Journal of Optimization Theory and Applications (2025) 206:3
https://doi.org/10.1007/s10957-025-02681-w

Non-linear Non-zero-Sum Dynkin Games with Bermudan Strategies

Miryana Grigorova[1] · **Marie-Claire Quenez**[2] · **Peng Yuan**[1]

Received: 10 September 2024 / Accepted: 30 March 2025 / Published online: 2 May 2025
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Abstract
In this paper, we study a non-zero-sum game with two players, where each of the players plays what we call Bermudan strategies and optimizes a general non-linear assessment functional of the pay-off. By using a recursive construction, we show that the game has a Nash equilibrium point.

Keywords Non-zero-sum game · Non-linear operator · Dynkin game · Optimal stopping · Bermudan strategy · Non-linear expectation · Nash equilibrium

Mathematics Subject Classification 60G40 · 90C90 · 91A10 · 91A15 · 91G70

1 Introduction

Game problems with linear evaluations between a finite number of players are by now classical problems in stochastic control and optimal stopping (cf., e.g., [1, 4, 5, 10, 15–20, 26, 27, 29]) with various applications, in particular in economics and finance (cf., e.g., [15, 16, 21, 22, 24–26]). In the recent years game problems with *non-linear* evaluation functionals have attracted considerable interest: cf. [3] for the case of non-linear functionals of the form of worst case expectations over a set of possibly singular measures; [3, 7–9, 12, 31] for the case of non-linear functionals induced by backward stochastic differential equations (BSDEs). Most of the works dealing with *non-linear* games have focused on the *zero-sum* case (cf., e.g., [3, 8, 9, 33]). *Non-zero-sum* games are notoriously more intricate than their zero-sum counterparts even in the case of linear evaluations (cf., e.g., [16, 18, 19, 28, 30, 33]). Non-zero-sum games with non-linear functionals have been considered in [14] in the discrete-time framework and with non-linear functionals induced by Backward SDEs with Lipschitz driver, in

Communicated by Mehdi Salimi.

✉ Miryana Grigorova
miryana.grigorova@warwick.ac.uk

[1] Department of Statistics, University of Warwick, Coventry, UK

[2] Université Paris-Cité, Paris, France

[23] in the continuous time framework and with non-linear functionals of the form of expected exponential utilities.

In the current paper, we address the question of existence of a Nash equilibrium point in a framework with general non-linear evaluations and with a set of stopping strategies, called here the set of Bermudan stopping strategies,[1] which is in between the discrete time and the continuous time stopping strategies. In [13], we addressed the single-agent counterpart of this problem (that is, the single-agent optimal stopping problem with Bermudan strategies and general non-linear evaluations).

The stopping framework of the current paper is more general than the discrete-time framework. It allows for a finite or a possibly infinite countable number of stopping opportunities $\theta_0(\omega), \ldots, \theta_n(\omega), \theta_{n+1}(\omega), \ldots$, depending on the scenario (or state of nature) ω, while still remaining tractable from a mathematical point of view. Arguably, it is closer to the way agents make stopping decisions in "real life" than the discrete-time framework (where the number of stopping times and the stopping times themselves are deterministic), and than the continuous time (which, from an applied point of view, might appear as rather "idealistic"). In the current paper, we work with general (non-linear) evaluation operators $\rho_{S,\tau}$ depending on two time indices: the time index S, which is the time of evaluation (or pricing, or assessment) and the second time index τ, which is the time when the pay-off falls. Recently, some other authors have also emphasised the importance of "horizon risk" in evaluation operators (cf., e.g., [32]). Moreover, we do not assume any concavity or convexity of the evaluation operators and do not rely on duality results.

The paper is organised as follows: In Sect. 2, we introduce the framework, including the set of optimal stopping strategies of the agents (namely the Bermudan strategies), as well as the properties on the risk functionals ρ^1 and ρ^2 of agent 1 and agent 2. In Sect. 3, we introduce the game problem and we present our main results and show that the non-linear non-zero-sum game with Bermudan strategies has a Nash equilibrium point. In Sect. 4 we conclude. The Appendix contains the proof of an auxiliary result and examples of non-linear operators entering our framework.

2 The Framework

Let $T > 0$ be a **fixed finite** terminal horizon.

Let $(\Omega, \mathcal{F}, P)$ be a (complete) probability space equipped with a right-continuous complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$.

In the sequel, equalities and inequalities between random variables are to be understood in the P-almost sure sense. Equalities between measurable sets are to be understood in the P-almost sure sense. Let $\mathbb{N}$ be the set of natural numbers, including 0. Let $\mathbb{N}^*$ be the set of natural numbers, excluding 0.

