JP Journal of Algebra, Number Theory and Applications
© 2025 Pushpa Publishing House, Prayagraj, India
https://pphmjopenaccess.com
https://doi.org/10.17654/0972555525004 **P-ISSN: 0972-5555**
Volume 64, Number 1, January 2025, Pages 47-79 **E-ISSN: 3049-2084**

DYNAMICAL SYSTEMS FOR FIGURATE NUMBERS AND ELLIPTIC SURFACES

Hizuru Yamagishi

Division of Science
School of Science and Engineering
Tokyo Denki University
Hatoyama, Saitama, 350-0394
Japan
e-mail: hizuru@mail.dendai.ac.jp

Abstract

We consider the problem how long a linear iteration continues to produce figurate numbers. We reduce the problem to considering rational points on a certain elliptic surface. Next, we consider a similar problem given by replacing a quadrangular number with a value $g(x_0)$ obtained by putting $x = x_0$ in a quadratic $g(x) = cx^2 + dx + e$. It is reduced to investigate rational points on an analogous algebraic variety. As a result, we show that there are infinitely many triangular numbers that satisfy this condition.

Received: August 6, 2024; Accepted: November 9, 2024

2020 Mathematics Subject Classification: 11B37, 11B83, 14J27, 14G25.

Keywords and phrases: figurate number, recurrence, quadrangular number, elliptic surface, triangular number, cubic surface.

Communicated by K. K. Azad

How to cite this article: Hizuru Yamagishi, Dynamical systems for figurate numbers and elliptic surfaces, JP Journal of Algebra, Number Theory and Applications 64(1) (2025), 47-79. https://doi.org/10.17654/0972555525004

This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

Published Online: December 5, 2024

1. Introduction

We propose the following problem.

Problem 1.1. Let k be a field, $f(x) \in k[x]$, and $g(x)$ be an expression which represents a figurate number. Then, what are x_i, and $f(x)$ which satisfy

$$f(g(x_i)) = g(x_{i+1}) \quad (i = 0, 1, 2, ...)?$$

Especially what are they in the case that k is the rational number field? Moreover, what is it in the case that x_i are integers?

As it seems difficult for all of these to be treated simultaneously, we consider the basic cases of this problem. In this paper, we treat the case when $f(x) = ax + b$ and $g(x)$ is the expression for a quadrangular number or a triangular number. As a result, we prove that there are infinitely many solutions for each $g(x)$, such that x_i are integers and a, b are rational numbers. In the case of quadrangular numbers, we investigate the problem through the arithmetic of an elliptic surface whose rank of the group of sections is found to be at least one. In contrast to the result in [8] or [1], which deal with generic cases, we need here to construct explicit rational solutions. This task is accomplished through a specialization homomorphism to a multiplicative fiber of the elliptic surface at a primitive 12-th root of unity.

As a result we see that all of these problems with rational solutions $(x_0, x_1, x_2, x_3, a, b)$ for $g(x)$ representing figurate numbers are reduced to the case of quadrangular numbers.

Next, we consider the solutions $(x_0, x_1, x_2, x_3, a, b)$, where $x_i\,(i = 0, 1, 2, 3)$ are integers and a, b are rational numbers of the problem for triangular numbers. We will see that we can relate the problem to the geometry of a certain cubic surface S. We find unexpectedly that S contains a double line ℓ. By parametrizing the planes, which contain the line ℓ, we

prove that there exist two infinite families of integer solutions of the problem.

The calculations in this paper include calculations using Maple [3], and PARI/GP [4].

We will deal with other figurate numbers elsewhere.

2. The Case of Quadrangular Numbers

We prove the following theorem.

Theorem 2.1. *Let* $f(x) = ax + b$ *with* $a, b \in \mathbf{Q}$. *Then there are infinitely many rational solutions* $(x_0, x_1, x_2, x_3, a, b)$ *of the simultaneous equations*

$$f(x_i^2) = x_{i+1}^2 \quad (i = 0, 1, 2). \tag{1}$$

Furthermore, there are also infinitely many solutions such that x_i $(i = 0, 1, 2, 3)$ *are integers.*

Theorem 2.1 will be proved later as a consequence of Theorem 2.3. We obtain the values of

$$a = \frac{x_1^2 - x_2^2}{x_0^2 - x_1^2}, \quad b = \frac{x_0^2 x_2^2 - x_1^4}{x_0^2 - x_1^2} \tag{2}$$

from (1) for $i = 0, 1,$ and put them to (1) for $i = 2.$ Then it becomes

$$\frac{(x_1^2 - x_2^2)x_2^2}{x_0^2 - x_1^2} + \frac{x_0^2 x_2^2 - x_1^4}{x_0^2 - x_1^2} = x_3^2. \tag{3}$$

Let V be the variety in $\mathbf{P}^3$ defined by the equation

$$V : x_0^2 x_2^2 - x_0^2 x_3^2 - x_1^4 + x_1^2 x_2^2 + x_1^2 x_3^2 - x_2^4 = 0. \tag{4}$$

Equation (4) is obtained by cleaning the fractions of (3). It is crucial to note that (4) is expressed as a determinantal equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x_0^2 & x_1^2 & x_2^2 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = 0.$$

From this expression, we can observe that there are 12 lines $\ell_1, \ldots, \ell_{12}$ on V as follows:

$$\ell_1 : x_0 = x_2, \quad x_1 = x_3, \qquad \ell_2 : x_0 = x_2, \quad x_1 = -x_3,$$

$$\ell_3 : x_0 = -x_2, \quad x_1 = x_3, \quad \ell_4 : x_0 = -x_2, \quad x_1 = -x_3,$$

$$\ell_5 : x_1 = x_0 = x_2, \qquad \ell_6 : x_1 = x_0 = -x_2,$$

$$\ell_7 : x_1 = -x_0 = x_2, \qquad \ell_8 : x_1 = -x_0 = -x_2,$$

$$\ell_9 : x_1 = x_2 = x_3, \qquad \ell_{10} : x_1 = x_2 = -x_3,$$

$$\ell_{11} : x_1 = -x_2 = x_3, \qquad \ell_{12} : x_1 = -x_2 = -x_3.$$

Moreover, V has 10 singular points $(1, \pm 1, \pm 1, \pm 1)$, $(1, 0, 0, 0)$, and $(0, 0, 0, 1)$. Next, we show that V is birational to an elliptic surface.

Theorem 2.2. *The surface in* $\mathbf{A}^3$ *which is obtained by putting* $x_1 = 1$, $x_2 = t$ *in* (4) *is birational to the elliptic curve*

$$E_t : y^2 = x^3 - (t^4 - 6t^2 + 1)x^2 - 8t^2(t^2 - 1)^2 x + 4t^2(t^2 - 1)^4$$

over $\mathbf{Q}(t)$.

Proof. The map φ from $V|_{x_1=1,\, x_2=t}$ to E_t defined by

$$\varphi(x_0, x_3) = \left(\frac{2(t^2 - 1)^2}{x_0 + 1}, \frac{2(t^2 - 1)^2 (x_0 - 1) x_3}{x_0 + 1} \right)$$

is a birational map defined over $\mathbf{Q}(t)$, whose inverse map is

$$\varphi^{-1}(x, y) = \left(\frac{2(t^2 - 1)^2}{x} - 1, \frac{y}{2((t^2 - 1)^2 - x)} \right). \qquad \square$$

We compute the singular fibers of E_t using Tate's algorithm (e.g., [5, 7]). The value of its discriminant is $256t^2(t^4 - t^2 + 1)(t-1)^8(t+1)^8$, and the j-invariant is

$$\frac{16(t^8 + 12t^6 - 10t^4 + 12t^2 + 1)^3}{t^2(t^4 - t^2 + 1)(t-1)^8(t+1)^8}.$$

Then we obtain the following Table 1.

Table 1. Singular fibers

t	Kodaira symbol
$t = 1, -1$	I_8
$t = 0$	I_2
$t^4 - t^2 + 1 = 0$	I_1

The lines $\ell_1, ..., \ell_4$ on V correspond to sections $P_1, ..., P_4$ on E_t respectively. Through the map φ, we find that they are expressed as follows:

$$\begin{aligned} P_1 &= (2(t-1)^2(t+1),\ 2(t-1)^3(t+1)), \\ P_2 &= (2(t-1)^2(t+1),\ -2(t-1)^3(t+1)), \\ P_3 &= (-2(t-1)(t+1)^2,\ 2(t-1)(t+1)^3), \\ P_4 &= (-2(t-1)(t+1)^2,\ -2(t-1)(t+1)^3). \end{aligned} \tag{5}$$

(The lines $\ell_5, ..., \ell_{12}$ are contained in the exceptional locus of φ). Furthermore, we can check that there is a section $Q_2 = ((t^2-1)^2, 0)$ which corresponds to a 2-division point. Among those the following relations hold:

$$P_1 = -P_2, \quad P_3 = -P_4, \quad P_1 + P_3 = Q_2.$$

Now, we obtain the following theorem.

