + M_2^2M_1^2 + \\
&M_3^2M_1^2 - M_2^2M_3^2)s_{02} + (D-1)s_{11}^2 s_{12}^3 s_{02} + \left(2M_1^2 + 2M_3^2\right) s_{11}s_{12}^3 s_{01} \\
&+s_{11}^3\left(-DM_1^4 + 2DM_3^2M_1^2 - DM_3^4 + 2M_1^4 - 4M_3^2M_1^2 + 2M_3^4\right) s_{02} \\
&+\left(-M_1^4 + 2M_3^2M_1^2 - M_3^4\right) s_{11}^2 s_{12}s_{01} - s_{12}^5 s_{01}\Big)\,, \qquad (3.37b)
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{3\to 3;\widehat{2}} = &\frac{1}{(D-1)\left(M_1^2(s_{11}-s_{12}) - M_3^2 s_{11} + s_{12}\left(M_2^2 - s_{11} + s_{12}\right)\right)^2} \times \\
&\Big(s_{11}^2\left(DM_1^2 - DM_3^2 - M_1^2 + M_3^2\right) s_{01} + s_{11}(DM_1^4 - DM_2^2M_1^2 - \\
&DM_3^2M_1^2 + DM_2^2M_3^2 - M_1^4 + M_2^2M_1^2 + M_3^2M_1^2 - M_2^2M_3^2)s_{01} \\
&+s_{12}s_{11}\left(-2DM_1^2 + 2DM_2^2 - 4M_2^2\right) s_{01} + s_{12}^2(DM_1^2 - DM_2^2 - \\
&M_1^2 + M_2^2)s_{01} + s_{12}(-DM_1^4 + 2DM_2^2M_1^2 - DM_2^4 + 2M_1^4 \\
&-4M_2^2M_1^2 + 2M_2^4)s_{01} + (2-D)s_{12}s_{11}^2 s_{01} + (D-1)s_{12}^2 s_{11}s_{01} \\
&+\left(2M_1^2 + 2M_2^2\right) s_{11}^2 s_{02} + \left(-M_1^4 + 2M_2^2M_1^2 - M_2^4\right) s_{11}s_{02} \\
&-s_{11}^3 s_{02}\Big)\,, \qquad (3.37c)
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{3\to 3;\widehat{12}} &= -\frac{(D-2)\left(s_{12}s_{01} - s_{11}s_{02}\right)}{(D-1)s_{12}\left(M_1^2(s_{11}-s_{12}) - M_3^2 s_{11} + s_{12}\left(M_2^2 - s_{11} + s_{12}\right)\right)}\,, \\
F^{(1)}_{3\to 3;\widehat{01}} &= \frac{(D-2)s_{11}\left(s_{11}s_{02} - s_{12}s_{01}\right)}{(D-1)\left(s_{11}-s_{12}\right)s_{12}\left(M_1^2(s_{12}-s_{11}) + M_3^2 s_{11} - s_{12}\left(M_2^2 - s_{11} + s_{12}\right)\right)}\,, \\
F^{(1)}_{3\to 3;\widehat{02}} &= \frac{(D-2)\left(s_{12}s_{01} - s_{11}s_{02}\right)}{(D-1)\left(s_{11}-s_{12}\right)\left(M_1^2(s_{12}-s_{11}) + M_3^2 s_{11} - s_{12}\left(M_2^2 - s_{11} + s_{12}\right)\right)}\,. \qquad (3.37d)
\end{aligned}
$$

One interesting point is that expressions of the first three coefficients are complicated while the last three are relatively simple. The reason is that I_{n+1} is reduced to combinations of I_n only (see (3.19)) at zeroth order, thus coefficients of $I_{m\leq n-1}$ are not affected.

In this section, we present examples up to the triangle. A complete result of $\mathcal{B}$ is listed in the Mathematica file in the supplementary material. For box and pentagon, we will give some numerical results in appendix A.

3.4 Two-loop example: sunset

Our idea to deal with degenerate Gram determinant is simple and it should apply to higher loops. To demonstrate this point, in this subsection we consider the simplest two-loop integrals, i.e., the integrals with sunset topology. Involving two auxiliary vectors R_1, R_2 we consider the tensor reduction of following integrals

$$
I^{(r_1,r_2)}_{a_1,a_2,a_3} \equiv \int \frac{d^D\ell_1}{i\pi^{D/2}}\frac{d^D\ell_2}{i\pi^{D/2}}\frac{(2\ell_1\cdot R_1)^{r_1}(2\ell_2\cdot R_2)^{r_2}}{D_1^{a_1}D_2^{a_2}D_3^{a_3}}\,, \qquad (3.38)
$$

where the propagators are

$$
D_1 \equiv \ell_1^2 - M_1^2\,, \quad D_2 \equiv \ell_2^2 - M_2^2\,, \quad D_3 \equiv (\ell_1+\ell_2-K)^2 - M_3^2\,. \qquad (3.39)
$$

JHEP08(2022)110

It is found that the integrals (3.38) can be reduced to seven master integrals,[6] i.e.,

$$I^{(r_1,r_2)}_{1,1,1} = \mathbf{C}^{(r_1,r_2)}\mathbf{J} = \sum_{i_1,i_2,j} s_{01}^{i_1} s_{0'1}^{i_2} s_{00'}^{j} s_{00}^{\frac{r_1-i_1-j}{2}} s_{0'0'}^{\frac{r_2-i_2-j}{2}} \vec{\alpha}^{(r_1,r_2)}_{i_1,i_2,j}\ \mathbf{J}\,. \tag{3.40}$$

where the seven master integrals are (we have written $\mathbf{C}$, $\mathbf{J}$ to emphasize they are vector)

$$\begin{aligned}\mathbf{J} = \Bigg\{ &\int \frac{1}{D_1D_2D_3}, \int \frac{(2\ell_1\cdot K)}{D_1D_2D_3}, \int \frac{(2\ell_2\cdot K)}{D_1D_2D_3}, \int \frac{(2\ell_1\cdot K)(2\ell_2\cdot K)}{D_1D_2D_3} \\ &\int \frac{1}{D_2D_3}, \int \frac{1}{D_1D_3}, \int \frac{1}{D_1D_2} \Bigg\},\end{aligned} \tag{3.41}$$

and kinematic variables are

$$s_{01} = R_1\cdot K\,,\ s_{0'1} = R_2\cdot K\,,\ s_{11} = K^2\,,\ s_{00} = R_1^2\,,\ s_{0'0'} = R_2^2\,,\ s_{00'} = R_1\cdot R_2\,. \tag{3.42}$$

All the reduction coefficients could be found in [41], where we have generalized the improved PV-reduction method to the two-loop sunset topology. For example:

$$\begin{aligned}\mathbf{C}^{(1,0)} &= \left\{0, \frac{s_{01}}{s_{11}}, 0, 0, 0, 0, 0\right\}\,, \\ \mathbf{C}^{(0,1)} &= \left\{0, 0, \frac{s_{0'1}}{s_{11}}, 0, 0, 0, 0\right\}\,.\end{aligned} \tag{3.43}$$

To study the reduction for degenerate Gram determinant ($s_{11}\to 0$), we follow the same logic, i.e., the basis with the highest topology will be decomposed to the linear combinations of other integrals in the basis. However, there is something different between two-loop and one-loop integrals. For one-loop integrals, there is only one master integral with the highest topology in the basis. For two-loop integrals, in general, there are more than one highest-topology master integrals. For example, for the sunset topology, there are four master integrals J_1, J_2, J_3, J_4 as given in (3.41). Now we face the problem: whether just one or several master integrals of them should be degenerate in the limit? In fact, from the tensor reduction results given in (3.43), we can see J_2 and J_3 should be degenerate. There is also a brute-force algorithm to determine the number of degenerate master integrals. Starting with just one degenerate master integral, we solve the decomposition coefficients using the idea given in the last subsection for various choices of tensor rank. If it is not the right degenerate pattern, we will find that it is impossible to cancel all divergences coming from $s_{11}\to 0$. Then we take two master integrals to be degenerate, repeat the same computations for several tensor structures and check the consistency. Interacting the procedure until reaching the point that all divergences have been canceled consistently, we get the correct degenerate pattern.

For our sunset topology, because there are five independent Lorentz invariant contractions $R_i\cdot R_j, R_i\cdot K$ comparing to two $R^2, R\cdot K$ for one-loop bubble, there are three master

[6] For our purpose, we consider only the case $a_i = 1$.

JHEP08(2022)110

integrals degenerated under the limit, which can be taken as J_2, J_3, J_4, i.e., we have the expansion

$$J_i = \sum_{k=0}^{\infty} \mathcal{B}^{(k)}_{i\to j} s_{11}^k J_j\,, \quad i \in \text{deI}\,,\ j \in \text{reI}\,, \tag{3.44}$$

where $\text{deI} = \{2,3,4\}$ denotes the degenerate master integrals and $\text{reI} = \{1,5,6,7\}$ denotes the remaining master integrals. Expansion coefficients $\mathcal{B}^{(k)}_{i\to j}$ are determined by requiring

$$F_j^{(r_1,r_2)} \equiv \sum_{i\in\text{deI}} \sum_k C_i^{(r_1,r_2)} \mathcal{B}^{(k)}_{i\to j} s_{11}^k + C_j^{(r_1,r_2)}, \qquad \forall j \in \text{reI}\,. \tag{3.45}$$

to be finite. The $F_j^{(r_1,r_2)}$ is nothing but the reduction coefficients for the degenerate case. Above relation can be written more compactly as

$$C_i^{(r_1,r_2)}\Big|_{s_{11}^{-k}} \mathcal{B}^{(k)}_{i\to j} = -C_j^{(r_1,r_2)}\Big|_{s_{11}^{-k}} \tag{3.46}$$

where we have treated $C_{j=1,5,6,7}$ and $C_{i=2,3,4}$ as row vector and $\mathcal{B}$ as a matrix. The symbol $\Big|_{s_{11}^{-k}}$ means to take the coefficients of the k-th pole s_{11}^{-k}.

Now we determine decomposition coefficients using the results from rank one to three given in [41].

- For the tensor rank level one, i.e., $r_1 + r_2 = 1$, there are two cases given in (3.43). With the only divergent pole s_{11}^{-1}, by (3.46) we get following equations:

$$(1,0): \quad \begin{pmatrix} s_{01} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{B}^{(0)}_{2\to1} & \mathcal{B}^{(0)}_{2\to5} & \mathcal{B}^{(0)}_{2\to6} & \mathcal{B}^{(0)}_{2\to7} \\ \mathcal{B}^{(0)}_{3\to1} & \mathcal{B}^{(0)}_{3\to5} & \mathcal{B}^{(0)}_{3\to6} & \mathcal{B}^{(0)}_{3\to7} \\ \mathcal{B}^{(0)}_{4\to1} & \mathcal{B}^{(0)}_{4\to5} & \mathcal{B}^{(0)}_{4\to6} & \mathcal{B}^{(0)}_{4\to7} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.47}$$

  and

$$(0,1): \quad \begin{pmatrix} 0 & s_{0'1} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{B}^{(0)}_{2\to1} & \mathcal{B}^{(0)}_{2\to5} & \mathcal{B}^{(0)}_{2\to6} & \mathcal{B}^{(0)}_{2\to7} \\ \mathcal{B}^{(0)}_{3\to1} & \mathcal{B}^{(0)}_{3\to5} & \mathcal{B}^{(0)}_{3\to6} & \mathcal{B}^{(0)}_{3\to7} \\ \mathcal{B}^{(0)}_{4\to1} & \mathcal{B}^{(0)}_{4\to5} & \mathcal{B}^{(0)}_{4\to6} & \mathcal{B}^{(0)}_{4\to7} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.48}$$

  thus we can solve

$$\mathcal{B}^{(0)}_{i\to j} = 0\,, \quad i = 2,3; \quad j = 1,5,6,7\,. \tag{3.49}$$

- For the tensor rank level two, i.e., $r_1 + r_2 = 2$, there are three cases $(r_1,r_2) = (2,0),(1,1),(0,2)$. There are also two divergent poles $\frac{1}{s_{11}^2}$ and $\frac{1}{s_{11}}$. Explicitly, we have

$$\begin{pmatrix} C_2^{(2,0)} & C_2^{(1,1)} & C_2^{(0,2)} \\ C_3^{(2,0)} & C_3^{(1,1)} & C_3^{(0,2)} \\ C_4^{(2,0)} & C_4^{(1,1)} & C_4^{(0,2)} \end{pmatrix}\Bigg|_{s_{11}^{-2}} = \frac{D}{D-1} \begin{pmatrix} M_{12,3} s_{01}^2 & 0 & 2M_2^2 s_{0'1}^2 \\ 2M_1^2 s_{01}^2 & 0 & M_{12,3} s_{0'1}^2 \\ -2s_{01}^2 & s_{01}s_{0'1} & -2s_{0'1}^2 \end{pmatrix}, \tag{3.50}$$

$$\begin{pmatrix} C_2^{(2,0)} & C_2^{(1,1)} & C_2^{(0,2)} \\ C_3^{(2,0)} & C_3^{(1,1)} & C_3^{(0,2)} \\ C_4^{(2,0)} & C_4^{(1,1)} & C_4^{(0,2)} \end{pmatrix}\Bigg|_{s_{11}^{-1}} = \begin{pmatrix} \frac{3Ds_{01}^2 - M_{12,3}s_{00}}{D-1} & -\frac{2s_{01}s_{0'1}}{D-1} & \frac{2\left(Ds_{0'1}^2 - M_2^2 s_{0'0'}\right)}{D-1} \\ \frac{2\left(Ds_{01}^2 - M_1^2 s_{00}\right)}{D-1} & -\frac{2s_{01}s_{0'1}}{D-1} & \frac{3Ds_{0'1}^2 - M_{12,3}s_{0'0'}}{D-1} \\ \frac{2s_{00}}{D-1} & -\frac{s_{0'0}}{D-1} & \frac{2s_{0'0'}}{D-1} \end{pmatrix} \tag{3.51}$$

JHEP08(2022)110

where we have defined

$$M_{12,3} \equiv M_1^2 + M_2^2 - M_3^2 \tag{3.52}$$

for the left hand side of (3.46). The corresponding data for the right hand side of (3.46) is following

$$\begin{pmatrix} C_1^{(2,0)} & C_1^{(1,1)} & C_1^{(0,2)} \\ C_5^{(2,0)} & C_5^{(1,1)} & C_5^{(0,2)} \\ C_6^{(2,0)} & C_6^{(1,1)} & C_6^{(0,2)} \\ C_7^{(2,0)} & C_7^{(1,1)} & C_7^{(0,2)} \end{pmatrix}\Bigg|_{s_{11}^{-2}}, \begin{pmatrix} C_1^{(2,0)} & C_1^{(1,1)} & C_1^{(0,2)} \\ C_5^{(2,0)} & C_5^{(1,1)} & C_5^{(0,2)} \\ C_6^{(2,0)} & C_6^{(1,1)} & C_6^{(0,2)} \\ C_7^{(2,0)} & C_7^{(1,1)} & C_7^{(0,2)} \end{pmatrix}\Bigg|_{s_{11}^{-1}} = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix},$$

$$\begin{pmatrix} -\frac{2\left((D+2)M_1^2+D\left(M_2^2-M_3^2\right)\right)s_{01}^2}{D-1} & \frac{2\left(M_1^2+M_2^2-M_3^2\right)s_{01}s_{0'1}}{D-1} & -\frac{2\left(DM_1^2+(D+2)M_2^2-DM_3^2\right)s_{0'1}^2}{D-1} \\ -\frac{4s_{01}^2}{D-1} & \frac{2s_{01}s_{0'1}}{D-1} & -\frac{2Ds_{0'1}^2}{D-1} \\ -\frac{2Ds_{01}^2}{D-1} & \frac{2s_{01}s_{0'1}}{D-1} & -\frac{4s_{0'1}^2}{D-1} \\ \frac{2Ds_{01}^2}{D-1} & -\frac{2s_{01}s_{0'1}}{D-1} & \frac{2Ds_{0'1}^2}{D-1} \end{pmatrix}. \tag{3.53}$$

Now we can use the (3.46) to solve wanted decomposition coefficients.