We first define the so-called **Bermudan stopping strategies** (introduced in [13]). Let $(\theta_k)_{k\in\mathbb{N}}$ be a sequence of stopping times satisfying the following properties:

(a) The sequence $(\theta_k)_{k\in\mathbb{N}}$ is non-decreasing, i.e. for all $k \in \mathbb{N}$, $\theta_k \leq \theta_{k+1}$, a.s.
(b) $\lim_{k\to\infty} \uparrow \theta_k = T$ a.s.

[1] This set is more general than the set of exercise times of Bermudan options in finance.

Moreover, we set $\theta_0 = 0$.

We note that the family of σ-algebras $(\mathcal{F}_{\theta_k})_{k\in\mathbb{N}}$ is non- decreasing (as the sequence (θ_k) is non-decreasing). We denote by Θ the set of stopping times τ of the form

$$\tau = \sum_{k=0}^{+\infty} \theta_k \mathbf{1}_{A_k} + T\mathbf{1}_{\bar{A}}, \tag{1}$$

where $\{(A_k)_{k=0}^{+\infty}, \bar{A}\}$ form a partition of Ω such that, for each $k \in \mathbb{N}$, $A_k \in \mathcal{F}_{\theta_k}$, and $\bar{A} \in \mathcal{F}_T$.

The set Θ can also be described as the set of stopping times τ such that for almost all $\omega \in \Omega$, either $\tau(\omega) = T$ or $\tau(\omega) = \theta_k(\omega)$, for some $k = k(\omega) \in \mathbb{N}$.

Note that the set Θ is closed under concatenation, that is, for each $\tau \in \Theta$ and each $A \in \mathcal{F}_\tau$, the stopping time $\tau\mathbf{1}_A + T\mathbf{1}_{A^c} \in \Theta$. More generally, for each $\tau \in \Theta$, $\tau' \in \Theta$ and each $A \in \mathcal{F}_{\tau\wedge\tau'}$, the stopping time $\tau\mathbf{1}_A + \tau'\mathbf{1}_{A^c}$ is in Θ. The set Θ is also closed under pairwise minimization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \wedge \tau' \in \Theta$) and under pairwise maximization (that is, for each $\tau \in \Theta$ and $\tau' \in \Theta$, we have $\tau \vee \tau' \in \Theta$). Moreover, the set Θ is closed under monotone limit, that is, for each non-decreasing (resp. non-increasing) sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \in \Theta^{\mathbb{N}}$, we have $\lim_{n\to+\infty} \tau_n \in \Theta$.

We note also that all stopping times in Θ are bounded from above by T.

Remark 2.1 We recall from [13] the following *canonical* writing of the sets in (1):

$$\begin{aligned}
&A_0 = \{\tau = \theta_0\};\\
&A_{n+1} = \{\tau = \theta_{n+1}, \theta_{n+1} < T\}\backslash(A_n \cup \ldots \cup A_0); \text{ for all } n \in \mathbb{N}^*\\
&\bar{A} = (\cup_{k=0}^{+\infty} A_k)^c
\end{aligned}$$

From this writing, we have: if $\omega \in A_{k+1} \cap \{\theta_k < T\}$, then $\omega \notin \{\tau = \theta_k\}$.

For each $\tau \in \Theta$, we denote by Θ_τ the set of stopping times $\nu \in \Theta$ such that $\nu \geq \tau$ a.s. The set Θ_τ satisfies the same properties as the set Θ. We will refer to the set Θ as the set of **Bermudan stopping strategies**, and to the set Θ_τ as the set of Bermudan stopping strategies, greater than or equal to τ (or the set of Bermudan stopping strategies from time τ perspective). For simplicity, the set Θ_{θ_k} will be denoted by Θ_k.

Definition 2.1 We say that a family $\phi = (\phi(\tau),\ \tau \in \Theta)$ is *admissible* if it satisfies the following conditions

1. for all $\tau \in \Theta$, $\phi(\tau)$ is a real valued random variable, which is $\mathcal{F}_\tau$-measurable.
2. for all $\tau, \tau' \in \Theta$, $\phi(\tau) = \phi(\tau')$ a.s. on $\{\tau = \tau'\}$.

Moreover, for $p \in [1, +\infty]$ fixed, we say that an admissible family ϕ is p-integrable, if for all $\tau \in \Theta$, $\phi(\tau)$ is in L^p.