Theorem 2.3. *The point P_1 is a generator of the free part of $E_t(\mathbf{Q}(t))$. Therefore, the rank of the group of sections on E_t is at least* 1.

Proof. Let t_0 be a primitive 12-th root of unity which satisfies $t_0^4 - t_0^2 + 1 = 0$, and specialize E_t at $t = t_0$. As $t_0^6 = -1$, the coefficients of the right hand side of E_{t_0} are simplified to

$$-(t_0^4 - 6t_0^2 + 1) = -(t_0^2 - 1 - 6t_0^2 + 1) = 5t_0^2,$$

$$-8t_0^2(t_0^2 - 1)^2 = -8t_0^2 t_0^8 = 8t_0^4,$$

$$4t_0^2(t_0^2 - 1)^4 = 4t_0^2 t_0^{16} = 4t_0^6.$$

Hence, we have

$$E_{t_0} : y^2 = x^3 + 5t_0^2 x^2 + 8t_0^4 x + 4t_0^6 = (x + t_0^2)(x + 2t_0^2)^2,$$

which shows that the fiber at $t = t_0$ is a multiplicative fiber. We consider how the section P_1 is specialized on the fiber E_{t_0}. Let ψ be the map from E_{t_0} to $\mathbf{Q}(t_0)$ defined by $\psi(x, y) = \dfrac{y - it_0(x + 2t_0^2)}{y + it_0(x + 2t_0^2)}$, which maps the slopes of the tangent lines at the node of E_{t_0} to 0 and ∞ on $\mathbf{P}^1(\mathbf{Q}(t_0))$. As $P_1 = (2(t_0 - 1)^2(t_0 + 1),\ 2(t_0 - 1)^3(t_0 + 1))$ by (5), we see that

$$\psi(P_1) = \frac{2(t_0 - 1)^3(t_0 + 1) - it_0(2(t_0 - 1)^2(t_0 + 1) + 2t_0^2)}{2(t_0 - 1)^3(t_0 + 1) + it_0(2(t_0 - 1)^2(t_0 + 1) + 2t_0^2)}$$

$$= \frac{(t_0 - 1)^3(t_0 + 1) - it_0((t_0 - 1)^2(t_0 + 1) + t_0^2)}{(t_0 - 1)^3(t_0 + 1) + it_0((t_0 - 1)^2(t_0 + 1) + t_0^2)}. \tag{6}$$

We can simplify the numerator and the denominator as follows. First we calculate the numerator.

Noting that $t_0^2 - 1 = t_0^4$,

$$
\begin{aligned}
&(t_0-1)^3(t_0+1)-it_0((t_0-1)^2(t_0+1)+t_0^2)\\
&=(t_0-1)^2(t_0^2-1)-it_0((t_0-1)(t_0^2-1)+t_0^2)\\
&=(t_0-1)^2t_0^4-it_0((t_0-1)t_0^4+t_0^2)\\
&=(t_0-1)^2t_0^4-i(t_0^6-t_0^5+t_0^3)\\
&=(t_0-1)^2t_0^4-i(-1-t_0^3(t_0^2-1))\\
&=(t_0-1)^2t_0^4-i(-1-t_0^7)\\
&=(t_0-1)^2t_0^4-i(-1+t_0)\\
&=(t_0-1)((t_0-1)t_0^4-i).
\end{aligned}
$$

On the other hand, the denominator is

$$(t_0-1)^3(t_0+1)+it_0((t_0-1)^2(t_0+1)+t_0^2)=(t_0-1)((t_0-1)t_0^4+i).$$

Hence, it follows from (6) that

$$\psi(P_1)=\frac{(t_0-1)((t_0-1)t_0^4-i)}{(t_0-1)((t_0-1)t_0^4+i)}=\frac{(t_0-1)t_0^4-i}{(t_0-1)t_0^4+i}. \tag{7}$$

Now put $t_0=\dfrac{\sqrt{3}+i}{2}$ which is one of the primitive 12-th roots of unity. As $t_0^4=\dfrac{-1+\sqrt{3}i}{2}$,

$$
\begin{aligned}
(t_0-1)t_0^4&=\left(\frac{\sqrt{3}+i}{2}-1\right)\times\frac{-1+\sqrt{3}i}{2}\\
&=\frac{(\sqrt{3}+i-2)(-1+\sqrt{3}i)}{4}\\
&=\frac{1}{4}((-\sqrt{3}+2-\sqrt{3})+(-1+3-2\sqrt{3})i)\\
&=\frac{1}{2}(1-\sqrt{3}+(1-\sqrt{3})i),
\end{aligned}
\tag{8}
$$

and hence

$$(t_0-1)t_0^4 - i = \frac{1}{2}(1-\sqrt{3}+(1-\sqrt{3})i) - i \qquad \text{by (8)}$$

$$= \frac{1}{2}(1-\sqrt{3}+(-1-\sqrt{3})i), \tag{9}$$

$$(t_0-1)t_0^4 + i = \frac{1}{2}(1-\sqrt{3}+(1-\sqrt{3})i) + i \qquad \text{by (8)}$$

$$= \frac{1}{2}(1-\sqrt{3}+(3-\sqrt{3})i). \tag{10}$$

It follows from (7), (9), and (10) that

$$\psi(P_1)$$

$$= \frac{(t_0-1)t_0^4 - i}{(t_0-1)t_0^4 + i} \qquad \text{by (7)}$$

$$= \frac{\frac{1}{2}(1-\sqrt{3}+(-1-\sqrt{3})i)}{\frac{1}{2}(1-\sqrt{3}+(3-\sqrt{3})i)} \qquad \text{by (9) and (10)}$$

$$= \frac{(1-\sqrt{3}+(-1-\sqrt{3})i)(1-\sqrt{3}-(3-\sqrt{3})i)}{(1-\sqrt{3})^2+(3-\sqrt{3})^2}$$

$$= \frac{(1-\sqrt{3})^2+(-1-\sqrt{3})(3-\sqrt{3})+(1-\sqrt{3})((-1-\sqrt{3})-(3-\sqrt{3}))i}{16-8\sqrt{3}}$$

$$= \frac{4-4\sqrt{3}-4(1-\sqrt{3})i}{16-8\sqrt{3}}$$

$$= \frac{1-\sqrt{3}-(1-\sqrt{3})i}{4-2\sqrt{3}}.$$

Since $\mathbf{Q}(t_0) = \mathbf{Q}(\sqrt{3})(i)$, we see that the norm $N_{\mathbf{Q}(t_0)/\mathbf{Q}(\sqrt{3})}(\psi(P_1))$ of $\psi(P_1)$ is given by

$$N_{\mathbf{Q}(t_0)/\mathbf{Q}(\sqrt{3})}(\psi(P_1))$$

$$= \left(\frac{1-\sqrt{3}-(1-\sqrt{3})i}{4-2\sqrt{3}}\right)\left(\frac{1-\sqrt{3}+(1-\sqrt{3})i}{4-2\sqrt{3}}\right)$$

$$= \frac{(1-\sqrt{3})^2+(1-\sqrt{3})^2}{4(2-\sqrt{3})^2}$$

$$= \frac{(1-\sqrt{3})^2}{2(2-\sqrt{3})^2}$$

$$= \frac{(1-\sqrt{3})^2(2+\sqrt{3})^2}{2(2-\sqrt{3})^2(2+\sqrt{3})^2}$$

$$= \frac{(4-2\sqrt{3})(7+4\sqrt{3})}{2}$$

$$= (2-\sqrt{3})(7+4\sqrt{3})$$

$$= 2+\sqrt{3}.$$

The number $2+\sqrt{3}$ is a generator of the unit group of $\mathbf{Q}(\sqrt{3})$. Hence, it follows from the fact that specialization map is a homomorphism, P_1 is free and primitive. Therefore, P_1 is a generator of the free part of $E_t(\mathbf{Q}(t))$ and the rank of the group of sections on E_t is atleast 1. □

Proof of Theorem 2.1. By Theorem 2.3 and using Silverman's specialization theorem, the rank of Mordell-Weil group over $\mathbf{Q}$ of elliptic curve E_{t_1} which is specialized E_t at a rational number t_1 is atleast one for all but finitely many $t_1 \in \mathbf{Q}$. These rational points on E_{t_1} except exceptional points of φ give rational points (x_0, x_1, x_2, x_3) on V. Since (x_0, x_1, x_2, x_3) is in $\mathbf{P}^3$, it also gives an integral point. Moreover, by using (x_0, x_1, x_2, x_3) and (2), we obtain rational solutions a and b. □

For example, we specialize E_t at $t = 3$. Then it becomes

$$E_3 : y^2 = x^3 - 28x^2 - 4608x + 147456,$$

and the five points P_1, ..., P_4, and Q_2 are respectively specialized to

$$P_1 = (32, 64), \quad P_2 = (32, -64), \quad P_3 = (-64, 256),$$

$$P_4 = (-64, -256), \quad Q_2 = (64, 0).$$

The minimal form $\widetilde{E}_3$ of E_3 is

$$\widetilde{E}_3 : y^2 + xy + y = x^3 + x^2 - 19x + 17.$$

According to [2], it is the curve of LMFDB label 438.e3 (Cremona label 438f1) and of rank 1 over **Q**, and the generator of free part is $(x, y) = (1, 0)$. We can check that this point corresponds to P_1 on E_3. These are in line with Theorem 20.3 in [6].