– For the pole s_{11}^{-2} using the (3.46) we can solve

$$\mathcal{B}_{i\to j}^{(0)} = 0\,, \quad i = 2,3,4\,. \tag{3.54}$$

It is worth to notice that for $i = 2,3$ it is consistent with the result (3.49), but for $i = 4$ it is the new solution.

– For the pole s_{11}^{-1}, using the data (3.51) and (3.53), by (3.46) we can solve

$$\begin{aligned}
\mathcal{B}_{2\to1}^{(1)} &= \frac{2\left(2DM_1^2M_{12,3} - DM_{12,3}^2 - 6M_1^2M_{12,3} + 2M_{12,3}^2 + 4M_2^2M_1^2\right)}{D\left(4M_1^2M_2^2 - M_{12,3}^2\right)}, \\
\mathcal{B}_{3\to1}^{(1)} &= \frac{2\left(2DM_2^2M_{12,3} - DM_{12,3}^2 - 6M_2^2M_{12,3} + 2M_{12,3}^2 + 4M_1^2M_2^2\right)}{D\left(4M_1^2M_2^2 - M_{12,3}^2\right)}, \\
\mathcal{B}_{2\to5}^{(1)} &= \frac{4(D-2)M_1^2}{D\left(4M_1^2M_2^2 - M_{12,3}^2\right)}, \qquad \mathcal{B}_{3\to5}^{(1)} = \frac{2(D-2)M_{12,3}}{D\left(M_{12,3}^2 - 4M_1^2M_2^2\right)}, \\
\mathcal{B}_{2\to6}^{(1)} &= \frac{2(D-2)M_{12,3}}{D\left(M_{12,3}^2 - 4M_1^2M_2^2\right)}, \qquad \mathcal{B}_{3\to6}^{(1)} = \frac{4(D-2)M_2^2}{D\left(4M_1^2M_2^2 - M_{12,3}^2\right)}, \\
\mathcal{B}_{2\to7}^{(1)} &= \frac{2(D-2)\left(2M_1^2 - M_{12,3}\right)}{D\left(M_{12,3}^2 - 4M_1^2M_2^2\right)}, \qquad \mathcal{B}_{3\to7}^{(1)} = \frac{2(D-2)\left(2M_2^2 - M_{12,3}\right)}{D\left(M_{12,3}^2 - 4M_1^2M_2^2\right)}, \\
\mathcal{B}_{4\to1}^{(1)} &= -\frac{2M_{12,3}}{D}\,, \quad \mathcal{B}_{4\to5}^{(1)} = -\frac{2}{D}\,, \quad \mathcal{B}_{4\to6}^{(1)} = -\frac{2}{D}\,, \quad \mathcal{B}_{4\to7}^{(1)} = \frac{2}{D}\,.
\end{aligned} \tag{3.55}$$

There is one thing we need to emphasize. The expressions of C's have different tensor structures of R while the coefficients $\mathcal{B}$ are R independent. Thus the

JHEP08(2022)110

same solution (3.55) solving (3.46) for various tensor structures is very nontrivial thing. Thus for practical purpose, we can focus on one tensor structure with the simplest expressions to solve $\mathcal{B}$.

- For the tensor rank level 3: $r_1 + r_2 = 3$, there are four cases $(r_1, r_2) = (3,0), (2,1), (1,2), (0,3)$. Now the expressions for C_i, C_j of (3.46) becomes complicated because of various tensor structures. As we mentioned before, we will give only the part of a particular tensor structure. For example, for $C_i^{(3,0)}$ we give the part of the tensor structure s_{01}^3 as following

$$\left.\begin{pmatrix} C_2^{(3,0)} \\ C_3^{(3,0)} \\ C_4^{(3,0)} \end{pmatrix}\right|_{s_{11}^{-3}} = \begin{pmatrix} \frac{(D+2)\left(4(1-2D)M_2^2M_{12,3}+(3D-2)M_{12,3}^2+8(D-1)M_1^2M_2^2\right)s_{01}^3}{(D-1)(3D-2)} \\ \frac{2(D+2)M_1^2\left((5D-4)M_{12,3}+4(1-2D)M_2^2\right)s_{01}^3}{(D-1)(3D-2)} \\ -\frac{2(D+2)\left((4D-2)M_{12,3}+(D-2)M_1^2+4(1-2D)M_2^2\right)s_{01}^3}{(D-1)(3D-2)} \end{pmatrix} \tag{3.56}$$

while for $C_i^{(1,2)}$ we give the part of the tensor structure $s_{01}^2 s_{0'1}$ as following

$$\left.\begin{pmatrix} C_2^{(1,2)} \\ C_3^{(1,2)} \\ C_4^{(1,2)} \end{pmatrix}\right|_{s_{11}^{-3}} = \begin{pmatrix} \frac{2(D+2)M_2^2\left((2D-1)M_{12,3}-2(D-1)M_1^2\right)s_{01}^2 s_{0'1}}{(D-1)(3D-2)} \\ -\frac{2(D+2)M_1^2\left((D-1)M_{12,3}+2(1-2D)M_2^2\right)s_{01}^2 s_{0'1}}{(D-1)(3D-2)} \\ \frac{(D+2)\left(DM_{12,3}+4(D-1)M_1^2+4(1-2D)M_2^2\right)s_{01}^2 s_{0'1}}{(D-1)(3D-2)} \end{pmatrix} \tag{3.57}$$

Expressions of C_i, C_j for other divergent poles and the chosen tensor structures are listed in appendix B.

  - For the pole s_{11}^{-3}, it produce the same results as given by (3.54).
  - For the pole s_{11}^{-2}, using (3.46) we find the same results as given in (3.55).
  - For the pole s_{11}^{-1}, using (3.46) we find expansion coefficients $\mathcal{B}_{i\to j}^{(2)}$. Since their expressions are very long, we have collected them into a Mathematica file in the supplementary material. To give a flavor, we present $\mathcal{B}_{4\to 1}^{(2)}$ as following

$$\begin{aligned} \mathcal{B}_{4\to 1}^{(2)} = & \frac{1}{D^2(D+2)\left(M_{12,3}^2-4M_1^2M_2^2\right)^2} \\ & \Big[\left(16D^3-208D^2+528D-288\right)M_2^2M_1^2M_{12,3}^2 \\ & +(4D^3-14D^2+28D-48)M_{12,3}^4 \\ & -(8D^3-40D^2+80D-96)(M_1^2+M_2^2)M_{12,3}^3 \\ & +\left(128D^2-416D+96\right)(M_2^2M_1^4+M_1^2M_2^4)M_{12,3} \\ & +\left(384D-96D^2\right)M_2^4M_1^4\Big]. \end{aligned} \tag{3.58}$$

JHEP08(2022)110

Now we can calculate the final reduction coefficients by (3.45). For example, the reduction coefficients for rank $(0,1)$ is given by

$$
\begin{aligned}
F_1^{(0,1)} &= \frac{2s_{0'1}\left(-2DM_2^2M_{12,3}+DM_{12,3}^2+6M_2^2M_{12,3}-2M_{12,3}^2-4M_1^2M_2^2\right)}{D\left(M_{12,3}^2-4M_1^2M_2^2\right)}\,,\\
F_1^{(1,0)} &= \frac{2s_{01}\left(2DM_1^2M_{12,3}-DM_{12,3}^2-6M_1^2M_{12,3}+2M_{12,3}^2+4M_2^2M_1^2\right)}{D\left(4M_1^2M_2^2-M_{12,3}^2\right)}\,,\\
F_5^{(0,1)} &= \frac{2(D-2)M_{12,3}s_{0'1}}{D\left(M_{12,3}^2-4M_1^2M_2^2\right)}\,, \qquad F_5^{(1,0)} = \frac{4(D-2)M_1^2s_{01}}{D\left(4M_1^2M_2^2-M_{12,3}^2\right)}\,,\\
F_6^{(0,1)} &= \frac{4(D-2)M_2^2s_{0'1}}{D\left(4M_1^2M_2^2-M_{12,3}^2\right)}\,, \qquad F_6^{(1,0)} = \frac{2(D-2)M_{12,3}s_{01}}{D\left(M_{12,3}^2-4M_1^2M_2^2\right)}\,,\\
F_7^{(0,1)} &= \frac{2(D-2)s_{0'1}\left(2M_2^2-M_{12,3}\right)}{D\left(M_{12,3}^2-4M_1^2M_2^2\right)}\,, \quad F_7^{(1,0)} = \frac{2(D-2)s_{01}\left(2M_1^2-M_{12,3}\right)}{D\left(M_{12,3}^2-4M_1^2M_2^2\right)}\,.
\end{aligned}
\tag{3.59}
$$

In the Mathematica file in the supplementary material we have present reduction coefficients for other tensor ranks up to rank level two.

4 Self-consistence

From the previous section we see that although the decomposition coefficients $\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}}$ are computed using the tensor reduction results in (3.15), by definition in (3.10), it should be independent of the auxiliary R and the tensor rank used in the formula. Thus to show the correctness of the formula (3.15), we should prove this point.

More explicitly, there are two facts we want to prove. The first one is that to get $\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}}$ up to $|G|^k$ order, we require m in (3.15) to satisfy condition $m > k$. One can choose different m_i . As long as $m_i > k$, result given by (3.15) should be the same up to $|G|^k$. The second one is that although at the r.h.s. of (3.15), $C^{(m)}_{n+1\to n+1;\widehat{\mathbf{b}_j}}$, $C^{(m)}_{n+1\to n+1}$ are functions of R, as long as $m > k$, the series of $|G|$ should be independent of R up to $|G|^k$.

In this section, we will prove these crucial consistencies using two different methods.

4.1 Proof with recursion relation

The crucial input of our proof is the rather non-trivial relation for one-loop integrals[7]

$$
\begin{aligned}
I_n^{(m)} &= \frac{1}{|G|}\left[A_mI_n^{(m-1)}+B_mI_n^{(m-2)}+\text{Lower Terms}\right]\\
&= \frac{1}{|G|}\left[A_mI_n^{(m-1)}+B_mI_n^{(m-2)}+\mathcal{O}(1)\right]\,.
\end{aligned}
\tag{4.1}
$$

[7]This relation is observed in [41]. Here we rephrase it in the language of projective space [39].

JHEP08(2022)110

where "Lower Terms" means the contributions from lower topologies, we also denote it as $\mathcal{O}(1)$ for it contains no poles of $|G|$, and

$$\begin{aligned} A_m &= a_m(VQ^*L) = \frac{2(D+2m-n-2)}{D+m-n-1}(VQ^*L)\,, \\ B_m &= b_m\left[(\det Q)R^2 - VQ^*V)\right] = \frac{4(m-1)((\det Q)R^2 - VQ^*V)}{D+m-n-1}\,. \end{aligned} \tag{4.2}$$

Here Q^* is the adjoint matrix of Q with component $Q_{ij} \equiv \frac{M_i^2+M_j^2-(K_i-K_j)^2}{2}, i,j = 0,\ldots,n-1, K_0 = 0$. The two vectors in the expression of A_m, B_m are $L = \{1,1,\ldots,1\}$, $V = \{0, s_{01}, s_{02}, \ldots, s_{0,n-1}\}$. One can prove the relation (4.1) by showing that it satisfies the $\mathcal{D}$-type and $\mathcal{T}$-type differential relations.[8] First, acting with $\mathcal{D}$-type operator, we have

$$\mathcal{D}_i I_n^{(m)} = m(f_i I_n^{(m-1)} + I_{n;\widehat{0}}^{(m)} - I_{n;\widehat{i}}^{(m)}) = m f_i I_n^{(m-1)} + \mathcal{O}(1) \tag{4.3}$$

at one side, while applying $\mathcal{D}_i$ on the r.h.s. of (4.1), we find

$$\begin{aligned} &\frac{1}{|G|}\mathcal{D}_i\left[A_m I_n^{(m-1)} + B_m I_n^{(m-2)} + \mathcal{O}(1)\right] \\ &= \frac{1}{|G|}\left[(\mathcal{D}_i A_m) I_n^{(m-1)} + ((m-1)f_i + \mathcal{D}_i B_m) I_n^{(m-2)} + B_m(m-2) f_i I_n^{(m-3)} + \mathcal{O}(1)\right] \end{aligned} \tag{4.4}$$

at another side. Identifying (4.4) and (4.3), we have

$$I_n^{(m-1)} = \frac{1}{m f_i\,|G| - \mathcal{D}_i A_m}\left[((m-1)f_i + \mathcal{D}_i B_m) I_n^{(m-2)} + B_m(m-2) f_i I_n^{(m-3)} + \mathcal{O}(1)\right]. \tag{4.5}$$

By our recursive assumption of the form (4.1), we just need to prove

$$\begin{aligned} (m f_i\,|G| - \mathcal{D}_i A_m) A_{m-1} &= |G|\,((m-1)f_i + \mathcal{D}_i B_m), \\ (m f_i\,|G| - \mathcal{D}_i A_m) B_{m-1} &= |G|\,B_m(m-2) f_i\,, \end{aligned} \tag{4.6}$$

which can be checked using the identities

$$\begin{aligned} \mathcal{D}_i A_m &= \frac{2(D+2m-n-2)}{D+m-n-1}\sum_{ab} s_{ia} Q^*_{ab} = \frac{2(D+2m-n-2)}{D+m-n-1} f_i\,|G|\,, \\ \mathcal{D}_i B_m &= \frac{4(m-1)(2(\det Q)s_{0i} - 2\sum_{jk} s_{ik} Q^*_{kj} s_{0j})}{D+m-n-1} = \frac{4(m-1) f_i (VQ^*L)}{D+m-n-1}\,. \end{aligned} \tag{4.7}$$

Next we consider the $\mathcal{T}$ operator. First, we have

$$\mathcal{T} I_n^{(m)} = 4m(m-1)M_0^2 I_n^{(m-2)} + 4m(m-2) I_{n;\widehat{0}}^{(m-2)} = 4m(m-1)M_0^2 I_n^{(m-2)} + \mathcal{O}(1)\,. \tag{4.8}$$

[8] Since we have shown the reduction coefficients can be completely determined by $\mathcal{D}$-type and $\mathcal{T}$-type differential relations in [1, 2], checking relation (4.1) satisfies these two recursions will be sufficient for our proof.