Example Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space satisfying the usual assumptions. Let $p \in [1, +\infty]$. Let $(\xi_t)_{t\in[0,T]}$ be an optional process. Then, the family $\phi = (\phi(\tau),\ \tau \in \Theta)$, defined by $\phi(\tau) := \xi_\tau$ for each $\tau \in \Theta$, is admissible. If moreover, $\xi_\tau \in L^p$, for all $\tau \in \Theta$, then the family ϕ is p-integrable.

Let $\phi = (\phi(\tau),\ \tau \in \Theta)$ be an admissible family. For a stopping time τ of the form (1), we have

$$\phi(\tau) = \sum_{k=0}^{+\infty} \phi(\theta_k)\mathbf{1}_{A_k} + \phi(T)\mathbf{1}_{\bar{A}} \quad \text{a.s.} \tag{2}$$

Given two admissible families $\phi = (\phi(\tau),\ \tau \in \Theta)$ and $\phi' = (\phi'(\tau),\ \tau \in \Theta)$, we say that ϕ is *equal to* ϕ' and write $\phi = \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) = \phi'(\tau)$ a.s. We say that ϕ *dominates* ϕ' and write $\phi \geq \phi'$ if, for all $\tau \in \Theta$, $\phi(\tau) \geq \phi'(\tau)$ a.s.

Let $p \in [1, +\infty]$. We introduce the following properties on the non-linear operators $\rho_{S,\tau}[\cdot]$, which will appear in the sequel.

For $S \in \Theta$, $S' \in \Theta$, $\tau \in \Theta$, for η, η_1 and η_2 in $L^p(\mathcal{F}_\tau)$, for $\xi = (\xi(\tau))$ an admissible p-integrable family:

(i) $\rho_{S,\tau} : L^p(\mathcal{F}_\tau) \longrightarrow L^p(\mathcal{F}_S)$
(ii) *(admissibility)* $\rho_{S,\tau}[\eta] = \rho_{S',\tau}[\eta]$ a.s. on $\{S = S'\}$.
(iii) *(knowledge preservation)* $\rho_{\tau,S}[\eta] = \eta$, for all $\eta \in L^p(\mathcal{F}_S)$, all $\tau \in \Theta_S$.
(iv) *(monotonicity)* $\rho_{S,\tau}[\eta_1] \leq \rho_{S,\tau}[\eta_2]$ a.s., if $\eta_1 \leq \eta_2$ a.s.
(v) *(consistency)* $\rho_{S,\theta}[\rho_{\theta,\tau}[\eta]] = \rho_{S,\tau}[\eta]$, for all S, θ, τ in Θ such that $S \leq \theta \leq \tau$ a.s.
(vi) *("generalized zero–one law")* $I_A \rho_{S,\tau}[\xi(\tau)] = I_A \rho_{S,\tau'}[\xi(\tau')]$, for all $A \in \mathcal{F}_S$, $\tau \in \Theta_S$, $\tau' \in \Theta_S$ such that $\tau = \tau'$ on A.
(vi bis) *("usual zero–one law")* $I_A \rho_{S,\tau}[\eta] = \rho_{S,\tau}[I_A \eta]$, for all $A \in \mathcal{F}_S$, $\tau \in \Theta_S$.
(vii) *(monotone Fatou property with respect to terminal condition)* $\rho_{S,\tau}[\eta] \leq \liminf_{n\to+\infty} \rho_{S,\tau}[\eta_n]$, for (η_n), η such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n\to+\infty} \uparrow \eta_n = \eta$ a.s.
(viii) (left-upper-semicontinuity (LUSC) along Bermudan stopping times with respect to the terminal condition and the terminal time), that is,

$$\limsup_{n\to+\infty} \rho_{S,\tau_n}[\phi(\tau_n)] \leq \rho_{S,\nu}[\limsup_{n\to+\infty} \phi(\tau_n)],$$

for each non-decreasing sequence $(\tau_n) \in \Theta_S^{\mathbb{N}}$ such that $\lim_{n\to+\infty} \uparrow \tau_n = \nu$ a.s., and for each p-integrable admissible family ϕ such that $\sup_{n\in\mathbb{N}} |\phi(\tau_n)| \in L^p$.
(ix) $\limsup_{n\to+\infty} \rho_{\theta_n,T}[\eta] \leq \rho_{T,T}[\eta]$, for all $\eta \in L^p(\mathcal{F}_T)$.