At the last of this section, we describe an example of Theorem 2.1. For $P_1 = (32, 64)$ on E_3, the point which corresponds to $2P_1 = (640, 15744)$ is $(x_0, x_1, x_2, x_3) = (-12, 15, 45, -205)$ on V, with $a = \frac{200}{9}$, $b = -2975$. Furthermore, we see that

$$f((-12)^2) = 15^2, \quad f(15^2) = 45^2, \quad f(45^2) = (-205)^2.$$

3. Integral Points on E_t

In the case of specialization of E_t at an integer t, P_1, ..., P_4 are integral points. There are other sections of E_t like these. In Tables 2 and 3, we write down these coordinates on E_t and name these sections. In addition, we write down their defining equations on V in Table 2. Especially the curve on V which corresponds to P_5 is the rational normal curve.

Table 2. Sections and their defining equations

Sections of E_t and its name	Corresponding defining equations on V
$P_5 = (2t(t-1)^2(t+1), -2t^2(t-1)^3(t+1))$	$x_0x_3 - x_1x_2 = 0,$ $x_0x_2 - x_1^2 = 0,$ $x_1x_3 - x_2^2 = 0$ (The rational normal curve)
$P_6 = (2t(t-1)(t+1)^2, 2t^2(t-1)(t+1)^3)$	$x_0x_3 - x_1x_2 = 0,$ $x_0x_2 + x_1^2 = 0,$ $x_1x_3 + x_2^2 = 0$
$P_7 = (2t(t-1)^2(t+1), 2t^2(t-1)^3(t+1))$	$x_0x_3 + x_1x_2 = 0,$ $x_0x_2 - x_1^2 = 0,$ $x_1x_3 + x_2^2 = 0$
$P_8 = (2t(t-1)(t+1)^2, -2t^2(t-1)(t+1)^3)$	$x_0x_3 + x_1x_2 = 0,$ $x_0x_2 + x_1^2 = 0,$ $x_1x_3 - x_2^2 = 0.$

Table 3. Sections

$P_9 = (-4t^2, 2t^5 - 2t)$
$P_{10} = (-4t^2, -2t^5 + 2t)$
$P_{11} = (t^6 - t^4 - t^2 + 1, t^9 - 2t^7 + 2t^5 - 2t^3 + t)$
$P_{12} = (t^6 - t^4 - t^2 + 1, -t^9 + 2t^7 - 2t^5 + 2t^3 - t)$
$Q_1 = (0, 2t^5 - 4t^3 + 2t)$
$Q_3 = (0, -2t^5 + 4t^3 - 2t)$

Pay attention to that Q_1 in Table 3 is a 4-division point. Table 4 displays how P_2, ..., P_{12}, Q_2, Q_3 are expressed by P_1 and Q_1.

Table 4. Sections expressed by P_1 and Q_1

$P_2 = -P_1$	$P_3 = -P_1 + 2Q_1$
$P_4 = P_1 + 2Q_1$	$P_5 = -P_1 - Q_1$
$P_6 = -P_1 + Q_1$	$P_7 = P_1 + Q_1$
$P_8 = P_1 - Q_1$	$P_9 = 2P_1 + Q_1$
$P_{10} = -2P_1 - Q_1$	$P_{11} = 2P_1$
$P_{12} = -2P_1$	$Q_2 = 2Q_1$
$Q_3 = -Q_1$	

Note: Each coordinate of $2P_1 + 2Q_1$, $2P_1 - Q_1$ is not an integral expression.

Table 3, Table 4, and Note use PARI/GP [4].

At the end of this section, we give a few comments on V.

(1) There is a rational point $(x_0, x_1, x_2, x_3) = (3, -7, 13, 23)$ on V. Rearranging these coordinates into –7, 3, 13, 23, it becomes an arithmetic sequence with the initial term –7 with common difference 10.

(2) In the case of $i = 0, 1, 2, 3, ..., n$ in equation (1), we can treat it similarly, and obtain the simultaneous equations:

$$V_n : x_0^2x_2^2 - x_0^2x_i^2 - x_1^4 + x_1^2x_{i-1}^2 + x_1^2x_i^2 - x_2^2x_{i-1}^2 = 0 \ (i = 3, 4, 5, ..., n).$$

Here V_3 is V. There is also a rational normal curve on V_n.

4. Rational Solutions for the Case of General Figurative Numbers

Since the n-th figurate number, e.g., n-th triangular number, quadrangular number, etc. is expressed by a quadratic in n, we deal with a general quadratic $cX^2 + dX + e$, where $c\,(\neq 0)$, d, e are rational numbers. In this case, we consider the similar equation to the one in the previous section:

$$f(cX_i^2 + dX_i + e) = cX_{i+1}^2 + dX_{i+1} + e \quad (i = 0, 1, 2). \tag{11}$$

Put $X_i = x_i - \frac{d}{2c}$, then $cX_i^2 + dX_i + e = cx_i^2 - \frac{d^2}{4c} + e$. Therefore, (11) becomes

$$a\left(cx_i^2 - \frac{d^2}{4c} + e\right) + b = cx_{i+1}^2 - \frac{d^2}{4c} + e \quad (i = 0, 1, 2). \tag{12}$$

By using the case of $i = 0, 1$, we obtain

$$a = \frac{x_1^2 - x_2^2}{x_0^2 - x_1^2}, \quad b = -\frac{x_1^2 - x_2^2}{x_0^2 - x_1^2}\left(cx_0^2 - \frac{d^2}{4c} + e\right) + cx_1^2 - \frac{d^2}{4c} + e.$$

We can check that the variety which is obtained by substituting these values to (12) for $i = 2$ is birational to *V*. Consequently, if $\frac{d}{2c}$ is an integer, we obtain the following by Theorem 2.1:

Corollary 4.1. *Let c, d, e be given rational numbers,* $f(x) = ax + b$ *with* $a, b \in \mathbf{Q}$, *and* $g(x) = cx^2 + dx + e$. *Then there are infinitely many rational solutions* $(X_0, X_1, X_2, X_3, a, b)$ *of the simultaneous equations*

$$f(g(X_i)) = g(X_{i+1}) \quad (i = 0, 1, 2).$$

Especially, if $\frac{d}{2c}$ *is an integer, then there are infinitely many solutions such as* X_i $(i = 0, 1, 2, 3)$ *are integers, and a, b are rational numbers.*

5. The Case of Triangular Numbers

We consider the case of triangular numbers, namely the case when $g(x) = \frac{x(x+1)}{2}$. This time $\frac{d}{2c} = \frac{1}{2}$ in the notation of Corollary 4.1. Since this is not an integer, we must find a different way from Corollary 4.1.

From now on, let

$$f(x) = ax + b \text{ and } g(x) = \frac{x(x+1)}{2}. \tag{13}$$

In this section, we prove that there are infinitely many solutions such as $X_i\,(i = 0, 1, 2, 3)$ are integers, and a, b are rational numbers. We will accomplish this task by relating the problem to the geometry of a certain cubic surface.