JHEP08(2022)110

Acting with $\mathcal{T}$ on the r.h.s. of (4.1), we get

$$\begin{aligned}
&\frac{\mathcal{T}}{|G|}\left[A_m I_n^{(m-1)} + B_m I_n^{(m-2)} + \mathcal{O}(1)\right] \\
&= \frac{1}{|G|}\left[2a_m(m-1)\sum_{i,j} Q_{ij}^* f_i + 2(D+2m-4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^*\right] I_n^{(m-2)} \\
&\quad + \frac{1}{|G|}\left[4(m-1)(m-2)M_0^2 A_m - 4b_m(m-2)\sum_{i,j=1}^{n-1} f_i Q_{ij}^* s_{0j}\right] I_n^{(m-3)} \\
&\quad + \frac{1}{|G|}\left[4b_m(m-2)(m-3)M_0^2(\det(Q)R^2 - (VQ^*V))\right] I_n^{(m-4)}
\end{aligned} \tag{4.9}$$

where we have used following algebraic results

$$\begin{aligned}
\mathcal{T}\left[(VQ^*L)I_n^{(m)}\right] &= 2\sum_{i,j} Q_{ij}^* m f_i I_n^{(m-1)} + 4m(m-1)M_0^2 (VQ^*L) I_n^{(m-2)} + \mathcal{O}(1)\,, \\
\mathcal{T}\left[R^2 I_n^{(m)}\right] &= 2(D+2m) I_n^{(m)} + 4m(m-1)M_0^2 R^2 I_n^{(m-2)} + \mathcal{O}(1)\,, \\
\mathcal{T}\left[(VQ^*V)I_n^{(m)}\right] &= 2\sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^* I_n^{(m)} + 4m \sum_{i,j=1}^{n-1} f_i Q_{ij}^* s_{0j} I_n^{(m-1)} \\
&\quad + 4m(m-1)M_0^2 (VQ^*V) I_n^{(m-2)} + \mathcal{O}(1)\,.
\end{aligned} \tag{4.10}$$

Comparing both sides, we just need to prove

$$\begin{aligned}
&4m(m-1)M_0^2\,|G| \\
&- \left[2a_m(m-1)\sum_{i,j} Q_{ij}^* f_i + 2(D+2m-4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^*\right] A_{m-2} \\
&= \frac{-4b_m(m-2)}{|G|} \sum_{i,j=1}^{n-1} f_i Q_{ij}^* s_{0j}
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&4m(m-1)M_0^2\,|G| \\
&- \left[2a_m(m-1)\sum_{i,j} Q_{ij}^* f_i + 2(D+2m-4)b_m \det(Q) - 2b_m \sum_{i,j=1}^{n-1} s_{ij} Q_{ij}^*\right] B_{m-2} \\
&= \left[4a_m(m-1)(m-2)M_0^2(VQ^*L) + 4b_m(m-2)(m-3)M_0^2(\det(Q)R^2 - (VQ^*V))\right].
\end{aligned} \tag{4.12}$$

One can check the two equations after some algebra.

Now we employ (4.1) to prove self-consistence, i.e., to show that

$$\mathcal{B}_{n;\widehat{\mathbf{b}_j}} = -\frac{C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n\to n}} + \mathcal{O}(|G|^m) \tag{4.13}$$

JHEP08(2022)110

is independent of the rank choice m and R. First let us show the independence of rank, which is equivalent to the relation

$$\frac{C^{(m+1)}_{n\to n;\widehat{\mathbf{b}_j}}}{C^{(m+1)}_{n\to n}} - \frac{C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n\to n}} = \mathcal{O}(|G|^m)\,. \tag{4.14}$$

To see it, first we rewrite (4.14) to

$$\frac{C^{(m+1)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m+1)}_{n\to n}}{C^{(m)}_{n\to n} C^{(m+1)}_{n\to n}} = \mathcal{O}(|G|^m)\,. \tag{4.15}$$

Using the divergence behavior of denominator

$$C^{(m)}_{n\to n} C^{(m+1)}_{n\to n} = \mathcal{O}(|G|^{-2m-1})\,, \tag{4.16}$$

we just need to show the leading divergent behavior of numerator is

$$C^{(m+1)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m+1)}_{n\to n} = \mathcal{O}(|G|^{-m-1})\,. \tag{4.17}$$

To prove it, we assume the equation holds for all $m < m'$, while $m' = 1, 2$ can be checked explicitly, then we prove it holds for $m = m'$. Using (4.1), the l.h.s. of (4.17) becomes

$$\begin{aligned}
&C^{(m'+1)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m')}_{n\to n} - C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m'+1)}_{n\to n} \\
&\frac{1}{|G|}\left[A_{m'+1} C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} + B_{m'+1} C^{(m'-1)}_{n\to n;\widehat{\mathbf{b}_j}} + \mathcal{O}(1)\right] C^{(m')}_{n\to n} \\
&- C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} \frac{1}{|G|}\left[A_{m'+1} C^{(m')}_{n\to n} + B_{m'+1} C^{(m'-1)}_{n\to n} + \mathcal{O}(1)\right] \\
&= \frac{B_{m'+1}}{|G|}\left[C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m'-1)}_{n\to n} - C^{(m'-1)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m')}_{n\to n}\right] + \mathcal{O}(|G|^{-1}) \\
&= \mathcal{O}(|G|^{-m'-1}) + \mathcal{O}(|G|^{-1}) = \mathcal{O}(|G|^{-m'-1})\,.
\end{aligned} \tag{4.18}$$

Having shown the independence of the rank choice, now we prove the first m terms in (4.13) don't rely on R, i.e., the derivation over R is zero up to order $\mathcal{O}(|G|^m)$:

$$\frac{\partial}{\partial R^\mu}\left[\frac{C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}}}{C^{(m)}_{n\to n}}\right] = \frac{\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n}}{(C^{(m)}_{n\to n})^2} \leq \mathcal{O}(|G|^m)\,. \tag{4.19}$$

Similarly, since the leading divergent behavior is $|G|^{-2m}$, we just need to show that

$$\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n} \leq \mathcal{O}(|G|^{-m})\,. \tag{4.20}$$

Due to reduction coefficients are functions of s_{0i}, s_{00}, after acting $\frac{\partial}{\partial R^\mu}$, the l.h.s. of (4.20) can be wrote as the form

$$L^\mu \equiv \sum_{i=1}^{n-1} \widetilde{\alpha}_i K_i^\mu + \widetilde{\alpha}_n R^\mu\,. \tag{4.21}$$

JHEP08(2022)110

Noticing the limit $|G| \to 0$ means these K_i are not linearly independent, thus the limit can be parameterized as

$$K_1 = \sum_{i=2}^{n-1} a_i K_i + tK_0 \tag{4.22}$$

where t is the same order of $|G|$ and K_0 is another linear independent momentum. Putting it back to (4.21) we get

$$L^\mu = \sum_{i=2}^{n-1} (\widetilde{\alpha}_i + \widetilde{\alpha}_1 a_i) K_i^\mu + \widetilde{\alpha}_1 t K_0^\mu + \widetilde{\alpha}_n R^\mu \equiv \sum_{i=2}^{n-1} \alpha_i K_i^\mu + \alpha_1 t K_0^\mu + \alpha_n R^\mu \,. \tag{4.23}$$

Let us contract L^μ with K_0, $K_i, i = 2, \ldots, n-1$ and R, we will get n equations in the matrix form as

$$G(K_0, K_2, \ldots, K_{n-1}, R) \begin{pmatrix} t\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{4.24}$$

where $G(K_0, K_2, \ldots, K_{n-1}, R)$ is the non-degenerate Gram matrix and

$$\begin{aligned} b_1 &= K_0^\mu \left[\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n} \right] \\ b_i &= K_i^\mu \left[\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n} \right] \\ &= \mathcal{D}_i C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \mathcal{D}_i C^{(m)}_{n\to n}, \quad i = 2, \ldots, n-1 \\ b_n &= R^\mu \left[\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n} \right] = 0 \end{aligned} \tag{4.25}$$

where we have used the homogeneity of R for b_n. Thus to show $t\alpha_1$ and all other $\alpha_i \leq O(|G|^{-m})$, we just need to show $b_i \leq \mathcal{O}(|G|^{-m})$.

We will prove it recursively by assuming the equation holds for all $m < m'$, where the $m' = 1, 2$ can be checked explicitly. For the case $m = m'$, using

$$\mathcal{D}_i C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} = m f_i C^{(m-1)}_{n\to n;\widehat{\mathbf{b}_j}} + \mathcal{O}(1) \,, \tag{4.26}$$

we have

$$\begin{aligned} b_i &= \mathcal{D}_i C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m')}_{n\to n} - C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} \mathcal{D}_i C^{(m')}_{n\to n} \\ &= m' f_i \left[C^{(m'-1)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m')}_{n\to n} - C^{(m')}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m'-1)}_{n\to n} \right] + \mathcal{O}(|G|^{-m'}) \\ &= \mathcal{O}(|G|^{-m'}) \end{aligned} \tag{4.27}$$

where in the second line we have used the result (4.17). Above argument has shown that $b_i \leq O(|G|^{-m})$ $i = 2, \ldots, n-1$ for $m = m'$. For b_1, we need to be more careful. Using (4.22), we have

$$b_1 = \frac{1}{t} \left(K_1 - \sum_{i=2}^{n-1} a_i K_i) \right) \left[\frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} C^{(m)}_{n\to n} - C^{(m)}_{n\to n;\widehat{\mathbf{b}_j}} \frac{\partial}{\partial R^\mu} C^{(m)}_{n\to n} \right] . \tag{4.28}$$

JHEP08(2022)110

Naively we have $\frac{1}{t}$ in the front, which will make the divergence higher. However, when $t \to 0$, the combination $\left(K_1 - \sum_{i=2}^{n-1} a_i K_i)\right) \sim t \to 0$, thus the divergent order estimation given in (4.27) is also true for b_1. Thus we finish the proof for R independence.

4.2 Another proof

Now we will give another proof from a different point of view. For simplification, we will constrict to the decomposition coefficient $\mathcal{B}_{n+1;\widehat{\mathbf{b}_j}}$ with $\mathbf{b}_j = \{r+1, \cdots, n\}$. For given tensor rank m, the reduction coefficients have the expansion

$$\begin{aligned} C^{(m)}_{n+1\to n+1;\widehat{r+1,r+2,\ldots,n}} &= \sum_{2a_0+\sum_{k=1}^{n} a_k = m} \left\{ c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m) (M_0^2)^{a_0+r-n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\} \\ C^{(m)}_{n+1\to n+1} &= \sum_{2a_0+\sum_{k=1}^{n} a_k = m} \left\{ c^{(0,1,\cdots,n)}_{a_1,\cdots,a_n}(m) (M_0^2)^{a_0+r-n} \prod_{k=0}^{n} s_{0k}^{a_k} \right\}. \end{aligned} \tag{4.29}$$

Thus we have the expansion of $\mathcal{F}$ in (3.12)

$$\begin{aligned} \mathcal{F}^{(m)}_{n+1\to n+1;\widehat{r+1,\ldots,n}} &= \sum_{2a_0+\sum a_i = m} \left\{ c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m) + c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m) \mathcal{B}_{n+1;\widehat{r+1,\ldots,n}} \right\} \\ &\quad \times (R\cdot R)^{a_0} \prod (R\cdot K_i)^{a_i} \\ &\equiv \sum_{2a_0+\sum a_i = m} \mathcal{F}^{(m)}_{n+1\to n+1;\widehat{r+1,\ldots,n}}(a_1,\ldots,a_n) \times (R\cdot R)^{a_0} \prod (R\cdot K_i)^{a_i}. \end{aligned} \tag{4.30}$$

Since $c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m) \sim |G|^{-\left(\frac{m+\sum a_i}{2}\right)}$, smoothness under the limit gives

$$\mathcal{B}_{n+1;\widehat{r+1,\ldots,n}} = -\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m)} + \mathcal{O}\left(|G|^{\frac{m+\sum a_i}{2}}\right). \tag{4.31}$$

The difference between (3.15) and (4.31) is that we have used expansion coefficients in (4.31) instead of the whole reduction coefficients in (3.15). Thus $\mathcal{B}_{n+1;\widehat{r+1,\ldots,n}}$ from (4.31) will be independent of R manifestly, but we need to show the same result will be obtained taking different a'_i and m. Overall, the irrelevancy of the tensor rank m and the expansion indices lead to following condition

$$\begin{aligned} &\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m)} - \frac{c^{(0,\ldots,r)}_{a'_1,\ldots,a'_n}(m')}{c^{(0,\ldots,n)}_{a'_1,\ldots,a'_n}(m')} \le |G|^{\left(\mathrm{Min}\left[\frac{m+\sum a_i}{2}, \frac{m'+\sum a'_i}{2}\right]\right)} \\ \Rightarrow\ & c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m) c^{(0,\ldots,n)}_{a'_1,\ldots,a'_n}(m') - c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m) c^{(0,\ldots,r)}_{a'_1,\ldots,a'_n}(m') \le |G|^{-\mathrm{Max}\left[\frac{m+\sum a_i}{2}, \frac{m'+\sum a'_i}{2}\right]} \end{aligned} \tag{4.32}$$

holds for any nonzero $c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m), c^{(0,\ldots,n)}_{a'_1,\ldots,a'_n}(m')$.