Let us provide some explanations on these properties. Properties (i) and (ii) (admissibility) ensure that for each $\eta \in L^p(\mathcal{F}_\tau)$, the family of random variables $(\rho_{S,\tau}(\eta))_S$ is admissible (in the sense of Definition 2.1). The knowledge preservation property (property (iii)) implies, in particular, that $\rho_{S,S}(\eta) = \eta$, for $\eta \in L^p(\mathcal{F}_S)$; in other words, if a random variable η is "known" at time S (in the sense that η is $\mathcal{F}_S$-measurable), then its "risk" (or "utility") $\rho_{S,S}(\eta)$ at time S is equal to η itself. The monotonicity (property (iv)) is the usual monotonicity of the operator $\rho_{S,\tau}$ in the a.s. sense. The consistency property (property (v)) can be seen as a generalization of the tower property of conditional expectations, to the case of families of operators depending on two time indices. It ensures the consistency of evaluation through time. The "usual

zero–one-law" (property (vi bis)) is often assumed in the risk management literature (sometimes, it is referred to as regularity property or also local property). We note that when ρ does not depend on the second time index, the "usual zero–one law" implies the "generalized zero–one law" (property (vi)). The "generalized zero–one law" is useful when the operators depend on both time indices (the time of evaluation and the time when the pay-off falls). Property (vii) is the Fatou property of the operator $\rho_{S,\tau}$ along non-decreasing sequences of pay-offs (η_n); Fatou property is often assumed in the risk measurement literature, especially in the case where $p = +\infty$. Note also that if ρ satisfies the monotonicity (iv) and the (monotone) Fatou property with respect to terminal condition (vii), then $\rho_{S,\tau}[\eta] = \lim_{n\to+\infty} \rho_{S,\tau}[\eta_n]$, for (η_n), η such that (η_n) is non-decreasing, $\eta_n \in L^p(\mathcal{F}_\tau)$, $\sup_n \eta_n \in L^p$, and $\lim_{n\to+\infty} \uparrow \eta_n = \eta$ a.s. This type of property is also known in the literature (e.g. risk measures) as continuity from below. We note that, properties (i), (ii), (iii), (iv), (v), (vi), (vii), together with appropriate integrability assumptions on the pay-off family, ensure that the value family of the one agent's non-linear optimal stopping problem with Bermudan strategies satisfies the dynamic programming principle (cf. Theorem 2.17 in [13]). Then, these properties, together with the upper-semicontinuity on the left-type properties (viii) and (ix) (and the upper-semicontinuity on the left of the pay-off family at the maturity T) ensure that the one-agent's non-linear optimal stopping problem with Bermudan strategies admits an optimal stopping time and that the first hitting time (when the value family "hits" the pay-off family) is optimal (cf. [13], Theorem 2.19 and Proposition 2.29 for more details). Note that in [13], the "usual zero–one law" (vi bis) is not needed. However, in the current paper, we use a localisation property for the value family of the single-agent optimal stopping problem (cf. Proposition A.1 in the Appendix), which holds under (vi bis).

Moreover, we introduce an additional assumption on ρ, which will be used later on (in Proposition 3.4) to "pass to the limit on the right":

$$(\mathrm{x}) \qquad \limsup_{n\to+\infty} \rho_{0,\nu_n}[\phi(\nu_n)] = \rho_{0,\nu}[\lim_{n\to+\infty} \phi(\nu_n)], \tag{3}$$

for any sequence $(\nu_n) \subset \Theta^{\mathbb{N}}$, $\nu \in \Theta$, such that $\nu_n \downarrow \nu$, and for any p-integrable admissible family ϕ such that $\sup_{n\in\mathbb{N}} |\phi(\tau_n)| \in L^p$.

Example We provide two examples of non-linear operators $\rho_{S,\tau}$, popular in stochastic control and mathematical finance, which enter our framework:

1. Non-linear evaluations induced by BSDEs.
2. Dynamic concave utilities.

More details about these examples can be found in the Appendix.

3 The Game Problem

We consider two agents, agent 1 and agent 2, whose pay-offs are defined via four admissible families $X^1 = (X^1(\tau))_{\tau\in\Theta}$, $X^2 = (X^2(\tau))_{\tau\in\Theta}$, $Y^1 = (Y^1(\tau))_{\tau\in\Theta}$ and $Y^2 = (Y^2(\tau))_{\tau\in\Theta}$. We assume that X^1, X^2, Y^1 and Y^2 are p-integrable families such that

(A1) $X^1 \leq Y^1,\ X^2 \leq Y^2$ (that is, for each $\tau \in \Theta$, $X^1(\tau) \leq Y^1(\tau)$, and $X^2(\tau) \leq Y^2(\tau)$).
(A2) $X^1(T) = Y^1(T),\ X^2(T) = Y^2(T)$.
(A3) $\text{ess sup}_{\tau\in\Theta} X^1(\tau) \in L^p$, $\text{ess sup}_{\tau\in\Theta} X^2(\tau) \in L^p$, $\text{ess sup}_{\tau\in\Theta} Y^1(\tau) \in L^p$ and $\text{ess sup}_{\tau\in\Theta} Y^2(\tau) \in L^p$.
(A4) $\limsup_{k\to+\infty} X^1(\theta_k) \leq X^1(T),\ \limsup_{k\to+\infty} X^2(\theta_k) \leq X^2(T)$.