Lemma 5.1. *The solution of simultaneous equations*

$$f(g(x_i)) = g(x_{i+1}) \quad (i = 0, 1) \tag{14}$$

for a, b is given by

$$a = \frac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1}, \tag{15}$$

$$b = \frac{1}{2}x_2(x_2 + 1) - \frac{1}{2}x_1(x_1 + 1)a \tag{16}$$

$$= \frac{-x_1^4 + x_0^2x_2^2 - 2x_1^3 + x_0^2x_2 + x_0x_2^2 - x_1^2 + x_0x_2}{2(x_0^2 - x_1^2 + x_0 - x_1)}. \tag{17}$$

Proof. Equations (14) are

$$a \times \frac{x_i(x_i + 1)}{2} + b = \frac{x_{i+1}(x_{i+1} + 1)}{2} \quad (i = 0, 1), \tag{18}$$

therefore

$$a = \frac{\dfrac{x_1(x_1+1)}{2} - \dfrac{x_2(x_2+1)}{2}}{\dfrac{x_0(x_0+1)}{2} - \dfrac{x_1(x_1+1)}{2}} = \frac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1}.$$

Substituting this into (18) for $i = 0$, we have

$$\frac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1} \times \frac{x_0(x_0+1)}{2} + b = \frac{x_1(x_1+1)}{2}.$$

Then

$$b = \frac{x_1(x_1+1)}{2} - \frac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1} \times \frac{x_0(x_0+1)}{2}$$

$$= \frac{x_1(x_1+1)(x_0^2 - x_1^2 + x_0 - x_1) - x_0(x_0+1)(x_1^2 - x_2^2 + x_1 - x_2)}{2(x_0^2 - x_1^2 + x_0 - x_1)}$$

$$= \frac{x_1(x_1+1)(x_0(x_0+1) - x_1(x_1+1))}{2(x_0^2 - x_1^2 + x_0 - x_1)}$$

$$- \frac{x_0(x_0+1)(x_1(x_1+1) - x_2(x_2+1))}{2(x_0^2 - x_1^2 + x_0 - x_1)}$$

$$= \frac{1}{2(x_0^2 - x_1^2 + x_0 - x_1)}$$

$$\times (x_0x_1(x_0+1)(x_1+1) - x_1^2(x_1+1)^2$$

$$- x_0x_1(x_0+1)(x_1+1) + x_0x_2(x_0+1)(x_2+1))$$

$$= \frac{-x_1^2(x_1+1)^2 + x_0x_2(x_0+1)(x_2+1)}{2(x_0^2 - x_1^2 + x_0 - x_1)}$$

$$= \frac{-x_1^4 - 2x_1^3 - x_1^2 + x_0^2x_2^2 + x_0^2x_2 + x_0x_2^2 + x_0x_2}{2(x_0^2 - x_1^2 + x_0 - x_1)}.$$

This coincides with (17). □

For later reference, we put

$$A_1 = x_1^2 - x_2^2 + x_1 - x_2 \text{ (numerator of (15))}, \tag{19}$$

$$A_2 = x_0^2 - x_1^2 + x_0 - x_1 \text{ (denominator of (15))}, \tag{20}$$

$$B_1 = -x_1^4 + x_0^2x_2^2 - 2x_1^3 + x_0^2x_2 + x_0x_2^2 - x_1^2 + x_0x_2 \text{ (numerator of (17))}. \tag{21}$$

Lemma 5.2. *Let* V_{tri} *be the variety in* $\mathbf{A}^4(x_0, x_1, x_2, x_3)$ *defined by*

$$\begin{aligned} V_{tri} : & x_0^2x_2^2 - x_0^2x_3^2 - x_1^4 + x_1^2x_2^2 + x_1^2x_3^2 - x_2^4 \\ & + x_0^2x_2 - x_0^2x_3 + x_0x_2^2 - x_0x_3^2 - 2x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 \\ & + x_1x_3^2 - 2x_2^3 + x_0x_2 - x_0x_3 - x_1^2 + x_1x_2 + x_1x_3 - x_2^2 = 0. \end{aligned} \tag{22}$$

Then the algebraic variety defined by the following equations (23) *is birational to* V_{tri} *over k.*

$$f(g(x_i)) = g(x_{i+1}) \quad (i = 0, 1, 2). \tag{23}$$

Proof. By definition, we have

$$a = \frac{A_1}{A_2},\ b = \frac{B_1}{2A_2} \text{ by (15), (17), (19), (20), and (21).}$$

Inserting these values to (23) for $i = 2$, we have

$$a \cdot \frac{x_2(x_2+1)}{2} + b = \frac{x_3(x_3+1)}{2}.$$

Multiplying both sides by $2A_2$, we obtain

$$x_2(x_2+1)A_1 + B_1 - x_3(x_3+1)A_2 = 0.$$

Substituting (19), (20), and (21), we have

$$\begin{aligned} & x_2(x_2+1)(x_1^2 - x_2^2 + x_1 - x_2) \\ & + (-x_1^4 + x_0^2x_2^2 - 2x_1^3 + x_0^2x_2 + x_0x_2^2 - x_1^2 + x_0x_2) \\ & - x_3(x_3+1)(x_0^2 - x_1^2 + x_0 - x_1) = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & x_2(x_1^2x_2 - x_2^3 + x_1^2 + x_1x_2 - 2x_2^2 + x_1 - x_2) \\ & + (-x_1^4 + x_0^2x_2^2 - 2x_1^3 + x_0^2x_2 + x_0x_2^2 - x_1^2 + x_0x_2) \\ & - x_3(x_0^2x_3 - x_1^2x_3 + x_0^2 + x_0x_3 - x_1^2 - x_1x_3 + x_0 - x_1) = 0. \end{aligned} \tag{24}$$

By expanding (24), we obtain (22) in the statement of Lemma 5.2. □

Lemma 5.3. *The intersection of* V_{tri} *and the plane defined by*

$$x_0 + x_1 - x_2 - x_3 = 0 \tag{25}$$

is the union of the surface in $\mathbf{A}^3(x_0, x_1, x_2)$

$$x_0^3 + 2x_0^2x_1 - x_0^2x_2 - x_0x_2^2 - 2x_1^3 + 2x_1^2x_2 - x_2^3$$
$$+ 2(x_0^2 + x_0x_1 - x_0x_2 - x_1^2 + x_1x_2 - x_2^2) + x_0 - x_2 = 0 \tag{26}$$

and $x_0 - x_2 = 0$.

Proof. Put

$$x_0 - x_2 = A,$$

and substitute $x_3 = x_0 + x_1 - x_2 = A + x_1$ to (22). Then the quartic part of (22) becomes

$$x_0^2x_2^2 - x_0^2x_3^2 - x_1^4 + x_1^2x_2^2 + x_1^2x_3^2 - x_2^4$$
$$= x_0^2x_2^2 - x_0^2(A + x_1)^2 - x_1^4 + x_1^2x_2^2 + x_1^2(A + x_1)^2 - x_2^4$$
$$= x_2^2(x_0^2 - x_2^2) - x_0^2(A + x_1)^2 - x_1^4 + x_1^2x_2^2 + x_1^2(A + x_1)^2. \tag{27}$$

Since $x_0 - x_2 = A$, this is equal to

$$x_2^2(x_0 + x_2)A - x_0^2(A^2 + 2x_1A + x_1^2) - x_1^4 + x_1^2x_2^2 + x_1^2(A^2 + 2x_1A + x_1^2)$$
$$= A(x_2^2(x_0 + x_2) - x_0^2(A + 2x_1) + x_1^2(A + 2x_1)) - x_0^2x_1^2 - x_1^4 + x_1^4 + x_1^2x_2^2$$
$$= A(x_0x_2^2 + x_2^3 - x_0^2A - 2x_0^2x_1 + x_1^2A + 2x_1^3 - x_1^2(x_0 + x_2))$$
$$= A(x_0x_2^2 + x_2^3 - x_0^2(x_0 - x_2) - 2x_0^2x_1 + x_1^2(x_0 - x_2) + 2x_1^3 - x_1^2(x_0 + x_2))$$
$$= -A(x_0^3 + 2x_0^2x_1 - x_0^2x_2 - x_0x_2^2 - 2x_1^3 + 2x_1^2x_2 - x_2^3).$$

The cubic part of (22) is computed to be

$$\begin{aligned}
&x_0^2x_2 - x_0^2x_3 + x_0x_2^2 - x_0x_3^2 - 2x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 - 2x_2^3 \\
&= x_0^2x_2 - x_0^2(A + x_1) + x_0x_2^2 - x_0(A + x_1)^2 \\
&\quad - 2x_1^3 + x_1^2x_2 + x_1^2(A + x_1) + x_1x_2^2 + x_1(A + x_1)^2 - 2x_2^3 \\
&= A(-x_0^2 - x_0A - 2x_0x_1 + x_1^2 + x_1A + 2x_1^2) \\
&\quad + x_0^2x_2 - x_0^2x_1 + x_0x_2^2 - x_0x_1^2 - 2x_1^3 + x_1^2x_2 + x_1^3 + x_1x_2^2 + x_1^3 - 2x_2^3 \\
&= A(-x_0^2 - x_0A - 2x_0x_1 + x_1^2 + x_1A + 2x_1^2) \\
&\quad + x_2(x_0^2 - x_2^2) - x_1(x_0^2 - x_2^2) + x_2^2(x_0 - x_2) - x_1^2(x_0 - x_2) \\
&= A(-x_0^2 - x_0A - 2x_0x_1 + x_1^2 + x_1A + 2x_1^2 \\
&\quad + x_2(x_0 + x_2) - x_1(x_0 + x_2) + x_2^2 - x_1^2) \\
&= -A(x_0^2 + x_0(x_0 - x_2) + 2x_0x_1 - x_1^2 - x_1(x_0 - x_2) - 2x_1^2 \\
&\quad - x_2(x_0 + x_2) + x_1(x_0 + x_2) - x_2^2 + x_1^2) \\
&= -2A(x_0^2 + x_0x_1 - x_0x_2 - x_1^2 + x_1x_2 - x_2^2). \qquad (28)
\end{aligned}$$