JHEP08(2022)110

Before to prove (4.32), let us list several useful relations. Using the $\mathcal{T}$ operator [1, 2] we have relation

$$\begin{aligned}4m(m+1)c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m-1) =& \left(m+1-\sum_{k=1}^{n}a_k\right)\left(D+m+\sum_{k=1}^{n}a_k-1\right)c^{(0,1,\cdots,r)}_{a_1,\cdots,a_n}(m+1)\\ &+\sum_{0<i<j}2(a_i+1)(a_j+1)\beta_{ij}c^{(0,1,\cdots,r)}_{a_1,\cdots,a_i+1,\cdots,a_j+1,\cdots,a_n}(m+1)\\ &+\sum_{a=1}^{n}(a_i+1)(a_i+2)\beta_{ii}c^{(0,1,\cdots,r)}_{a_1,\cdots,a_j+2,\cdots,a_n}(m+2).\end{aligned}\tag{4.33}$$

Using the $\mathcal{D}$-relation (2.12) twice, we have

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m+1)=\frac{(m+1)\left\{4m\cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)-\boldsymbol{\alpha}^T\widetilde{\boldsymbol{G}}^{-1}\boldsymbol{O}^{(0,\ldots,r)}(a_1,\ldots,a_n;m)\right\}}{(m+1-\sum_{k=1}^{n}a_k)(D+m-n-1)}.\tag{4.34}$$

where vector $\boldsymbol{O}^{(0,\ldots,r)}(a_1,\ldots,a_n;m)$ is given in (2.16). Combining with the component form of (2.12) we have

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)=\frac{1}{a_i+1}\sum_k\frac{\widetilde{\boldsymbol{G}}^*_{ik}}{|\widetilde{\boldsymbol{G}}|}[\boldsymbol{O}^{(0,\ldots,r)}(a_1,\ldots,a_n;m)]_k.\tag{4.35}$$

From (4.34) and (4.35) we can read

$$\begin{aligned}c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)=&\frac{N_i(a_1,\ldots,a_n;m)}{|G|}\left\{A_i\cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)+\sum_k B_{ik}\cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)\right\},\\ c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)=&\frac{N_i(a_1,\ldots,a_n;m)}{|G|}\left\{A_i\cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)+\sum_k B_{ik}\cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)\right\}\\ &+\frac{X^{\text{lower}}_i}{|G|},\\ c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1)=&\frac{M(a_1,\ldots,a_n;m)}{|G|}\left\{D\cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)+\sum_k E_k\cdot c^{(0,\ldots n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)\right\},\\ c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m+1)=&\frac{M(a_1,\ldots,a_n;m)}{|G|}\left\{D\cdot c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)+\sum_k E_k\cdot c^{(0,\ldots r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)\right\}\\ &+\frac{Y^{\text{lower}}}{|G|}.\end{aligned}\tag{4.36}$$

where A, B_k, D, E_k are finite under $|G|\sim|\widetilde{\boldsymbol{G}}|\to 0$. $X^{\text{lower}}_i, Y^{\text{lower}}$ are those contributions from the reduction of lower topologies, which are also finite under $|G|\to 0$. M_i, N are

JHEP08(2022)110

constants decided by $a_1,\ldots,a_n,m$. We also need to mention that under $|G|\to 0$:

$$\begin{aligned} A_i : A_j &= \left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{i\mu}\cdot\boldsymbol{\alpha}_\mu\right) : \left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{j\mu}\cdot\boldsymbol{\alpha}_\mu\right), \\ B_{ik_i} : B_{jk_2} &= \widetilde{\boldsymbol{G}}^*_{ik_1} : \widetilde{\boldsymbol{G}}^*_{ik_2}, \\ D : E_{k_1} : E_{k_2} &= \left(\sum_{\mu\nu} \boldsymbol{\alpha}_\mu\cdot\widetilde{\boldsymbol{G}}^*_{\mu\nu}\cdot\boldsymbol{\alpha}_\nu\right) : \left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{k_1\mu}\cdot\boldsymbol{\alpha}_\mu\right) : \left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{k_2\mu}\cdot\boldsymbol{\alpha}_\mu\right). \end{aligned} \tag{4.37}$$

There are also some useful properties of $\widetilde{\boldsymbol{G}}$. Since $|\widetilde{\boldsymbol{G}}|\to 0$ means the corank of $\widetilde{\boldsymbol{G}}$ is one, we have

$$\begin{aligned} \sum_\mu \widetilde{\boldsymbol{G}}^*_{i\mu}\cdot\boldsymbol{v_\mu}/\sum_\mu \widetilde{\boldsymbol{G}}^*_{j\mu}\cdot\boldsymbol{v_\mu} &= \widetilde{\boldsymbol{G}}^*_{ik}/\widetilde{\boldsymbol{G}}^*_{jk}, \\ \widetilde{\boldsymbol{G}}^*_{ik_1}\widetilde{\boldsymbol{G}}^*_{jk_2} - \widetilde{\boldsymbol{G}}^*_{ik_2}\widetilde{\boldsymbol{G}}^*_{jk_1} &= 0, \\ \left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{i\mu}\cdot\boldsymbol{v_\mu}\right)\cdot\left(\sum_\mu \widetilde{\boldsymbol{G}}^*_{j\mu}\cdot\boldsymbol{v_\mu}\right) &= \widetilde{\boldsymbol{G}}_{ij}\cdot\boldsymbol{\alpha}^T\widetilde{\boldsymbol{G}}^{-1}\boldsymbol{\alpha}. \end{aligned} \tag{4.38}$$

Having above preparations, we will prove (4.32) by induction with index $\kappa = \mathrm{Max}\left[\frac{m+\sum a_i}{2}, \frac{m'+\sum a'_i}{2}\right]$. Suppose it is true for $\kappa<\kappa'$, all we need to prove are following four relations:

- (A) With the same m but different a_i,a_j,a_0: $\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)} - \frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)} \leq |\widetilde{\boldsymbol{G}}|^{\left(\frac{m+\sum a_i-1}{2}\right)}$

  Proof of (A).
  Above relation is equal to

$$\begin{aligned} &c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m) - c^{(0,\ldots,r)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) \\ &\leq |\widetilde{\boldsymbol{G}}|^{\frac{m+\sum a_i+1}{2}}. \end{aligned} \tag{4.39}$$

  Using (4.36), we have

$$\begin{aligned} &c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m) - c^{(0,\ldots,r)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) \\ &= \frac{N_i(a_1,\ldots,a_n;m)}{|\widetilde{\boldsymbol{G}}|} \\ &\times\Bigg\{A_i\cdot[c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m) - c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)c^{(0,\ldots,r)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)] \\ &+\sum_k B_{ik}\cdot[c^{(0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m) - c^{(0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)c^{(0,\ldots,r)}_{a_1,\ldots,a_j-1,\ldots,a_n}]\Bigg\} \\ &+\frac{X^{\mathrm{lower}}\cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_j-1,\ldots,a_n}(m)}{|\widetilde{\boldsymbol{G}}|} \leq \frac{1}{|\widetilde{\boldsymbol{G}}|}|\widetilde{\boldsymbol{G}}|^{\frac{m+\sum a_i-1}{2}} \sim |\widetilde{\boldsymbol{G}}|^{\frac{m+\sum a_i+1}{2}}. \end{aligned} \tag{4.40}$$

JHEP08(2022)110

Thus we have done the proof.

- (B) With same a_0, but different m, a_i: $\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)} - \frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)} \leq |\widetilde{\boldsymbol{G}}|^{\left(\frac{m+\sum a_i}{2}\right)}$

Proof of (B):
Change $c^{(0,\ldots,r)/(0,\ldots,n)}_{a_1,\ldots a_j-1,\ldots,a_m}(m)$ in (4.40) into $c^{(0,\ldots,r)/(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)$, we will reach the proof of (B).

- (C) With the same m, a_0 but different a_i, a_j: $\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)} - \frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m)} \leq |\widetilde{\boldsymbol{G}}|^{\left(\frac{m+\sum a_i+1}{2}\right)}$.

Proof of (C):
By (4.36), we have

$$
\begin{aligned}
&c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m) - c^{(0,\ldots,r)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)\\
&= \frac{N_i(a_1,\ldots,a_n;m)N_j(a_1,\ldots,a_n;m)}{|\widetilde{\boldsymbol{G}}|^2} \times \Bigg\{ \sum_k [A_iB_{jk} - A_jB_{ik}]\\
&\quad \times [c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)c^{0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m) - c^{0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)]\\
&\quad + \sum_{k_1<k_2} [B_{ik_1}B_{jk_2} - B_{jk_1}B_{ik_2}]\\
&\quad \times [c^{(0,\ldots,r)}_{a_1,\ldots,a_{k_1}-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_{k_2}-1,\ldots,a_n}(m) - c^{(0,\ldots,r)}_{a_1,\ldots,a_{k_2-1},\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_{k_1}-1,\ldots,a_n}(m)] \Bigg\}\\
&\quad + \frac{1}{|\widetilde{\boldsymbol{G}}|} \times \left\{ X^{\text{lower}}_i C^{(0,\ldots,n)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m) - X^{\text{lower}}_j c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) \right\}.
\end{aligned}
\tag{4.41}
$$

From (4.37) and (4.38), we know

$$
\lim_{|\widetilde{\boldsymbol{G}}|\to 0} (A_iB_{jk} - A_jB_{ik}) = \lim_{|\widetilde{\boldsymbol{G}}|\to 0} (B_{ik_1}B_{jk_2} - B_{jk_1}B_{ik_2}) = 0. \tag{4.42}
$$

For the last line of (4.41), using the following expression of X^{lower}_i and $c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)$

$$
\begin{aligned}
c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) &= \frac{1}{(a_i+1)|\widetilde{\boldsymbol{G}}|} \sum_\mu \widetilde{\boldsymbol{G}}^*_{i\mu} \cdot \boldsymbol{O}^{(0,\ldots,n)}_\mu(a_1,\ldots,a_n;m),\\
X^{\text{lower}}_i &= -\frac{m}{a_i+1} \sum_\mu \widetilde{\boldsymbol{G}}^*_{i\mu} \cdot \delta_{0a_\mu} c^{(0,\ldots,r)}_{a1,\ldots,\widehat{a_\mu},\ldots,a_n}(m-1;\widehat{\mu}),
\end{aligned}
\tag{4.43}
$$

then by the first line of (4.38), we know

$$
\begin{aligned}
&X^{\text{lower}}_i C^{(0,\ldots,n)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m) - X^{\text{lower}}_j c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)\\
&= 0 \cdot |\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k+1}{2}} + \mathcal{O}\left(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\right).
\end{aligned}
\tag{4.44}
$$

JHEP08(2022)110

Thus we have

$$c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m) - c^{(0,\ldots,r)}_{a_1,\ldots,a_j+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)$$
$$\leq \frac{1}{|\widetilde{\boldsymbol{G}}|^2}\cdot\mathcal{O}(|\widetilde{\boldsymbol{G}}|)\cdot\mathcal{O}\Big(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\Big)+\frac{1}{|\widetilde{\boldsymbol{G}}|}\cdot\mathcal{O}\Big(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\Big)\sim|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k+1}{2}}. \tag{4.45}$$

- (D) With the same a_0, but different m, a_i: $\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)}-\frac{c^{(0,\ldots,r)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1)}{c^{(0,\ldots,n)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1)}\leq|\widetilde{\boldsymbol{G}}|^{\frac{m+\sum a_i+1}{2}}$

Proof of (D):

By (4.36), we have

$$\begin{aligned}
&c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1)-c^{(0,\ldots,r)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1)c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)\\
&=\frac{N_i(a_1,\ldots,a_n;m)M(a_1,\ldots,a_n;m)}{|\widetilde{\boldsymbol{G}}|^2}\times\Bigg\{\sum_{k_1<k_2}[E_{k_1}B_{ik_2}-E_{k_2}B_{ik_1}]\\
&\quad\times[c^{(0,\ldots,r)}_{a_1,\ldots,a_{k_1}-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_{k_2}-1,\ldots,a_n}(m)-c^{(0,\ldots,r)}_{a_1,\ldots,a_{k_2}-1,\ldots,a_n}(m)c^{(0,\ldots,n)}_{a_1,\ldots,a_{k_1}-1,\ldots,a_n}(m)]\\
&\quad+\sum_k[A_iE_k-D\cdot B_{ik}]\\
&\quad\times[c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)c^{(0,\ldots,r)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)-c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m-1)c^{(0,\ldots,n)}_{a_1,\ldots,a_k-1,\ldots,a_n}(m)]\Bigg\}\\
&\quad+\frac{1}{|\widetilde{\boldsymbol{G}}|}[Y^{\text{lower}}c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m)-X^{\text{lower}}_ic^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1)].
\end{aligned}\tag{4.46}$$

Similar to the proof of (C), we have

$$\lim_{|\widetilde{\boldsymbol{G}}|\to0}(E_{k_1}B_{ik_2}-E_{k_2}B_{ik_1})=\lim_{|\widetilde{\boldsymbol{G}}|\to0}(A_iE_k-D\cdot B_{ik})=0. \tag{4.47}$$

Then using the following expressions for Y^{lower} and $c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1)$

$$\begin{aligned}
Y^{\text{lower}}=&\frac{m(m+1)}{(m+1-\sum a_\alpha)(D+m-n-1)}\\
&\cdot\sum_\mu\widetilde{\boldsymbol{G}}^*_{i\mu}\cdot\delta_{0a_\mu}c^{(0,\ldots,r)}_{a1,\ldots,\widehat{a_\mu},\ldots,a_n}(m-1;\widehat{\mu})\\
c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1)=&\frac{m+1}{(m+1-\sum a_\alpha)(D+m-n-1)|\widetilde{\boldsymbol{G}}|}\times\\
&\Bigg\{4m|\widetilde{\boldsymbol{G}}|\cdot c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m-1)-\sum_{\mu\nu}\alpha_\nu\widetilde{\boldsymbol{G}}^*_{\mu\nu}O^{(0,\ldots,n)}_\nu(a_1,\ldots,a_n;m)\Bigg\}
\end{aligned}\tag{4.48}$$

JHEP08(2022)110

and combining with (4.43) and the property (4.38), the leading order of last line of (4.46) is

$$\begin{aligned} &Y^{\text{lower}} c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) - X_i^{\text{lower}} c^{(0,\ldots,n)}_{a_1,\ldots,a_n}(m+1) \\ &= 0 \cdot |\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k+1}{2}} + \mathcal{O}\Big(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\Big). \end{aligned} \tag{4.49}$$

thus we arrive

$$\begin{aligned} &c^{(0,\ldots,r)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) c^{(0,\ldots,n)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1) - c^{(0,\ldots,r)}_{a_1,\ldots,a_i,\ldots,a_n}(m+1) c^{(0,\ldots,n)}_{a_1,\ldots,a_i+1,\ldots,a_n}(m) \\ &\leq \frac{1}{|\widetilde{\boldsymbol{G}}|^2} \cdot \mathcal{O}(|\widetilde{\boldsymbol{G}}|) \cdot \mathcal{O}\Big(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\Big) + \frac{1}{|\widetilde{\boldsymbol{G}}|} \cdot \mathcal{O}\Big(|\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k-1}{2}}\Big) \sim |\widetilde{\boldsymbol{G}}|^{-\frac{m+\sum a_k+1}{2}}. \end{aligned} \tag{4.50}$$

Having above four relations (A)(B)(C)(D), we can prove (4.32) iteratively. Eventually, we have proved that $\mathcal{B}_{n+1;\mathbf{b}_j}$ is independent of the rank m and R by the recursion relation of $c^{(0,\ldots,r)}_{a_1,\ldots,a_n}(m)$.