The quadratic part of (22) becomes

$$\begin{aligned}
&x_0x_2 - x_0x_3 - x_1^2 + x_1x_2 + x_1x_3 - x_2^2 \\
&= x_0x_2 - x_0(A + x_1) - x_1^2 + x_1x_2 + x_1(A + x_1) - x_2^2 \\
&= A(-x_0 + x_1) + x_0x_2 - x_0x_1 + x_1x_2 - x_2^2 \\
&= A(-x_0 + x_1) + x_2(x_0 - x_2) - x_1(x_0 - x_2) \\
&= -A(x_0 - x_1 - x_2 + x_1) \\
&= -A^2. \qquad (29)
\end{aligned}$$

Therefore, using (27), (28), and (29), the equation which is multiplied both hand sides of (22) by (–1) is

$$(x_0 - x_2) \times (x_0^3 + 2x_0^2x_1 - x_0^2x_2 - x_0x_2^2 - 2x_1^3 + 2x_1^2x_2 - x_2^3 + 2(x_0^2 + x_0x_1 - x_0x_2 - x_1^2 + x_1x_2 - x_2^2) + x_0 - x_2) = 0. \quad (30)$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let S be the cubic surface in* $\mathbf{A}^3$ *which is defined by the second factor of* (30), *that is*

$$S : x_0^3 + 2x_0^2x_1 - x_0^2x_2 - x_0x_2^2 - 2x_1^3 + 2x_1^2x_2 - x_2^3 + 2(x_0^2 + x_0x_1 - x_0x_2 - x_1^2 + x_1x_2 - x_2^2) + x_0 - x_2 = 0. \quad (31)$$

Then the line $L : (x_0, x_1, x_2) = (t, -t-1, -t-1)$ *is a singular line on S.*

Proof. To get singular points on *S*, put the left hand side of (31) as $h(x_0, x_1, x_2)$. We find the derivatives of $h(x_0, x_1, x_2)$ as follows:

$$h_{x_0} = 3x_0^2 + 4x_0x_1 - 2x_0x_2 - x_2^2 + 2(2x_0 + x_1 - x_2) + 1, \quad (32)$$

$$h_{x_1} = 2x_0^2 - 6x_1^2 + 4x_1x_2 + 2(x_0 - 2x_1 + x_2), \quad (33)$$

$$h_{x_2} = -x_0^2 - 2x_0x_2 + 2x_1^2 - 3x_2^2 + 2(-x_0 + x_1 - 2x_2) - 1. \quad (34)$$

Since h_{x_1} is linear in x_2, the equation $h_{x_1} = 0$ gives us

$$x_2 = -\frac{x_0^2 - 3x_1^2 + x_0 - 2x_1}{2x_1 + 1} \quad \left(\text{where } x_1 \neq -\frac{1}{2}\right). \quad (35)$$

Inserting (35) to h_{x_0}, we have

$$h_{x_0} = -x_2^2 - 2(x_0 + 1)x_2 + 3x_0^2 + 4x_0x_1 + 4x_0 + 2x_1 + 1$$

$$= -\left(-\frac{x_0^2 - 3x_1^2 + x_0 - 2x_1}{2x_1 + 1}\right)^2 - 2(x_0 + 1)\left(-\frac{x_0^2 - 3x_1^2 + x_0 - 2x_1}{2x_1 + 1}\right)$$

$$+ 3x_0^2 + 4x_0x_1 + 4x_0 + 2x_1 + 1$$

$$= (-(x_0^2 - 3x_1^2 + x_0 - 2x_1)^2$$

$$+ 2(x_0 + 1)(x_0^2 - 3x_1^2 + x_0 - 2x_1)(2x_1 + 1)$$

$$+ (3x_0^2 + 4x_0x_1 + 4x_0 + 2x_1 + 1)(2x_1 + 1)^2)/(2x_1 + 1)^2. \quad (36)$$

Let $\bar{h}_{x_0}$ be the numerator of (36), i.e.,

$$\bar{h}_{x_0}(x_0, x_1) = -(x_0^2 - 3x_1^2 + x_0 - 2x_1)^2$$

$$+ 2(x_0 + 1)(x_0^2 - 3x_1^2 + x_0 - 2x_1)(2x_1 + 1)$$

$$+ (3x_0^2 + 4x_0x_1 + 4x_0 + 2x_1 + 1)(2x_1 + 1)^2.$$

If we regard $\bar{h}_{x_0}(x_0, x_1)$ as a polynomial in x_1, its constant term is obtained by putting $x_1 = 0$:

$$\bar{h}_{x_0}(x_0, 0) = -(x_0^2 + x_0)^2 + 2(x_0 + 1)(x_0^2 + x_0) + (3x_0^2 + 4x_0 + 1)$$

$$= -(x_0^4 + 2x_0^3 + x_0^2) + 2(x_0^3 + 2x_0^2 + x_0) + (3x_0^2 + 4x_0 + 1)$$

$$= -x_0^4 + 6x_0^2 + 6x_0 + 1$$

$$= -(x_0 + 1)(x_0^3 - x_0^2 - 5x_0 - 1).$$

Inspired by this, if we substitute $x_1 = -(x_0 + 1)$ to $\bar{h}_{x_0}(x_0, x_1)$, we obtain

$$\bar{h}_{x_0}(x_0, -(x_0 + 1))$$

$$= -(x_0^2 - 3(x_0 + 1)^2 + x_0 + 2(x_0 + 1))^2$$

$$+ 2(x_0 + 1)(x_0^2 - 3(x_0 + 1)^2 + x_0 + 2(x_0 + 1))(-2(x_0 + 1) + 1)$$

$$+ (3x_0^2 - 4x_0(x_0 + 1) + 4x_0 - 2(x_0 + 1) + 1)(-2(x_0 + 1) + 1)^2$$

$$= -(-2x_0^2 - 3x_0 - 1)^2 + 2(x_0 + 1)(-2x_0^2 - 3x_0 - 1)(-2x_0 - 1)$$

$$+ (-x_0^2 - 2x_0 - 1)(-2x_0 - 1)^2$$

$$= -(2x_0 + 1)^2(x_0 + 1)^2$$

$$- (2x_0 + 1)(2(x_0 + 1)(-2x_0^2 - 3x_0 - 1) + (x_0 + 1)^2(2x_0 + 1))$$

$$= -(2x_0 + 1)^2(x_0 + 1)^2$$

$$- (2x_0 + 1)(x_0 + 1)(-2(2x_0 + 1)(x_0 + 1) + (x_0 + 1)(2x_0 + 1))$$

$$= -(2x_0 + 1)^2(x_0 + 1)^2 + (2x_0 + 1)^2(x_0 + 1)^2$$

$$= 0.$$

Hence $\bar{h}_{x_0}(x_0, x_1)$ has $x_0 + x_1 + 1$ as a factor. Consequently if $x_1 \neq -\frac{1}{2}$, then $x_0 + x_1 + 1$ is a factor of $h_{x_0}(x_0, x_1)$. Similarly, we can see that $h_{x_2}(x_0, x_1)$ also has this factor. Therefore, put

$$x_0 + x_1 + 1 = 0.$$

Then by $x_1 = -(x_0 + 1),$ and from (35),

$$x_2 = -\frac{x_0^2 - 3(x_0 + 1)^2 + x_0 + 2(x_0 + 1)}{-2(x_0 + 1) + 1} = -\frac{-2x_0^2 - 3x_0 - 1}{-2x_0 - 1} = -(x_0 + 1).$$