5 Discussion

In this paper, we show how to do the reduction for one-loop tensor integrals when Gram determinant degenerates. In this case, the highest topological master basis will be reduced to an expansion of the basis with lower topologies. By demanding the cancellation of divergent parts, we can solve the decomposition coefficients. We have proved that the results are independent of the tensor rank m and the auxiliary vector R, thus shown our method's self-consistency.

One advantage of our method is that we have the series expansion around the degenerate point $|G|$, which will be helpful for some situations, for example, the improvement of numerical accuracy. The same idea can also be used to deal with other kinds of singularities, such as soft limits or massless limits. Furthermore, as the improved PV-reduction method has the applied to two-loop sunset topology recently in [41], we show that the same method works well for the sunset topology in the limit $K^2 \to 0$ as given in the section 3.4.

Acknowledgments

This work is supported by Chinese NSF funding under Grant No.11935013, No.11947301, No.12047502 (Peng Huanwu Center).

A Expansion for degenerate Gram determinant

In this appendix, we will give the decomposition for box and pentagon as the series of $|G|$. Since the analytic expressions with higher and higher orders of $|G|$ are complicated, we will present only numerical results to demonstrate the independence of decomposition coefficients for the rank choice and R.

JHEP08(2022)110

In following, we list some numerical results for box and pentagon. Here we set $K_0 = 0$, and choose the Gram Matrices as:

$$\text{box} : \mathbf{G} = \frac{1}{71}\begin{pmatrix} 2 & 5 & & 11 \\ 5 & 17 & & 23 \\ 11 & 23 & 65 + \frac{357911}{9}t & \end{pmatrix},$$
$$\text{pentagon} : \mathbf{G} = \frac{1}{71}\begin{pmatrix} 2 & 5 & 11 & & 41 \\ 5 & 17 & 23 & & 47 \\ 11 & 23 & 31 & & 59 \\ 41 & 47 & 59 & \frac{1283}{17} + \frac{25411681}{306}t & \end{pmatrix}, \tag{A.1}$$

where $\mathbf{G}_{ij} = s_{ij} = K_i \cdot K_j$. With this parameterization, we have $\det(\mathbf{G}) = t$. The chosen squares of masses are

$$\text{box} : \left(M_0^2,\, M_1^2,\, M_2^2,\, M_3^2\right) = \frac{1}{71}\left(3,\, 7,\, 13,\, 19\right),$$
$$\text{pentagon} : \left(M_0^2,\, M_1^2,\, M_2^2,\, M_3^2,\, M_4^2\right) = \frac{1}{71}\left(3,\, 7,\, 13,\, 19,\, 29\right). \tag{A.2}$$

Firstly we present some reduction coefficients needed in (3.15) and (3.18). For $I_4^{(2)}$ we have

$$\begin{aligned} C_{4\to4}^{(2)} = {} & \frac{s_{03}s_{01}}{384300851763(D-3)t^2} \times (-12041426688574Dt^2 - 32933538576Dt - \\ & 20155392D + 29975466437514t^2 + 73134102096t + 40310784) \\ & + \frac{s_{03}^2}{128100283921(D-3)t^2} \times (128100283921Dt^2 + 463852656Dt + \\ & 419904D - 256200567842t^2 - 856838934t - 839808) \\ & + \frac{s_{02}s_{03}}{384300851763(D-3)t^2} \times (2818206246262Dt^2 + 6493937184Dt + \\ & 2519424D - 7686017035260t^2 - 16273497348t - 5038848) \\ & + \frac{s_{01}^2}{10376122997601(D-3)t^2} \times (2546761744633401Dt^2 + \\ & 4709032163712Dt + 2176782336D + 779242512213633527t^3 - \\ & 3722722351028181t^2 - 8800212589632t - 4353564672) \\ & + \frac{s_{02}s_{01}}{10376122997601(D-3)t^2} \times (-1192101242168826Dt^2 - \\ & 1690742931120Dt - 544195584D - 45848500718449031 0t^3 + \\ & 1496467516765122t^2 + 2876350319856t + 1088391168) \\ & + \frac{s_{02}^2}{10376122997601(D-3)t^2} \times (139501209189969Dt^2 + \\ & 137764238832Dt + 34012224D + 9169700143689806 2t^3 - \\ & 253638562163 58t^2 - 69403953654t - 68024448) \\ & + \frac{s_{00}\left(-128100283921t^2 - 534719034t - 419904\right)}{228705129(D-3)t}, \end{aligned} \tag{A.3a}$$

JHEP08(2022)110

$$
\begin{aligned}
C^{(2)}_{4\to 4;\widehat{3}} = & \frac{s_{02}s_{01}}{16238064159(D-3)t^2}\times\big(2145318534Dt+839808D+\\
& 1281002839210t^2-3652839666t-1679616\big)+\frac{(357911t+648)s_{00}}{3t57911(D-3)}\\
& +\frac{s_{03}s_{01}}{1804229351(D-3)t^2}\times(100930902Dt+93312D-251253522t-\\
& 186624)+\frac{s_{02}^2}{16238064159(D-3)t^2}\times\big(-183608343Dt-52488D-\\
& 256200567842t^2+57981582t+104976\big)+\frac{s_{01}^2}{16238064159(D-3)t^2}\times\\
& \big(-5411614320Dt-3359232D-2177704826657t^2+10436684760t+\\
& 6718464\big)+\frac{s_{03}^2(-3221199Dt-5832D+6442398t+11664)}{1804229351(D-3)t^2}\\
& +\frac{s_{02}s_{03}(-23622126Dt-11664D+64423980t+23328)}{1804229351(D-3)t^2}\,, \qquad \text{(A.3b)}\\
C^{(2)}_{4\to 4;\widehat{23}} = & \frac{s_{01}^2(48317985t+10368)}{71t(715822t+81)}-\frac{32s_{02}s_{01}}{71t}+\frac{s_{03}s_{01}(-7158220t-2592)}{71t(715822t+81)}+\frac{2s_{02}^2}{71t}\\
& +\frac{162s_{03}^2}{71t(715822t+81)}+\frac{4s_{02}s_{03}}{71t}\,. \qquad \text{(A.3c)}
\end{aligned}
$$

For $I^{(2)}_5$, we have

$$
\begin{aligned}
C^{(2)}_{5\to 5} = & \frac{s_{04}s_{01}}{32933430093533811(D-4)t^2}\times\big(143357283936558942Dt^2+23554798645968Dt+\\
& 801715968D-514019810871625756t^2-65771935818336t-2405147904\big)\\
& +\frac{s_{04}^2}{645753531245761(D-4)t^2}\times\big(645753531245761Dt^2+150030564624Dt+\\
& 8714304D-19372605937372283t^2-2909129240881t-26142912\big)\\
& +\frac{s_{00}}{552094181406(D-4)t}\times\big(645753531245761t^2+309209334408t+8714304\big)\\
& +\frac{s_{02}s_{04}}{32933430093533811(D-4)t^2}\times\big(484315148434320750Dt^2+100670508862704Dt+\\
& 5158867968D-1554974503239792488t^2-219630109481280t-15476603904\big)\\
& -\frac{s_{03}s_{04}}{109778100311779377(D-4)t^2}\times\big(167895918123897860Dt^2+36157366074384Dt+\\
& 1934575488D-528226388559032498t^2-75984992058960t-5803726464\big)\\
& +\frac{s_{01}^2}{302328888258640384 98(D-4)t^2}\times\big(143213926652622383058Dt^2+\\
& 13789009133462592Dt+331910410752D-164096827406408111342 41t^3-\\
& 5561177750805994071 12t^2-436003165183034 88t-995731232256\big)\\
& +\frac{s_{02}s_{01}}{151164444129320192 49(D-4)t^2}\times\big(483830833285886429250Dt^2+\\
& 67656883239451296Dt+2135771338752D+804074454291399745577809t^3-\\
& 139203797223586207248 0t^2-183993225050551104t-6407314016256\big)
\end{aligned}
$$

JHEP08(2022)110

$$
\begin{aligned}
&+\frac{s_{03}s_{01}}{1679604934770224361(D-4)t^2}\times\big(-55909340735257987380Dt^2-\\
&823712808955147 2Dt-266971417344D-65638730962563244536964t^3+\\
&17242394188499313613 2t^2+225897947357423 04t+800914252032\big)\\
&+\frac{s_{02}^2}{6046577765172807699 6(D-4)t^2}\times\big(32691272519316650625 00Dt^2+\\
&5995221362375040 00Dt+27486448533504D-96817128169780785692021 9t^3-\\
&1150328292667973757309 6t^2-1435853145074044608t-82459345600512\big)\\
&+\frac{s_{03}^2}{6718419739080897444(D-4)t^2}\times\big(39287644840992099240 0Dt^2+\\
&7793787771087552 0Dt+3865281825024D+16409682740640811134241t^3-\\
&1277220411366240783636t^2-174378410197442208t-11595845475072\big)\\
&+\frac{s_{02}s_{03}}{3359209869540448722(D-4)t^2}\times\big(-37776581577877018500 0Dt^2-\\
&721091902752331 20Dt-3435806066688D+16409682740640811134241t^3+\\
&1264318255811950478856t^2+166745623518606528t+10307418200064\big)\,,
\end{aligned}
\tag{A.4a}
$$

$$
\begin{aligned}
C^{(2)}_{5\to 5;\widehat{4}}=&\frac{s_{01}^2}{1391553384233823(D-4)t^2}\times\big(3703193448768Dt+112435776D-\\
&645753531245761t^2-13919908971456t-337307328\big)\\
&+\frac{s_{02}s_{01}}{1391553384233823(D-4)t^2}\times\big(33381800628840Dt+1446999552D+\\
&632838460620845 78t^2-113719508702928t-4340998656\big)\\
&+\frac{s_{03}s_{01}}{154617042692647(D-4)t^2}\times\big(-4023685569540Dt-180874944D-\\
&516602824996608 8t^2+13937595501432t+542624832\big)\\
&+\frac{s_{04}s_{01}(33848359092Dt+1629504D-121366188456t-4888512)}{9095120158391(D-4)t^2}\\
&+\frac{s_{02}^2}{2783106768467646(D-4)t^2}\times\big(122937240222144Dt+9311127552D-\\
&380994583434998 99t^2-4160219482151 28t-27933382656\big)\\
&+\frac{s_{03}^2}{309234085385294(D-4)t^2}\times\big(15130216514124Dt+1309377312D+\\
&645753531245761t^2-49174347196548t-3928131936\big)\\
&+\frac{s_{04}^2(7775974386Dt+903312D-23327923158t-2709936)}{9095120158391(D-4)t^2}\\
&+\frac{s_{02}s_{03}}{154617042692647(D-4)t^2}\times\big(-14408118186828Dt-1163890944D+\\
&645753531245761t^2+47688373738392t+3491672832\big)\\
&+\frac{s_{02}s_{04}(114352564500Dt+10485504D-367147967088t-31456512)}{9095120158391(D-4)t^2}\\
&+\frac{s_{03}s_{04}(-118926667080Dt-11796192D+374161591044t+35388576)}{9095120158391(D-4)t^2}\\
&+\frac{s_{00}(25411681t+2952)}{25411681(D-4)t}\,,
\end{aligned}
\tag{A.4b}
$$

JHEP08(2022)110

$$
\begin{aligned}
C^{(2)}_{5\to 5;\widehat{34}} = &\frac{s_{01}^2(1568612244768t-8455536)}{85697t(25411681t+443556)}+\frac{s_{02}s_{01}(-760215848796t-108819072)}{85697t(25411681t+443556)}\\
&+\frac{276s_{03}s_{01}}{85697t}-\frac{999s_{03}^2}{85697t}+\frac{1776s_{02}s_{03}}{85697t}+\frac{s_{04}s_{01}(-3659282064t-1102896)}{5041t(25411681t+443556)}\\
&+\frac{s_{02}^2(70517414775t-350113536)}{85697t(25411681t+443556)}-\frac{611388s_{04}^2}{5041t(25411681t+443556)}\\
&+\frac{s_{02}s_{04}(4574102 58t-7096896)}{5041t(25411681t+443556)}+\frac{18s_{03}s_{04}}{5041t}\,.
\end{aligned}
\tag{A.4c}
$$

When $t=\det(\mathbf{G})\to 0$, the basis with the highest typology is decomposed to the combinations of lower ones. Let us consider the coefficients $\mathcal{B}_{4;\widehat{3}}$ and $\mathcal{B}_{5;\widehat{4}}$ by (3.15) for different m:

- $\mathcal{B}_{4;\widehat{3}}$
  - $m=2$:

$$
\mathcal{B}_{4;\widehat{3}}=-\frac{C^{(2)}_{4\to 4;\widehat{3}}}{C^{(2)}_{4\to 4}}+\mathcal{O}\left(t^2\right)=\frac{71}{72}-\frac{25411681(36D-61)t}{1679616(D-2)}+\mathcal{O}\left(t^2\right)\,. \tag{A.5}
$$

  - $m=3$:

$$
\begin{aligned}
\mathcal{B}_{4;\widehat{3}}=&-\frac{C^{(3)}_{4\to 4;\widehat{3}}}{C^{(3)}_{4\to 4}}+\mathcal{O}\left(t^3\right)=\frac{71}{72}-\frac{25411681(36D-61)t}{1679616(D-2)}\\
&+\frac{9095120158391\left(432D^2-468D+121\right)t^2}{13060694016((D-2)D)}+\mathcal{O}\left(t^3\right)\,.
\end{aligned}
\tag{A.6}
$$

  - $m=4$:

$$
\begin{aligned}
\mathcal{B}_{4;\widehat{3}}=&-\frac{C^{(4)}_{4\to 4;\widehat{3}}}{C^{(4)}_{4\to 4}}+\mathcal{O}\left(t^4\right)=\frac{71}{72}-\frac{25411681(36D-61)t}{1679616(D-2)}\\
&+\frac{9095120158391\left(432D^2-468D+121\right)t^2}{13060694016((D-2)D)}\\
&-\frac{3255243551009881201\left(15552D^3+28512D^2+16596D+50215\right)t^3}{304679870005248D\left(D^2-4\right)}\\
&+\mathcal{O}\left(t^4\right)\,.
\end{aligned}
\tag{A.7}
$$

- $\mathcal{B}_{5;\widehat{4}}$
  - $m=2$:

$$
\mathcal{B}_{5;\widehat{4}}=-\frac{C^{(2)}_{5\to 5;\widehat{4}}}{C^{(2)}_{5\to 5}}+\mathcal{O}\left(t^2\right)=\frac{1207}{164}-\frac{30671898967(41D-36)t}{19849248(D-3)}+\mathcal{O}\left(t^2\right)\,. \tag{A.8}
$$

JHEP08(2022)110

order / rank	t^0	t^1	t^2	t^3
$m=2$	$-\frac{71}{72}$	$\frac{2109169523}{3359232}$	/	/
$m=3$	$-\frac{71}{72}$	$\frac{2109169523}{3359232}$	$-\frac{46939915137455951}{104485552128}$	/
$m=4$	$-\frac{71}{72}$	$\frac{2109169523}{3359232}$	$-\frac{46939915137455951}{104485552128}$	$\frac{5104609261966063899030919}{14624633760251904}$

Table 1. $\mathcal{B}_{4;\widehat{3}}$.

order / rank	t^0	t^1	t^2	t^3
$m=2$	$\frac{1207}{164}$	$-\frac{122687595868}{620289}$	/	/
$m=3$	$\frac{1207}{164}$	$-\frac{122687595868}{620289}$	$\frac{15550298443174202497177}{2402394183936}$	/
$m=4$	$\frac{1207}{164}$	$-\frac{122687595868}{620289}$	$\frac{15550298443174202497177}{2402394183936}$	$-\frac{625201865608355590831056722887 37}{290766572870141952}$

Table 2. $\mathcal{B}_{5;\widehat{4}}$.