$\left(\text{Note that if } x_1 \neq -\frac{1}{2}, \text{ then } x_0 \neq -\frac{1}{2}\right)$. Now letting $x_2 = x_1$, $x_0 = -(x_1 + 1),$ we have

$$h(-(x_1 + 1), x_1, x_1)$$

$$= -(x_1 + 1)^3 + 2(x_1 + 1)^2 x_1 - (x_1 + 1)^2 x_1 + (x_1 + 1)x_1^2 - 2x_1^3 + 2x_1^3 - x_1^3$$

$$+ 2((x_1+1)^2 - (x_1+1)x_1 + (x_1+1)x_1 - x_1^2 + x_1^2 - x_1^2) - (x_1+1) - x_1$$

$$= -(x_1+1)^3 + (x_1+1)^2 x_1 + (x_1+1)x_1^2 - x_1^3$$

$$+ 2((x_1+1)^2 - x_1^2) - (x_1+1) - x_1$$

$$= -(x_1+1)^2 + x_1^2 + 2(2x_1+1) - 2x_1 - 1$$

$$= 0.$$

Now put $x_0 = t$, then $x_1 = x_2 = -x_0 - 1 = -t - 1$. Thus $(x_0, x_1, x_2) = (t, -t-1, -t-1)$ is a singular point on S, where $x_1 \neq -\frac{1}{2}$, i.e., $t \neq -\frac{1}{2}$. Since if $x_1 = -\frac{1}{2}$ then $t = -\frac{1}{2}$, the point $(x_0, x_1, x_2) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ is on the line $L : (x_0, x_1, x_2) = (t, -t-1, -t-1)$. Moreover, by (31), (32), (33), and (34), we have

$$h\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{8} \times (1 + 2 - 1 - 1 - 2 + 2 - 1)$$

$$+ \frac{1}{2} \times (1 + 1 - 1 - 1 + 1 - 1) - \frac{1}{2} \times (1 - 1) = 0,$$

$$h_{x_0}\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} \times (3 + 4 - 2 - 1) - (2 + 1 - 1) + 1 = 0,$$

$$h_{x_1}\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} \times (2 - 6 + 4) - (1 - 2 + 1) = 0,$$

$$h_{x_2}\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{4} \times (-1 - 2 + 2 - 3) - (-1 + 1 - 2) - 1 = 0.$$

Consequently, $(x_0, x_1, x_2) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ is also a singular point, and L is a singular line on S. □

Theorem 5.5. *The points on S are parametrized as follows*:

$$\begin{cases} x_0 = \bar{t}, \\ x_1 = \dfrac{s^3\bar{t} + s^3 + 2s^2\bar{t} + s^2 + 2s\bar{t} + 2\bar{t}}{s^3 - 2s - 2}, \\ x_2 = \dfrac{s^3\bar{t} + 4s^2\bar{t} + 2s^2 + 6s\bar{t} + 4s + 2\bar{t} + 2}{s^3 - 2s - 2}. \end{cases} \tag{37}$$

Proof. Let $\bar{a}x_0 + \bar{b}x_1 + \bar{c}x_2 + \bar{d} = 0$ be the equation of a plane which contains the singular line L. Then

$$\bar{a}t + \bar{b} \times (-t - 1) + \bar{c} \times (-t - 1) + \bar{d} = 0,$$

$$(\bar{a} - \bar{b} - \bar{c})t + (-\bar{b} - \bar{c} + \bar{d}) = 0.$$

Therefore, the solution of the simultaneous equations

$$\begin{cases} \bar{a} - \bar{b} - \bar{c} = 0, \\ -\bar{b} - \bar{c} + \bar{d} = 0, \end{cases}$$

is $\bar{a} = \bar{b} + \bar{c}$, $\bar{d} = \bar{b} + \bar{c}$. Put $b = s$, $c = 1$, then the equation of the plane which contains L is expressed as

$$(s + 1)x_0 + sx_1 + x_2 + (s + 1) = 0.$$

It follows that

$$x_2 = -(s + 1)x_0 - sx_1 - (s + 1) = -(sB + x_0 + 1), \tag{38}$$

where we put $B = x_0 + x_1 + 1$. Therefore, rearranging (26) for x_2, and substituting (38) into it, we see that

$$-x_2^3 - (x_0 + 2)x_2^2 + (-x_0^2 + 2x_1^2 - 2x_0 + 2x_1 - 1)x_2$$

$$+ x_0^3 + 2x_0^2x_1 - 2x_1^3 + 2x_0^2 + 2x_0x_1 - 2x_1^2 + x_0 = 0,$$

and

$$(sB + x_0 + 1)^3 - (x_0 + 2)(sB + x_0 + 1)^2$$
$$- (-x_0^2 + 2x_1^2 - 2x_0 + 2x_1 - 1)(sB + x_0 + 1)$$
$$+ x_0^3 + 2x_0^2x_1 - 2x_1^3 + 2x_0^2 + 2x_0x_1 - 2x_1^2 + x_0 = 0. \tag{39}$$

Then regarding this as an equation for B, we find that the constant term of the left hand side is

$$(x_0 + 1)^3 - (x_0 + 2)(x_0 + 1)^2 - (-x_0^2 + 2x_1^2 - 2x_0 + 2x_1 - 1)(x_0 + 1)$$
$$+ x_0^3 + 2x_0^2x_1 - 2x_1^3 + 2x_0^2 + 2x_0x_1 - 2x_1^2 + x_0$$
$$= x_0^3 + 3x_0^2 + 3x_0 + 1 - (x_0^3 + 4x_0^2 + 5x_0 + 2)$$
$$- (-x_0^3 + 2x_0x_1^2 - 3x_0^2 + 2x_0x_1 + 2x_1^2 - 3x_0 + 2x_1 - 1)$$
$$+ x_0^3 + 2x_0^2x_1 - 2x_1^3 + 2x_0^2 + 2x_0x_1 - 2x_1^2 + x_0$$
$$= 2x_0^3 + 2x_0^2x_1 - 2x_0x_1^2 - 2x_1^3 + 4x_0^2 - 4x_1^2 + 2x_0 - 2x_1$$
$$= 2(x_0^3 - x_1^3 + x_0x_1(x_0 - x_1) + 2(x_0^2 - x_1^2) + x_0 - x_1)$$
$$= 2(x_0 - x_1)(x_0^2 + x_0x_1 + x_1^2 + x_0x_1 + 2(x_0 + x_1) + 1)$$
$$= 2(x_0 - x_1)(x_0 + x_1 + 1)^2$$
$$= 2(x_0 - x_1)B^2. \tag{40}$$

Hence (39) becomes

$$s^3B^3 + 3(x_0 + 1)s^2B^2 + 3(x_0 + 1)^2sB - (x_0 + 2)(s^2B^2 + 2(x_0 + 1)sB)$$
$$- (-x_0^2 + 2x_1^2 - 2x_0 + 2x_1 - 1)sB + 2(x_0 - x_1)B^2 = 0$$

by (40). Recalling that $B = x_0 + x_1 + 1$, we have

$$\begin{aligned}
&s^3B^3 + (3s^2(x_0+1) - (x_0+2)s^2 + 2(x_0 - x_1))B^2 \\
&\quad + (3(x_0+1)^2 - 2(x_0+2)(x_0+1) \\
&\quad - (-x_0^2 + 2x_1^2 - 2x_0 + 2x_1 - 1))sB = 0, \\
&s^3B^3 + ((2x_0+1)s^2 + 2(x_0 - x_1))B^2 + 2(x_0^2 - x_1^2 + x_0 - x_1)sB = 0, \\
&s^3B^3 + (2x_0s^2 + s^2 + 2x_0 - 2x_1)B^2 + 2(x_0 - x_1)(x_0 + x_1 + 1)sB = 0, \\
&s^2B^3 + (2s^2x_0 + 2x_0 - 2x_1 + s^2)B^2 + 2(x_0 - x_1)sB^2 = 0, \\
&B^2(s^3B + 2s^2x_0 + 2x_0 - 2x_1 + s^2 + 2sx_0 - 2sx_1) = 0, \\
&B^2(s^3x_0 + 2s^2x_0 + 2sx_0 + 2x_0 + s^3x_1 - 2sx_1 - 2x_1 + s^3 + s^2) = 0.
\end{aligned}$$

The second factor on the left hand side of the last equality is the equation of the line which is parameterized by s. Therefore, we have

$$x_1 = -\frac{s^3x_0 + 2s^2x_0 + 2sx_0 + 2x_0 + s^3 + s^2}{s^3 - 2s - 2}. \tag{41}$$

Then by (38) x_2 becomes

$$\begin{aligned}
x_2 &= -(s+1)x_0 + s \times \frac{s^3x_0 + 2s^2x_0 + 2sx_0 + 2x_0 + s^3 + s^2}{s^3 - 2s - 2} - (s+1) \\
&= \frac{1}{s^3 - 2s - 2} \\
&\quad \times (-(s+1)x_0(s^3 - 2s - 2) + s(s^3x_0 + 2s^2x_0 + 2sx_0 + 2x_0 + s^3 + s^2) \\
&\quad - (s+1)(s^3 - 2s - 2)) \\
&= \frac{(s^3 + 4s^2 + 6s + 2)x_0 + 2(s+1)^2}{s^3 - 2s - 2}.
\end{aligned} \tag{42}$$

By putting $x_0 = \bar{t}$, and according to (41) and (42), we obtain (37) of the statement in this theorem. □

From this, as a preliminary step of our aim, the next result is obtained as follows:

Theorem 5.6. *As* in (13), *let* $f(x) = ax + b$ *and* $g(x) = \frac{x(x+1)}{2}$. *Use* x_0, x_1, x_2 *in* (37), *a in* (15), *and b in* (16). *Put* x_3 *as follows*:

$$\begin{cases} x_0 = \bar{t}, \\ x_1 = -\dfrac{s^3\bar{t} + s^3 + 2s^2\bar{t} + s^2 + 2s\bar{t} + 2\bar{t}}{s^3 - 2s - 2}, \\ x_2 = \dfrac{s^3\bar{t} + 4s^2\bar{t} + 2s^2 + 6s\bar{t} + 4s + 2\bar{t} + 2}{s^3 - 2s - 2}, \\ x_3 = -\dfrac{s^3\bar{t} + s^3 + 6s^2\bar{t} + 3s^2 + 10s\bar{t} + 4s + 6\bar{t} + 2}{s^3 - 2s - 2}, \\ a = \dfrac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1}, \\ b = \dfrac{1}{2}x_2(x_2 + 1) - \dfrac{1}{2}x_1(x_1 + 1)a. \end{cases}$$

Then the tuple $(x_0, x_1, x_2, x_3, a, b)$ *is a solution of* (23), *i.e.*,

$$f(g(x_i)) = g(x_{i+1}) \quad (i = 0, 1, 2).$$

Proof. It is enough to find the expression for x_3. It follows from (25) that

$$\begin{aligned} x_3 &= x_0 + x_1 - x_2 \\ &= \bar{t} - \frac{s^3\bar{t} + s^3 + 2s^2\bar{t} + s^2 + 2s\bar{t} + 2\bar{t}}{s^3 - 2s - 2} \\ &\quad - \frac{s^3\bar{t} + 4s^2\bar{t} + 2s^2 + 6s\bar{t} + 4s + 2\bar{t} + 2}{s^3 - 2s - 2} \\ &= -\frac{s^3\bar{t} + s^3 + 6s^2\bar{t} + 3s^2 + 10s\bar{t} + 4s + 6\bar{t} + 2}{s^3 - 2s - 2}. \end{aligned}$$

□

To obtain integer points on S, put $s = -\frac{m}{m+1}$. Let m and $\bar{t}$ be integers. Then the denominator of both x_1 and x_2 in (37) is

$$\begin{aligned} s^3 - 2s - 2 &= \left(-\frac{m}{m+1}\right)^3 - 2\left(-\frac{m}{m+1}\right) - 2 \\ &= \frac{-m^3 + 2m(m+1)^2 - 2(m+1)^3}{(m+1)^3} \\ &= \frac{-m^3 - 2m^2 - 4m - 2}{(m+1)^3}. \end{aligned} \tag{43}$$

The numerator of x_1 is obtained as follows:

$$\begin{aligned} & -(s^3\bar{t} + s^3 + 2s^2\bar{t} + s^2 + 2s\bar{t} + 2\bar{t}) \qquad \text{by (37)} \\ &= -(s^3(\bar{t}+1) + s^2(2\bar{t}+1) + 2s\bar{t} + 2\bar{t}) \\ &= -\left(\left(-\frac{m}{m+1}\right)^3(\bar{t}+1) + \left(-\frac{m}{m+1}\right)^2(2\bar{t}+1) + 2\left(-\frac{m}{m+1}\right)\bar{t} + 2\bar{t}\right) \\ &= -\frac{-m^3(\bar{t}+1) + m^2(m+1)(2\bar{t}+1) - 2m(m+1)^2\bar{t} + 2(m+1)^3\bar{t}}{(m+1)^3} \\ &= -\frac{\bar{t}m^3 + (4\bar{t}+1)m^2 + 4\bar{t}m + 2\bar{t}}{(m+1)^3}. \end{aligned} \tag{44}$$

Similarly, we get the numerator of x_2 as follows:

$$\begin{aligned} & s^3\bar{t} + 4s^2\bar{t} + 2s^2 + 6s\bar{t} + 4s + 2\bar{t} + 2 \qquad \text{by (37)} \\ &= s^3\bar{t} + s^2(4\bar{t}+2) + s(6\bar{t}+4) + 2\bar{t} + 2 \\ &= \left(-\frac{m}{m+1}\right)^3\bar{t} + \left(-\frac{m}{m+1}\right)^2(4\bar{t}+2) + \left(-\frac{m}{m+1}\right)(6\bar{t}+4) + 2\bar{t} + 2 \end{aligned}$$

$$= \frac{-m^3\bar{t} + m^2(m+1)(4\bar{t}+2) - m(m+1)^2(6\bar{t}+4) + (m+1)^3(2\bar{t}+2)}{(m+1)^3}$$

$$= \frac{-\bar{t}m^3 - 2\bar{t}m^2 + 2m + 2\bar{t} + 2}{(m+1)^3}. \tag{45}$$

Thus

$$x_1 = \frac{\bar{t}m^3 + (4\bar{t}+1)m^2 + 4\bar{t}m + 2\bar{t}}{m^3 + 2m^2 + 4m + 2} \quad \text{by (43) and (44)}$$

$$= \bar{t} + \frac{(2\bar{t}+1)m^2}{m^3 + 2m^2 + 4m + 2}, \tag{46}$$

$$x_2 = \frac{\bar{t}m^3 + 2\bar{t}m^2 - 2m - 2\bar{t} - 2}{m^3 + 2m^2 + 4m + 2} \quad \text{by (43) and (45)}$$

$$= \bar{t} + \frac{-\bar{t}(4m+2) - 2m - 2\bar{t} - 2}{m^3 + 2m^2 + 4m + 2}$$

$$= \bar{t} - \frac{2(2\bar{t}+1)(m+1)}{m^3 + 2m^2 + 4m + 2}. \tag{47}$$

From these, if $2\bar{t}+1$ is a multiple of $m^3 + 2m^2 + 4m + 2$, then both x_1 and x_2 are integers. Therefore, we consider the following equation:

$$2\bar{t} + 1 = n(m^3 + 2m^2 + 4m + 2) \text{ where } m, n, \text{ and } \bar{t} \text{ are integers.}$$

It follows that both m and n are odd, and hence we can put $m = 2\overline{m} + 1$, $n = 2\overline{n} + 1$ (where $\overline{m}$, $\overline{n}$ are integers). Then

$$2\bar{t} + 1 = (2\overline{n}+1)((2\overline{m}+1)^3 + 2(2\overline{m}+1)^2 + 4(2\overline{m}+1) + 2)$$

$$2\bar{t} + 1 = (2\overline{n}+1)(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9)$$

$$2\bar{t} = 2\overline{n}(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 8$$

$$\bar{t} = \overline{n}(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 4\overline{m}^3 + 10\overline{m}^2 + 11\overline{m} + 4. \tag{48}$$

Note that

$$\frac{2\bar{t}+1}{m^3+2m^2+4m+2} = n = 2\overline{n}+1.$$

Therefore,

$$\begin{aligned} x_1 &= \bar{t} + (2\overline{n}+1)(2\overline{m}+1)^2 \qquad \text{by (46)} \\ &= \overline{n}\,(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 4\overline{m}^3 + 10\overline{m}^2 + 11\overline{m} + 4 \\ &\quad + (2\overline{n}+1)(4\overline{m}^2 + 4\overline{m} + 1) \\ &= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 30\overline{m}\,\overline{n} + 15\overline{m} + 11\overline{n} + 5, \qquad (49) \\ x_2 &= \bar{t} - 2(2\overline{n}+1)(2\overline{m}+2) \qquad \text{by (47)} \\ &= \overline{n}\,(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 4\overline{m}^3 + 10\overline{m}^2 + 11\overline{m} + 4 \\ &\quad - 4(2\overline{m}\,\overline{n} + \overline{m} + 2\overline{n} + 1) \\ &= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 20\overline{m}^2\overline{n} + 10\overline{m}^2 + 14\overline{m}\,\overline{n} + 7\overline{m} + \overline{n}. \qquad (50) \end{aligned}$$

From $x_0 = \bar{t}$, (48), (49), and (50), we obtain integer points on S. Moreover, by (25), (48), (49), and (50), we see that

$$\begin{aligned} x_3 &= x_0 + x_1 - x_2 \qquad \text{by (25)} \\ &= \overline{n}\,(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 4\overline{m}^3 + 10\overline{m}^2 + 11\overline{m} + 4 \\ &\quad + 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 30\overline{m}\,\overline{n} + 15\overline{m} + 11\overline{n} + 5 \\ &\quad - (8\overline{m}^3\overline{n} + 4\overline{m}^3 + 20\overline{m}^2\overline{n} + 10\overline{m}^2 + 14\overline{m}\,\overline{n} + 7\overline{m} + \overline{n}) \\ &\qquad\qquad \text{by (48), (49), and (50)} \\ &= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 38\overline{m}\,\overline{n} + 19\overline{m} + 19\overline{n} + 9. \end{aligned}$$

Therefore, we obtain the first half of the following theorem.