– $m=3$:

$$\mathcal{B}_{5;\widehat{4}} = -\frac{C^{(3)}_{5\to 5;\widehat{4}}}{C^{(3)}_{5\to 5}} + \mathcal{O}\left(t^3\right) = \frac{1207}{164} - \frac{30671898967(41D-36)t}{19849248(D-3)} + \frac{779424512213633527\left(1681D^2+3977D+17049\right)t^2}{2402394183936(D-3)(D-1)} + \mathcal{O}\left(t^3\right). \tag{A.9}$$

– $m=4$:

$$\mathcal{B}_{5;\widehat{4}} = -\frac{C^{(4)}_{5\to 5;\widehat{4}}}{C^{(4)}_{5\to 5}} + \mathcal{O}\left(t^4\right) = \frac{1207}{164} - \frac{30671898967(41D-36)t}{19849248(D-3)} + \frac{779424512213633527\left(1681D^2+3977D+17049\right)t^2}{2402394183936(D-3)(D-1)} - \frac{19806487067953459039028887t^3}{290766572870141952((D-3)(D-1)(D+1))} \times \left(68921D^3+670719D^2+4586014D+13861761\right) + \mathcal{O}\left(t^4\right). \tag{A.10}$$

To make them look more intuitive, we choose $D=4$, and here we list some results in tables 1 and 2.

From these numerical results, one can see that as the series of t, $\mathcal{B}_{4;\widehat{3}}$ and $\mathcal{B}_{5;\widehat{4}}$ are independent of the choice of rank m. And they are also independent of the auxiliary vector R, although $s_{0i}=R\cdot K_i$ and $s_{00}=R\cdot R$ appear in the expression of $C^{(3)}_{4\to 4;\widehat{3}}$, $C^{(3)}_{4\to 4}$, etc.

JHEP08(2022)110

We can also use (3.18) to derive $F^{(m)}_{n+1;\widehat{\mathbf{b}_j}}$. Some results are:

$$
\begin{aligned}
F^{(3)}_{4;\widehat{3}} = &\frac{\left(-1239447D^2-1774432D+652916\right)s_{03}s_{01}^2}{23328D\left(D^2-4\right)}+\frac{(-197593D-206184)s_{03}^2s_{01}}{46656D\left(D^2-4\right)}\\
&+\frac{\left(862224D^2+1492491D-274912\right)s_{02}s_{03}s_{01}}{23328D\left(D^2-4\right)}-\frac{94501s_{03}^3}{559872D\left(D^2-4\right)}\\
&+\frac{(274912D+455323)s_{02}s_{03}^2}{186624D\left(D^2-4\right)}+\frac{\left(-1199616D^2-2674144D-644325\right)s_{02}^2s_{03}}{186624D\left(D^2-4\right)}\\
&+\frac{\left(-7774713D^3-9053352D^2+13507608D+2542936\right)s_{01}^3}{17496D\left(D^2-4\right)}\\
&+\frac{\left(10816992D^3+12443673D^2-19600544D-3195852\right)s_{02}s_{01}^2}{23328D\left(D^2-4\right)}\\
&+\frac{\left(-7524864D^3-10046784D^2+11389039D+2955304\right)s_{02}^2s_{01}}{46656D\left(D^2-4\right)}\\
&+\frac{\left(10469376D^3+17994240D^2-7538496D-3393445\right)s_{02}^3}{559872D\left(D^2-4\right)}\\
&+\frac{(759D-242)s_{00}s_{01}}{108(D-2)D}+\frac{(121-1056D)s_{00}s_{02}}{432(D-2)D}+\frac{121s_{00}s_{03}}{432(D-2)D}\,,
\end{aligned}
\tag{A.11a}
$$

$$
\begin{aligned}
F^{(3)}_{4;\widehat{23}} = &\frac{\left(-222655929D^2+225327659D+51781152\right)s_{02}s_{03}s_{01}}{2916D\left(D^2-4\right)}\\
&+\frac{\left(159295600D^2-46664537D-85762533\right)s_{02}s_{03}^2}{23328D\left(D^2-4\right)}+\frac{(28045D-22791)s_{00}s_{03}}{54(D-2)D}\\
&+\frac{\left(887216D^2-259919001D+121362075\right)s_{02}^2s_{03}}{23328D\left(D^2-4\right)}+\frac{(781D-22791)s_{00}s_{02}}{54(D-2)D}\\
&+\frac{\left(1053906747D^3-444555708D^2-186234704D-478975656\right)s_{01}^3}{2187D\left(D^2-4\right)}\\
&+\frac{\left(-8000067D^3+607223737D^2-1031570076D+601955892\right)s_{02}s_{01}^2}{2916D\left(D^2-4\right)}\\
&+\frac{\left(-752051667D^3+144661577D^2+657043940D-122980236\right)s_{03}s_{01}^2}{2916D\left(D^2-4\right)}\\
&+\frac{\left(5565264D^3+9169579D^2+362503351D-556647384\right)s_{02}^2s_{01}}{5832D\left(D^2-4\right)}\\
&+\frac{\left(269794320D^3+32842115D^2-349326177D+38835864\right)s_{03}^2s_{01}}{5832D\left(D^2-4\right)}\\
&+\frac{\left(-7742976D^3-15727920D^2-3271609D+639173595\right)s_{02}^3}{69984D\left(D^2-4\right)}\\
&+\frac{\left(-193574400D^3-97997040D^2+264828935D+17799771\right)s_{03}^3}{69984D\left(D^2-4\right)}\\
&+\frac{(91164-79378D)s_{00}s_{01}}{27(D-2)D}\,,
\end{aligned}
\tag{A.11b}
$$

JHEP08(2022)110

$$
\begin{aligned}
F^{(3)}_{4;\widehat{123}} = & \frac{\left(117036897D^2+80887886D+52970828\right)s_{01}^3}{486D(D+2)} - \frac{5041 s_{00}s_{01}}{3D} \\
& + \frac{\left(-8231953D^2+40086032D-66571446\right)s_{02}s_{01}^2}{648D(D+2)} + \frac{5041 s_{00}s_{02}}{24D} \\
& + \frac{\left(-79814153D^2-62992336D+13600618\right)s_{03}s_{01}^2}{648D(D+2)} + \frac{5041 s_{00}s_{03}}{24D} \\
& + \frac{\left(22906304D^2+79098331D+246242768\right)s_{02}^2 s_{01}}{5184D(D+2)} \\
& + \frac{\left(114531520D^2+130637515D-17179728\right)s_{03}^2 s_{01}}{5184D(D+2)} \\
& + \frac{\left(-1206875892D^2-5693290277D-9613489460\right)s_{02}^3}{2115072D(D+2)} \\
& + \frac{\left(-10269182412D^2-15210859589D-1023625460\right)s_{03}^3}{8087040D(D+2)} \\
& + \frac{\left(-10021508D^2+107731211D+75877132\right)s_{02}s_{03}^2}{41472D(D+2)} \\
& + \frac{\left(10021508D^2+56192027D-107373300\right)s_{02}^2 s_{03}}{41472D(D+2)} \\
& + \frac{(-101288813D-22906304)s_{02}s_{03}s_{01}}{2592D(D+2)}\,, \qquad \text{(A.11c)}
\end{aligned}
$$

$$
\begin{aligned}
F^{(3)}_{5;\widehat{4}} = & \frac{s_{01}^3}{88197652332(D-3)\left(D^2-1\right)} \times \big(-1343847503375D^3+10417729271235D^2- \\
& 17337620658673D-1794280981875\big) + \frac{s_{02}s_{01}^2}{29399217444(D-3)\left(D^2-1\right)} \times \\
& \big(-1633291581025D^3+12077972337027D^2-68514754134455D+ \\
& 93475183460901\big) + \frac{s_{03}s_{01}^2}{9799739148(D-3)\left(D^2-1\right)} \times \big(392816962525D^3 \\
& -4603678621302D^2+27041270862119D-33850184121582\big) \\
& + \frac{\left(14623481275D^2-89044148172D+94194083969\right)s_{04}s_{01}^2}{192151748(D-3)\left(D^2-1\right)} \\
& + \frac{s_{02}^2 s_{01}}{29399217444(D-3)\left(D^2-1\right)} \times \big(-1985077460015D^3+30582780834201D^2- \\
& 243862525327933D+854168652203259\big) + \frac{(580029-231855D)s_{00}s_{01}}{28577(D-3)(D-1)} \\
& + \frac{(5019844059-1432168335D)s_{04}^2 s_{01}}{5651522(D-3)\left(D^2-1\right)} + \frac{s_{03}^2 s_{01}}{3266579716(D-3)\left(D^2-1\right)} \times \\
& \big(-114823419815D^3+2531886930939D^2-34017449607547D+ \\
& 122326329454647\big) + \frac{s_{02}s_{03}s_{01}}{4899869574(D-3)\left(D^2-1\right)} \times \big(477423692915D^3- \\
& 8829762425970D^2+90569439300937D-324098013014706\big) \\
& + \frac{\left(17773154165D^2-276974011536D+978449946715\right)s_{02}s_{04}s_{01}}{96075874(D-3)\left(D^2-1\right)}
\end{aligned}
$$

JHEP08(2022)110

$$+\frac{\left(-12823668195D^2+313094031390D-1106429051883\right)s_{03}s_{04}s_{01}}{96075874(D-3)\left(D^2-1\right)}$$
$$+\frac{s_{02}^3}{88197652332(D-3)\left(D^2-1\right)}\times\big(-2412632605249D^3+76689368763465D^2-$$
$$1007290897198703D+6188766114569991\big)+\frac{s_{03}^3}{3266579716(D-3)\left(D^2-1\right)}\times$$
$$\big(33563768869D^3-1300394788050D^2+30312717181523D-352004287844976\big)$$
$$+\frac{2384439363s_{04}^3}{5651522(D-3)\left(D^2-1\right)}+\frac{s_{02}s_{03}^2}{3266579716(D-3)\left(D^2-1\right)}\times$$
$$\big(-139554617929D^3+5018030800863D^2-98440205350997D+$$
$$919471445452659\big)+\frac{(39776661783-1740635361D)s_{02}s_{04}^2}{5651522(D-3)\left(D^2-1\right)}$$
$$+\frac{(1255901463D-45451057824)s_{03}s_{04}^2}{5651522(D-3)\left(D^2-1\right)}+\frac{(2522217-281793D)s_{00}s_{02}}{28577(D-3)(D-1)}$$
$$+\frac{s_{02}^2s_{03}}{9799739148(D-3)\left(D^2-1\right)}\times\big(580253411389D^3-19518756588588D^2+$$
$$316580122515497D-2391505497411054\big)+\frac{(203319D-2723796)s_{00}s_{03}}{28577(D-3)(D-1)}$$
$$+\frac{\left(21601218139D^2-722503219422D+7448642198855\right)s_{02}^2s_{04}}{192151748(D-3)\left(D^2-1\right)}$$
$$+\frac{\left(11245370571D^2-552200043258D+9805499547885\right)s_{03}^2s_{04}}{192151748(D-3)\left(D^2-1\right)}$$
$$+\frac{\left(-15585689037D^2+639672478488D-8561581798443\right)s_{02}s_{03}s_{04}}{96075874(D-3)\left(D^2-1\right)}$$
$$+\frac{22707s_{00}s_{04}}{1681(D-3)(D-1)}\,, \tag{A.11d}$$

$$F^{(3)}_{5;\widehat{34}}=\frac{s_{01}^3}{36248334956758812852(D-3)\left(D^2-1\right)}\times\big(-829943177997755662953 7D^3+$$
$$22726783007443171457382D^2+12750306686537356621621D-$$
$$25457369478990513141690\big)+\frac{s_{02}s_{01}^2}{81640394046754083(D-3)\left(D^2-1\right)}\times$$
$$\big(2847559757703331789 9D^3-906076046925498827 10D^2+$$
$$1071851353796273667 77D-85969113572839070718\big)$$
$$+\frac{s_{03}s_{01}^2}{72569239152670296(D-3)\left(D^2-1\right)}\times\big(-5435043687063540225D^3+$$
$$34743610215547035316D^2-13370364909404029514 7D+$$
$$13835351482253987674 16\big)+\frac{s_{04}s_{01}^2}{473834443879200168(D-3)\left(D^2-1\right)}\times$$
$$\big(20277430345339806 25D^3-47755513091825901372D^2+$$
$$14394505490676887919 5D-11850914018892087691 2\big)$$
$$+\frac{s_{02}^2s_{01}}{4412994272797518(D-3)\left(D^2-1\right)}\times\big(95273008629079876D^3-$$
$$54590077794224849D^2+34251474265548951092D-68798575805871096567\big)$$

JHEP08(2022)110

$$+\frac{s_{03}^2 s_{01}}{217925643101112(D-3)\,(D^2-1)}\times\big(14312658358171485D^3-$$
$$308183885271005346D^2+2120141865244853789D-4345132269866178904\big)$$
$$+\frac{s_{04}^2 s_{01}}{9290871448611768(D-3)\,(D^2-1)}\times\big(-8402381828589725D^3-$$
$$228340458232788810D^2+2175040781006931191D-4392791821329079248\big)$$
$$+\frac{(20997586789D-30664084565)s_{00}s_{01}}{1906477614(D-3)(D-1)}+\frac{s_{02}s_{03}s_{01}}{980665393955004(D-3)\,(D^2-1)}\times$$
$$\big(-178531580572981155D^3+27607447924355845 46D^2-$$
$$16899392445525068877D+34463496611388041614\big)$$
$$+\frac{s_{02}s_{04}s_{01}}{6403168160529732(D-3)\,(D^2-1)}\times\big(44888698974740075D^3-$$
$$22814816765125941 42D^2+17131774468846926949D-34844020585587415026\big)$$
$$+\frac{s_{03}s_{04}s_{01}}{355731564473874(D-3)\,(D^2-1)}\times\big(134329873732898840D^2-$$
$$1075850083783373373D+2183705381853832549\big)$$
$$+\frac{s_{02}^3}{178905173221521(D-3)\,(D^2-1)}\times\big(10367549408524951D^3-$$
$$2443952492174483 76D^2+24932179826901414 65D-6695874076061568024\big)$$
$$-\frac{s_{03}^3}{654431360664(D-3)\,(D^2-1)}\times\big(12563664070467D^3-480258600687861D^2+$$
$$11117705113357806D-37576216670725964\big)+\frac{(286248073-76106817D)s_{00}s_{03}}{5725158(D-3)(D-1)}$$
$$+\frac{s_{04}^3}{9290871448611768(D-3)\,(D^2-1)}\times\big(591889764258953D^3+$$
$$18615866319928209D^2+4380177956395647 94D-2088530127407051460\big)$$
$$+\frac{s_{02}s_{03}^2}{5889882245976(D-3)\,(D^2-1)}\times\big(470145534426423D^3-$$
$$16661705329346940D^2+27990496279149825 7D-883445043516938732\big)$$
$$+\frac{s_{02}s_{04}^2}{251104633746264(D-3)\,(D^2-1)}\times\big(-186005811461743D^3-$$
$$122956210936788D^2+2387756626662638 95D-941769244106471412\big)$$
$$+\frac{\big(-378033081851837D^2-28380809735275779D+119557455493102352\big)\,s_{03}s_{04}^2}{27900514860696(D-3)\,(D^2-1)}$$
$$+\frac{s_{02}^2 s_{03}}{53008940213784(D-3)\,(D^2-1)}\times\big(-5864446929424329D^3+$$
$$18307874648786232 4D^2-2359327728622119111D+6895583935338277052\big)$$
$$+\frac{s_{02}^2 s_{04}}{346117197866472(D-3)\,(D^2-1)}\times\big(993713336121961D^3-$$
$$10474391761875915 6D^2+21918398979696013 19D-7146277328453200380\big)$$
$$+\frac{\big(-467242505501283D^2+30910888602825155D-116182101038199856\big)\,s_{03}^2 s_{04}}{4273051825512(D-3)\,(D^2-1)}$$

JHEP08(2022)110

$$+\frac{\left(1961957761174202D^2-64920756785680566D+228237749746197820\right)s_{02}s_{03}s_{04}}{4807183303701(D-3)\left(D^2-1\right)}$$
$$+\frac{(64066921D-274208177)s_{00}s_{04}}{37381914(D-3)(D-1)}+\frac{(268004836D-1108569860)s_{00}s_{02}}{25763211(D-3)(D-1)}\,, \qquad \text{(A.11e)}$$

$$F^{(3)}_{5;\widehat{234}}=\frac{s_{01}^3}{16806910847891893632\left(D^2-1\right)}\times\big(459792839690092743005 59D^2+$$
$$1047032357312884852901D-46656230678599317469812\big)$$
$$+\frac{s_{02}s_{01}^2}{33767531499168\left(D^2-1\right)}\times\big(-74104146458248832D^2+$$
$$29506403105945531D+36745597196241525\big)$$
$$+\frac{s_{03}s_{01}^2}{1770919429734144\left(D^2-1\right)}\times\big(-90008546282088374 7D^2-$$
$$1780257440638046161D+3064860211892055028\big)$$
$$+\frac{\left(16379947016073091D^2+598222244103749657D-688591797903427356\right)s_{04}s_{01}^2}{3723697983359232\left(D^2-1\right)}$$
$$+\frac{\left(-46832281406738D^2+487441428989732D-919312210645983\right)s_{02}^2s_{01}}{114079498308\left(D^2-1\right)}$$
$$+\frac{\left(-5962412169596571D^2+57241427343714135D-108319299169698748\right)s_{03}^2s_{01}}{10636152731136\left(D^2-1\right)}$$
$$+\frac{\left(-9882296771733875D^2+317362498527208511D-611771709694482252\right)s_{04}^2s_{01}}{2482465322239488\left(D^2-1\right)}$$
$$+\frac{\left(188034785817840D^2-970303447937245D+1748833608329821\right)s_{02}s_{03}s_{01}}{101403998496\left(D^2-1\right)}$$
$$+\frac{\left(-114853210406800D^2+975040680624469D-1817277951053637\right)s_{02}s_{04}s_{01}}{662108460768\left(D^2-1\right)}$$
$$+\frac{\left(66634569689337D^2-976812872646037D+1863724399749532\right)s_{03}s_{04}s_{01}}{588540854016\left(D^2-1\right)}$$
$$+\frac{\left(-2271838998767D^2+36591900808730D-87535035935325\right)s_{02}^3}{4624844526\left(D^2-1\right)}$$
$$+\frac{\left(11671435834413D^2-1899510805030945D+5546345527961140\right)s_{03}^3}{191642391552\left(D^2-1\right)}$$
$$+\frac{\left(16098192433184611D^2+144789759004827209D-565364111032193004\right)s_{04}^3}{4964930644478976\left(D^2-1\right)}$$
$$+\frac{\left(-990342600288D^2+33466366766187D-91942066376347\right)s_{02}s_{03}^2}{1218066048\left(D^2-1\right)}$$
$$+\frac{\left(16181230755488D^2+500766696393835D-1664193721989627\right)s_{02}s_{04}^2}{882811281024\left(D^2-1\right)}$$
$$+\frac{\left(-31681674345033D^2-481937156164675D+1698523705808332\right)s_{03}s_{04}^2}{784721138688\left(D^2-1\right)}$$
$$+\frac{\left(772076303514D^2-17465983786156D+44961327781741\right)s_{02}^2s_{03}}{685162152\left(D^2-1\right)}$$
$$+\frac{\left(-317817093958D^2+16432322501716D-46449248275737\right)s_{02}^2s_{04}}{4473705816\left(D^2-1\right)}$$

JHEP08(2022)110

$$
\begin{aligned}
&+\frac{\left(829966618209D^2+30291424183827D-96937318177204\right)s_{03}^2 s_{04}}{7069559808\left(D^2-1\right)}\\
&+\frac{(94917907364339-31480868378483D)s_{02}s_{03}s_{04}}{3976627392\left(D^2-1\right)}-\frac{115943s_{00}s_{01}}{49284(D-1)}\\
&-\frac{5041s_{00}s_{02}}{333(D-1)}+\frac{5041s_{00}s_{03}}{296(D-1)}-\frac{85697s_{00}s_{04}}{32856(D-1)}\,,
\end{aligned}
\tag{A.11f}
$$

$$
F^{(3)}_{5;\widehat{1234}}=\frac{25411681(D-2)\left(46s_{01}+296s_{02}-333s_{03}+51s_{04}\right)^3}{13384745856(D+1)}\,.
\tag{A.11g}
$$

These results have been checked with FIRE6 [42].

B Divergence of reduction coefficients of two-loop sunset topology

In this part, we give the related C_i, C_j coefficients for rank level 3 with the chosen tensor structure, which have been used to solve $\mathcal{B}$ by (3.46).

- The pole of s_{11}^{-3}

$$
\left.\begin{pmatrix} C_2^{(3,0)}\\ C_3^{(3,0)}\\ C_4^{(3,0)}\end{pmatrix}\right|_{s_{11}^{-3}}=\begin{pmatrix} \frac{(D+2)\left(4(1-2D)M_2^2M_{12,3}+(3D-2)M_{12,3}^2+8(D-1)M_1^2M_2^2\right)s_{01}^3}{(D-1)(3D-2)}\\ \frac{2(D+2)M_1^2\left((5D-4)M_{12,3}+4(1-2D)M_2^2\right)s_{01}^3}{(D-1)(3D-2)}\\ -\frac{2(D+2)\left((4D-2)M_{12,3}+(D-2)M_1^2+4(1-2D)M_2^2\right)s_{01}^3}{(D-1)(3D-2)}\end{pmatrix},
$$

$$
\left.\begin{pmatrix} C_2^{(2,1)}\\ C_3^{(2,1)}\\ C_4^{(2,1)}\end{pmatrix}\right|_{s_{11}^{-3}}=\begin{pmatrix} \frac{2(D+2)M_2^2\left((2D-1)M_{12,3}-2(D-1)M_1^2\right)s_{01}^2s_{0'1}}{(D-1)(3D-2)}\\ -\frac{2(D+2)M_1^2\left((D-1)M_{12,3}+2(1-2D)M_2^2\right)s_{01}^2s_{0'1}}{(D-1)(3D-2)}\\ \frac{(D+2)\left(DM_{12,3}+4(D-1)M_1^2+4(1-2D)M_2^2\right)s_{01}^2s_{0'1}}{(D-1)(3D-2)}\end{pmatrix},
$$

$$
\left.\begin{pmatrix} C_2^{(1,2)}\\ C_3^{(1,2)}\\ C_4^{(1,2)}\end{pmatrix}\right|_{s_{11}^{-3}}=\begin{pmatrix} \frac{2(D+2)M_2^2\left((4D-2)M_1^2-(D-1)M_{12,3}\right)s_{01}s_{0'1}^2}{(D-1)(3D-2)}\\ -\frac{2(D+2)M_1^2\left((1-2D)M_{12,3}+2(D-1)M_2^2\right)s_{01}s_{0'1}^2}{(D-1)(3D-2)}\\ -\frac{(D+2)\left(-DM_{12,3}+(8D-4)M_1^2-4(D-1)M_2^2\right)s_{01}s_{0'1}^2}{(D-1)(3D-2)}\end{pmatrix},
$$

$$
\left.\begin{pmatrix} C_2^{(0,3)}\\ C_3^{(0,3)}\\ C_4^{(0,3)}\end{pmatrix}\right|_{s_{11}^{-3}}=\begin{pmatrix} -\frac{2(D+2)M_2^2\left((4-5D)M_{12,3}+(8D-4)M_1^2\right)s_{0'1}^3}{(D-1)(3D-2)}\\ \frac{(D+2)\left(4M_1^2\left((1-2D)M_{12,3}+2(D-1)M_2^2\right)+(3D-2)M_{12,3}^2\right)s_{0'1}^3}{(D-1)(3D-2)}\\ \frac{2(D+2)\left(2(1-2D)M_{12,3}+(8D-4)M_1^2-(D-2)M_2^2\right)s_{0'1}^3}{(D-1)(3D-2)}\end{pmatrix}.
\tag{B.1}
$$

$$
\left.\begin{pmatrix} C_1^{(r_1,r_2)}\\ C_5^{(r_1,r_2)}\\ C_6^{(r_1,r_2)}\\ C_7^{(r_1,r_2)}\end{pmatrix}\right|_{s_{11}^{-3}}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\forall r_1+r_2=3\,.
\tag{B.2}
$$

JHEP08(2022)110

- The pole of s_{11}^{-2}

$$\left(\begin{array}{c} C_2^{(3,0)} \\ C_3^{(3,0)} \\ C_4^{(3,0)} \end{array}\right)\Bigg|_{s_{01}^3 s_{11}^{-2}} = \left(\begin{array}{c} \frac{4\left(\left(-3D^2-5D+2\right)M_3^2+D(5D-2)M_1^2-D(D+2)M_2^2\right)}{(D-1)(3D-2)} \\ \frac{2(D+2)\left(2(5D-4)M_1^2+(4-5D)M_2^2-3DM_3^2\right)}{(D-1)(3D-2)} \\ -\frac{2\left(5D^2+6D-8\right)}{3D^2-5D+2} \end{array}\right),$$

$$\left(\begin{array}{c} C_2^{(2,1)} \\ C_3^{(2,1)} \\ C_4^{(2,1)} \end{array}\right)\Bigg|_{s_{01}^2 s_{0'1} s_{11}^{-2}} = \left(\begin{array}{c} -\frac{2(D+2)\left((2D-1)M_1^2+(2-4D)M_2^2-M_3^2\right)}{(D-1)(3D-2)} \\ \frac{2\left(\left(D^2-16D+12\right)M_1^2+(D+2)\left((4D-3)M_2^2+M_3^2\right)\right)}{(D-1)(3D-2)} \\ -\frac{D^2-10D+8}{3D^2-5D+2} \end{array}\right),$$

$$\left(\begin{array}{c} C_2^{(1,2)} \\ C_3^{(1,2)} \\ C_4^{(1,2)} \end{array}\right)\Bigg|_{s_{01} s_{0'1}^2 s_{11}^{-2}} = \left(\begin{array}{c} \frac{2\left(\left(4D^2+5D-6\right)M_1^2+\left(D^2-16D+12\right)M_2^2+(D+2)M_3^2\right)}{(D-1)(3D-2)} \\ \frac{2(D+2)\left((4D-2)M_1^2+(1-2D)M_2^2+M_3^2\right)}{(D-1)(3D-2)} \\ -\frac{D^2-10D+8}{3D^2-5D+2} \end{array}\right),$$

$$\left(\begin{array}{c} C_2^{(0,3)} \\ C_3^{(0,3)} \\ C_4^{(0,3)} \end{array}\right)\Bigg|_{s_{0'1}^3 s_{11}^{-2}} = \left(\begin{array}{c} -\frac{2(D+2)\left((5D-4)M_1^2+(8-10D)M_2^2+3DM_3^2\right)}{(D-1)(3D-2)} \\ -\frac{4\left(\left(3D^2+5D-2\right)M_3^2+D(D+2)M_1^2+(2-5D)DM_2^2\right)}{(D-1)(3D-2)} \\ -\frac{2\left(5D^2+6D-8\right)}{3D^2-5D+2} \end{array}\right). \tag{B.3}$$