Theorem 5.7 (Integer solutions)**.** *Let* $f(x) = ax + b$, $g(x) = \dfrac{x(x+1)}{2}$ *be the same as in* (13). *Then the following tuples* $(x_0, x_1, x_2, x_3, a, b)$ *are the solutions of* (23), *i.e.*

$$f(g(x_i)) = g(x_{i+1}) \quad (i = 0, 1, 2).$$

At this time, x_0, x_1, x_2, x_3 *are integers, and* a, b *are rational numbers. In order to obtain* x_0, x_1, x_2, x_3, *put*

$$\begin{cases} x_0 = \overline{n}\,(8\overline{m}^3 + 20\overline{m}^2 + 22\overline{m} + 9) + 4\overline{m}^3 + 10\overline{m}^2 + 11\overline{m} + 4, \\ x_1 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 30\overline{m}\,\overline{n} + 15\overline{m} + 11\overline{n} + 5, \\ x_2 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 20\overline{m}^2\overline{n} + 10\overline{m}^2 + 14\overline{m}\,\overline{n} + 7\overline{m} + \overline{n}, \\ x_3 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 38\overline{m}\,\overline{n} + 19\overline{m} + 19\overline{n} + 9, \end{cases} \tag{51}$$

or

$$\begin{cases} x_0 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 54\overline{m}\,\overline{n} + 27\overline{m} + 37\overline{n} + 18, \\ x_1 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 44\overline{m}^2\overline{n} + 22\overline{m}^2 + 70\overline{m}\,\overline{n} + 35\overline{m} + 41\overline{n} + 20, \\ x_2 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 22\overline{m}\,\overline{n} + 11\overline{m} + 11\overline{n} - 6, \\ x_3 = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 44\overline{m}^2\overline{n} + 22\overline{m}^2 + 102\overline{m}\,\overline{n} + 51\overline{m} + 89\overline{n} + 44, \end{cases} \tag{52}$$

for integers $\overline{m}, \overline{n}$, *moreover put*

$$\begin{cases} a = \dfrac{x_1^2 - x_2^2 + x_1 - x_2}{x_0^2 - x_1^2 + x_0 - x_1} \quad \textit{by } (15), \\ b = \dfrac{1}{2}x_2(x_2+1) - \dfrac{1}{2}x_1(x_1+1)a \quad \textit{by } (16) \end{cases}$$

in either case.

Proof. The second half comes from the case when

$$s = -\frac{m}{m+2}.$$

In a similar way as above, we see that

$$s^3 - 2s - 2 = -\frac{m^3 + 4m^2 + 16m + 16}{(m+2)^3},$$

$$x_1 = \bar{t} + \frac{2(2\bar{t}+1)m^2}{m^3 + 4m^2 + 16m + 16} \quad \text{by (37),} \tag{53}$$

$$x_2 = \bar{t} - \frac{8(2\bar{t}+1)(m+2)}{m^3 + 4m^2 + 16m + 16} \quad \text{by (37).} \tag{54}$$

Thus we are led to the equation

$$2\bar{t} + 1 = n(m^3 + 4m^2 + 16m + 16), \text{ where } m, n, \bar{t} \text{ are integers.}$$

From this, we can put $m = 2\overline{m} + 1$, $n = 2\overline{n} + 1$ for integers $\overline{m}, \overline{n}$ again, then

$$\bar{t} = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 54\overline{m}\,\overline{n} + 27\overline{m} + 37\overline{n} + 18. \tag{55}$$

Thus

$$\frac{2\bar{t}+1}{m^3 + 4m^2 + 16m + 16} = n = 2\overline{n} + 1.$$

From these, we obtain the following:

$$x_0 = \bar{t} = 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 54\overline{m}\,\overline{n} + 27\overline{m} + 37\overline{n} + 18$$

by (37) and (55).

$$x_1 = \bar{t} + \frac{2(2\bar{t}+1)m^2}{m^3 + 4m^2 + 16m + 16} \quad \text{by (53)}$$

$$= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 44\overline{m}^2\overline{n} + 22\overline{m}^2 + 70\overline{m}\,\overline{n} + 35\overline{m} + 41\overline{n} + 20$$

by (55).

$$x_2 = \bar{t} - \frac{8(2\bar{t}+1)(m+2)}{m^3 + 4m^2 + 16m + 16} \quad \text{by (54)}$$

$$= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 28\overline{m}^2\overline{n} + 14\overline{m}^2 + 22\overline{m}\,\overline{n} + 11\overline{m} - 11\overline{n} - 6 \quad \text{by (55).}$$

$x_3 = x_0 + x_1 - x_2$ by (25)

$= 8\overline{m}^3\overline{n} + 4\overline{m}^3 + 44\overline{m}^2\overline{n} + 22\overline{m}^2 + 102\overline{m}\,\overline{n} + 51\overline{m} + 89\overline{n} + 44.$

This completes the proof. □

Note on the two types of solutions (51) and (52).

Replace coefficients of x_0 (resp. x_1) of (51) with reminders when dividing them by 2, so it is $\overline{m} + \overline{n}$ (resp. $\overline{m} + \overline{n} + 1$). Therefore, when substituting integers for $\overline{m}$ and $\overline{n}$, the parities of x_0 and x_1 are different. Under similar considerations on x_0 and x_1 of (52), both x_0 and x_1 become $\overline{m} + \overline{n}$. Hence when substituting integers for $\overline{m}$ and $\overline{n}$, the parities of x_0 and x_1 coincide. Consequently (51) and (52) do give us different solutions.

6. Examples of the Case of Triangular Numbers

We give examples of the sequence of triangular numbers $T_{x_i} = \frac{x_i(x_i+1)}{2}$ $(i = 0, 1, 2, 3)$ which are constructed in the previous section.

Table 5. Examples from (51)

$\overline{m}$	$\overline{n}$	x_0	x_1	x_2	x_3	a	b	T_{x_0}	$f(T_{x_0}) = T_{x_1}$	$f(T_{x_1}) = T_{x_2}$	$f(T_{x_2}) = T_{x_3}$
1	1	88	115	64	139	$-\frac{5}{3}$	$\frac{39590}{3}$	3916	6670	2080	9730
1	2	147	192	107	232	$-\frac{5}{3}$	36658	10878	18528	5778	27028
2	1	295	370	259	406	$-\frac{7}{5}$	129759	43660	68635	33670	82621
2	2	492	617	432	677	$-\frac{7}{5}$	$\frac{1802211}{5}$	121278	190653	93528	229503

Table 6. Examples from (52)

$\overline{m}$	$\overline{n}$	x_0	x_1	x_2	x_3	a	b	T_{x_0}	$f(T_{x_0}) = T_{x_1}$	$f(T_{x_1}) = T_{x_2}$	$f(T_{x_2}) = T_{x_3}$
1	1	190	244	70	364	$-\frac{7}{3}$	$\frac{216685}{3}$	18145	29890	2485	66430
1	2	317	407	117	607	$-\frac{7}{3}$	200635	50403	83028	6903	184528
2	1	481	631	313	799	$-\frac{9}{5}$	$\frac{2040269}{5}$	115921	199396	49141	319600
2	2	802	1052	522	1332	$-\frac{9}{5}$	$\frac{5667417}{5}$	322003	553878	136503	887778

Acknowledgments

The author thanks Professor Fumio Hazama for his useful advice.

References

[1] F. Hazama, Picard numbers of certain complete intersection surfaces, in Algebraic cycles and related topics, edited by F. Hazama, World Scientific (1995), 23-36. https://doi.org/10.1142/2629.

[2] The *L*-functions and modular forms database (LMFDB), Last viewed date: August 6, 2024. https://www.lmfdb.org/EllipticCurve/Q/438/e/3.

[3] Maplesoft, Maple 2023, Version Maple 2023.2.

[4] The PARI group, PARI/GP, Version 2.15.4, https://pari.math.u-bordeaux.fr/.

[5] J. H. Silverman, Advanced topics in the arithmetic of elliptic curves, Graduate Texts in Mathematics 151, Springer-Verlag, New York, 1994. https://doi.org/10.1007/978-1-4612-0851-8.

[6] J. H. Silverman, The Arithmetic of elliptic curves, Graduate Texts in Mathematics 106, Springer-Verlag, New York, 1986.

[7] J. H. Silverman, Heights and the specialization map for families of abelian varieties, J. Reine Angew. Math. 342 (1983), 197-211.

[8] H. Yamagishi, On abelian surfaces with rank 12 associated to certain K3 surfaces, in Algebraic cycles and related topics, edited by F. Hazama, World Scientific (1995), 93-102. https://doi.org/10.1142/2629.