$$\left(\begin{array}{c} C_1^{(3,0)} \\ C_5^{(3,0)} \\ C_6^{(3,0)} \\ C_7^{(3,0)} \end{array}\right)\Bigg|_{s_{01}^3 s_{11}^{-2}} = \left(\begin{array}{c} -\frac{2(D+2)\left((7D-4)M_1^4+2M_1^2\left((D-4)M_2^2+(2-5D)M_3^2\right)-\left(M_2^2-M_3^2\right)\left((5D-4)M_2^2+3DM_3^2\right)\right)}{(D-1)(3D-2)} \\ -\frac{4(D+2)(2D-1)\left(DM_1^2-2\left(M_2^2+M_3^2\right)\right)}{(D-1)D(3D-2)} \\ -\frac{2(D+2)\left(\left(3D^2+2D-4\right)M_1^2+D\left((4-5D)M_2^2-3DM_3^2\right)\right)}{(D-1)D(3D-2)} \\ \frac{2(D+2)\left(\left(7D^2-10D+4\right)M_1^2+D\left((4-5D)M_2^2-3DM_3^2\right)\right)}{(D-1)D(3D-2)} \end{array}\right),$$

$$\left(\begin{array}{c} C_1^{(2,1)} \\ C_5^{(2,1)} \\ C_6^{(2,1)} \\ C_7^{(2,1)} \end{array}\right)\Bigg|_{s_{01}^2 s_{0'1} s_{11}^{-2}} = \left(\begin{array}{c} \frac{2(D+2)\left((2D-1)M_1^4-2M_1^2\left((D+1)M_2^2+DM_3^2\right)-\left(M_2^2-M_3^2\right)\left((4D-3)M_2^2+M_3^2\right)\right)}{(D-1)(3D-2)} \\ \frac{2(D+2)\left(D(2D-1)M_1^2+(4-7D)M_2^2-DM_3^2\right)}{(D-1)D(3D-2)} \\ -\frac{2(D+2)\left((4-5D)M_1^2+D\left((4D-3)M_2^2+M_3^2\right)\right)}{(D-1)D(3D-2)} \\ -\frac{2(D+2)\left((2D-1)M_1^2+(3-4D)M_2^2-M_3^2\right)}{(D-1)(3D-2)} \end{array}\right),$$

$$\left(\begin{array}{c} C_1^{(1,2)} \\ C_5^{(1,2)} \\ C_6^{(1,2)} \\ C_7^{(1,2)} \end{array}\right)\Bigg|_{s_{01} s_{0'1}^2 s_{11}^{-2}} = \left(\begin{array}{c} -\frac{2(D+2)\left((4D-3)M_1^4+2M_1^2\left((D+1)M_2^2-2(D-1)M_3^2\right)+(1-2D)M_2^4+2DM_2^2M_3^2-M_3^4\right)}{(D-1)(3D-2)} \\ -\frac{2(D+2)\left(D(4D-3)M_1^2+(4-5D)M_2^2+DM_3^2\right)}{(D-1)D(3D-2)} \\ -\frac{2(D+2)\left((7D-4)M_1^2+D\left((1-2D)M_2^2+M_3^2\right)\right)}{(D-1)D(3D-2)} \\ \frac{2(D+2)\left((4D-3)M_1^2+(1-2D)M_2^2+M_3^2\right)}{(D-1)(3D-2)} \end{array}\right),$$

$$\left(\begin{array}{c} C_1^{(0,3)} \\ C_5^{(0,3)} \\ C_6^{(0,3)} \\ C_7^{(0,3)} \end{array}\right)\Bigg|_{s_{0'1}^3 s_{11}^{-2}} = \left(\begin{array}{c} \frac{2(D+2)\left((5D-4)M_1^4-2M_1^2\left((D-4)M_2^2+(D-2)M_3^2\right)-\left(M_2^2-M_3^2\right)\left((7D-4)M_2^2-3DM_3^2\right)\right)}{(D-1)(3D-2)} \\ \frac{2(D+2)\left(\left(-3D^2-2D+4\right)M_2^2+3D^2M_3^2+D(5D-4)M_1^2\right)}{(D-1)D(3D-2)} \\ -\frac{4(D+2)(2D-1)\left(DM_2^2-2M_1^2-2M_3^2\right)}{(D-1)D(3D-2)} \\ -\frac{2(D+2)\left(\left(-7D^2+10D-4\right)M_2^2+3D^2M_3^2+D(5D-4)M_1^2\right)}{(D-1)D(3D-2)} \end{array}\right). \tag{B.4}$$

JHEP08(2022)110

- The pole of s_{11}^{-1}

$$\left.\begin{pmatrix} C_2^{(3,0)} \\ C_3^{(3,0)} \\ C_4^{(3,0)} \end{pmatrix}\right|_{s_{01}^3 s_{11}^{-1}} = \begin{pmatrix} \frac{13D^2+16D-20}{(D-1)(3D-2)} \\ \frac{2\left(5D^2+6D-8\right)}{(D-1)(3D-2)} \\ 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} C_2^{(2,1)} \\ C_3^{(2,1)} \\ C_4^{(2,1)} \end{pmatrix}\right|_{s_{01}^2 s_{0'1} s_{11}^{-1}} = \begin{pmatrix} \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} C_2^{(1,2)} \\ C_3^{(1,2)} \\ C_4^{(1,2)} \end{pmatrix}\right|_{s_{01} s_{0'1}^2 s_{11}^{-1}} = \begin{pmatrix} \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ 0 \end{pmatrix},$$

$$\left.\begin{pmatrix} C_2^{(0,3)} \\ C_3^{(0,3)} \\ C_4^{(0,3)} \end{pmatrix}\right|_{s_{0'1}^3 s_{11}^{-1}} = \begin{pmatrix} \frac{2\left(5D^2+6D-8\right)}{(D-1)(3D-2)} \\ \frac{13D^2+16D-20}{(D-1)(3D-2)} \\ 0 \end{pmatrix}. \tag{B.5}$$

$$\left.\begin{pmatrix} C_1^{(3,0)} \\ C_5^{(3,0)} \\ C_6^{(3,0)} \\ C_7^{(3,0)} \end{pmatrix}\right|_{s_{01}^3 s_{11}^{-1}} = \begin{pmatrix} -\frac{8(D+2)\left((3D-2)M_1^2+(1-2D)M_3^2\right)}{(D-1)(3D-2)} \\ \frac{8D^2-60D+40}{3D^2-5D+2} \\ -\frac{2\left(5D^2+6D-8\right)}{3D^2-5D+2} \\ \frac{2\left(5D^2+6D-8\right)}{(D-1)(3D-2)} \end{pmatrix},$$

$$\left.\begin{pmatrix} C_1^{(2,1)} \\ C_5^{(2,1)} \\ C_6^{(2,1)} \\ C_7^{(2,1)} \end{pmatrix}\right|_{s_{01}^2 s_{0'1} s_{11}^{-1}} = \begin{pmatrix} \frac{2\left(2\left(D^2-5D+2\right)M_3^2+4(3D-2)M_1^2+2(2-3D)DM_2^2\right)}{(D-1)(3D-2)} \\ -\frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ -\frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \end{pmatrix},$$

$$\left.\begin{pmatrix} C_1^{(1,2)} \\ C_5^{(1,2)} \\ C_6^{(1,2)} \\ C_7^{(1,2)} \end{pmatrix}\right|_{s_{01} s_{0'1}^2 s_{11}^{-1}} = \begin{pmatrix} \frac{2\left(2\left(D^2-5D+2\right)M_3^2+2(2-3D)DM_1^2+4(3D-2)M_2^2\right)}{(D-1)(3D-2)} \\ -\frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ -\frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \\ \frac{2\left(2D^2-9D+6\right)}{3D^2-5D+2} \end{pmatrix},$$

$$\left.\begin{pmatrix} C_1^{(0,3)} \\ C_5^{(0,3)} \\ C_6^{(0,3)} \\ C_7^{(0,3)} \end{pmatrix}\right|_{s_{0'1}^3 s_{11}^{-1}} = \begin{pmatrix} -\frac{8(D+2)\left((3D-2)M_2^2+(1-2D)M_3^2\right)}{(D-1)(3D-2)} \\ -\frac{2\left(5D^2+6D-8\right)}{3D^2-5D+2} \\ \frac{8D^2-60D+40}{3D^2-5D+2} \\ \frac{2\left(5D^2+6D-8\right)}{(D-1)(3D-2)} \end{pmatrix}. \tag{B.6}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

JHEP08(2022)110

References

[1] B. Feng, T. Li and X. Li, *Analytic tadpole coefficients of one-loop integrals*, *JHEP* **09** (2021) 081 [arXiv:2107.03744] [INSPIRE].

[2] C. Hu, T. Li and X. Li, *One-loop Feynman integral reduction by differential operators*, *Phys. Rev. D* **104** (2021) 116014 [arXiv:2108.00772] [INSPIRE].

[3] B. Feng, T. Li, H. Wang and Y. Zhang, *Reduction of general one-loop integrals using auxiliary vector*, *JHEP* **05** (2022) 065 [arXiv:2203.14449] [INSPIRE].

[4] L.M. Brown and R.P. Feynman, *Radiative corrections to Compton scattering*, *Phys. Rev.* **85** (1952) 231 [INSPIRE].

[5] D.B. Melrose, *Reduction of Feynman diagrams*, *Nuovo Cim.* **40** (1965) 181 [INSPIRE].

[6] G. Passarino and M.J.G. Veltman, *One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model*, *Nucl. Phys. B* **160** (1979) 151 [INSPIRE].

[7] G. 't Hooft and M.J.G. Veltman, *Scalar One Loop Integrals*, *Nucl. Phys. B* **153** (1979) 365 [INSPIRE].

[8] W.L. van Neerven and J.A.M. Vermaseren, *Large loop integrals*, *Phys. Lett. B* **137** (1984) 241 [INSPIRE].

[9] R.G. Stuart, *Algebraic Reduction of One Loop Feynman Diagrams to Scalar Integrals*, *Comput. Phys. Commun.* **48** (1988) 367 [INSPIRE].

[10] G.J. van Oldenborgh and J.A.M. Vermaseren, *New Algorithms for One Loop Integrals*, *Z. Phys. C* **46** (1990) 425 [INSPIRE].

[11] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett. B* **302** (1993) 299 [*Erratum ibid.* **318** (1993) 649] [hep-ph/9212308] [INSPIRE].

[12] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys. B* **412** (1994) 751 [hep-ph/9306240] [INSPIRE].

[13] J. Fleischer, F. Jegerlehner and O.V. Tarasov, *Algebraic reduction of one loop Feynman graph amplitudes*, *Nucl. Phys. B* **566** (2000) 423 [hep-ph/9907327] [INSPIRE].

[14] T. Binoth, J.P. Guillet and G. Heinrich, *Reduction formalism for dimensionally regulated one loop N point integrals*, *Nucl. Phys. B* **572** (2000) 361 [hep-ph/9911342] [INSPIRE].

[15] A. Denner and S. Dittmaier, *Reduction of one loop tensor five point integrals*, *Nucl. Phys. B* **658** (2003) 175 [hep-ph/0212259] [INSPIRE].

[16] G. Duplancic and B. Nizic, *Reduction method for dimensionally regulated one loop N point Feynman integrals*, *Eur. Phys. J. C* **35** (2004) 105 [hep-ph/0303184] [INSPIRE].

[17] A. Denner and S. Dittmaier, *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys. B* **734** (2006) 62 [hep-ph/0509141] [INSPIRE].

[18] R.K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002 [arXiv:0712.1851] [INSPIRE].

[19] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys. B* **763** (2007) 147 [hep-ph/0609007] [INSPIRE].

[20] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys. B* **435** (1995) 59 [hep-ph/9409265] [INSPIRE].

JHEP08(2022)110

[21] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159 [INSPIRE].

[22] F.V. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, *Phys. Lett. B* **100** (1981) 65 [INSPIRE].

[23] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys. B* **763** (2007) 147 [hep-ph/0609007] [INSPIRE].

[24] G. Ossola, C.G. Papadopoulos and R. Pittau, *Numerical evaluation of six-photon amplitudes*, *JHEP* **07** (2007) 085 [arXiv:0704.1271] [INSPIRE].

[25] R.K. Ellis, W.T. Giele and Z. Kunszt, *A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes*, *JHEP* **03** (2008) 003 [arXiv:0708.2398] [INSPIRE].

[26] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys. B* **425** (1994) 217 [hep-ph/9403226] [INSPIRE].

[27] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys. B* **725** (2005) 275 [hep-th/0412103] [INSPIRE].

[28] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev. D* **72** (2005) 065012 [hep-ph/0503132] [INSPIRE].

[29] J.M. Campbell, E.W.N. Glover and D.J. Miller, *One loop tensor integrals in dimensional regularization*, *Nucl. Phys. B* **498** (1997) 397 [hep-ph/9612413] [INSPIRE].

[30] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop amplitudes for e^+e^- to four partons*, *Nucl. Phys. B* **513** (1998) 3 [hep-ph/9708239] [INSPIRE].

[31] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and Reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277] [INSPIRE].

[32] R. Britto and E. Mirabella, *Single Cut Integration*, *JHEP* **01** (2011) 135 [arXiv:1011.2344] [INSPIRE].

[33] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett. B* **645** (2007) 213 [hep-ph/0609191] [INSPIRE].

[34] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev. D* **75** (2007) 105006 [hep-ph/0612089] [INSPIRE].

[35] R. Britto and B. Feng, *Integral coefficients for one-loop amplitudes*, *JHEP* **02** (2008) 095 [arXiv:0711.4284] [INSPIRE].

[36] R. Britto, B. Feng and P. Mastrolia, *Closed-Form Decomposition of One-Loop Massive Amplitudes*, *Phys. Rev. D* **78** (2008) 025031 [arXiv:0803.1989] [INSPIRE].

[37] R. Britto, B. Feng and G. Yang, *Polynomial Structures in One-Loop Amplitudes*, *JHEP* **09** (2008) 089 [arXiv:0803.3147] [INSPIRE].

[38] B. Feng and H. Wang, *Analytic structure of one-loop coefficients*, *JHEP* **05** (2013) 104 [arXiv:1301.7510] [INSPIRE].

[39] B. Feng, J. Gong and T. Li, *Universal Treatment of Reduction for One-Loop Integrals in Projective Space*, arXiv:2204.03190 [INSPIRE].

JHEP08(2022)110

[40] A. Denner, *Techniques for calculation of electroweak radiative corrections at the one loop level and results for W physics at LEP-200*, *Fortsch. Phys.* **41** (1993) 307 [arXiv:0709.1075] [INSPIRE].

[41] B. Feng and T. Li, *PV-Reduction of Sunset Topology with Auxiliary Vector*, arXiv:2203.16881 [INSPIRE].

[42] A.V. Smirnov and F.S. Chuharev, *FIRE6: Feynman Integral REduction with Modular Arithmetic*, *Comput. Phys. Commun.* **247** (2020) 106877 [arXiv:1901.07808] [INSPIRE].

JHEP08(2022)110