The set of stopping strategies of each agent at time 0 is the set Θ of Bermudan stopping times. If the first agent plays $\tau_1 \in \Theta$ and the second agent plays $\tau_2 \in \Theta$, the pay-off of agent 1 (resp. agent 2) at time $\tau_1 \wedge \tau_2$ is given by:

$$I^1(\tau_1, \tau_2) := X^1(\tau_1)\mathbb{1}_{\{\tau_1\leq\tau_2\}} + Y^1(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\}}$$
$$(\text{resp. } I^2(\tau_1, \tau_2) := X^2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\}} + Y^2(\tau_1)\mathbb{1}_{\{\tau_1\leq\tau_2\}}),$$

where we have adopted the convention: when $\tau_1 = \tau_2$, it is the first agent who is responsible for stopping the game. The agents evaluate their respective pay-offs via possibly different evaluation functionals. Let $\rho^1 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agent 1, and let $\rho^2 = (\rho_{S,\tau}[\cdot])$ be the family of evaluation operators of agents 2. If agent 1 plays $\tau_1 \in \Theta$, and agent 2 plays $\tau_2 \in \Theta$, then the assessment (or evaluation) of agent 1 (resp. agent 2) at time 0 of his/her pay-off is given by:

$$J_1(\tau_1, \tau_2) := \rho^1_{0,\tau_1\wedge\tau_2}[X^1(\tau_1)\mathbb{1}_{\{\tau_1\leq\tau_2\}} + Y^1(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\}}].$$
$$(\text{resp. } J_2(\tau_1, \tau_2) := \rho^2_{0,\tau_1\wedge\tau_2}[X^2(\tau_2)\mathbb{1}_{\{\tau_2<\tau_1\}} + Y^2(\tau_1)\mathbb{1}_{\{\tau_1\leq\tau_2\}}]).$$

We assume that both ρ^1 and ρ^2 satisfy the properties (i)–(x). We will investigate the problem of existence of a Nash equilibrium strategy (τ_1^*, τ_2^*).

Definition 3.1 A pair of Bermudan stopping times $(\tau_1^*, \tau_2^*) \in \Theta \times \Theta$ is called a Nash equilibrium strategy (or a Nash equilibrium point) for the above non-zero-sum non-linear Bermudan Dynkin game if: $J_1(\tau_1^*, \tau_2^*) \geq J_1(\tau_1, \tau_2^*)$, for any $\tau_1 \in \Theta$, and $J_2(\tau_1^*, \tau_2^*) \geq J_2(\tau_1^*, \tau_2)$, for any $\tau_2 \in \Theta$.

In other words, any unilateral deviation from the strategy (τ_1^*, τ_2^*) by one of the agents (the strategy of the other remaining fixed) does not render the deviating agent better off.

Theorem 3.1 *Under assumptions (i)–(x) on ρ^1 and ρ^2 and assumptions (A1)-(A4) on X^1, X^2, Y^1 and Y^2, there exists a Nash equilibrium point (τ_1^*, τ_2^*) for the game described above.*

The sequel is dedicated to the proof of this theorem. We follow an approach introduced in [16] (cf. also [18]) for the case of linear expectations and in continuous time. We will construct a sequence $(\tau_{2n+1}, \tau_{2n})_{n\in\mathbb{N}}$ (by induction), for which we will show that it converges to a Nash equilibrium point.

We set $\tau_1 := T$ and $\tau_2 := T$. We suppose that $\tau_{2n-1} \in \Theta$ and $\tau_{2n} \in \Theta$ have been defined. We set, for each $k \in \mathbb{N}$,

$$\xi^{2n+1}(\theta_k) := X^1(\theta_k)\mathbb{1}_{\{\theta_k<\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\leq\theta_k\}}. \tag{4}$$

Moreover, $\xi^{2n+1}(T) := Y^1(\tau_{2n})$. This definition is "consistent" with the above, as by (4), $\mathbb{1}_{\{\theta_k = T\}} \xi^{2n+1}(\theta_k) = \mathbb{1}_{\{\theta_k = T\}} Y^1(\tau_{2n})$.

[0.3cm] For $\tau \in \Theta$ of the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$, where $((A_k), \bar{A})$ is a partition, A_k is $\mathcal{F}_{\theta_k}$-measurable for each $k \in \mathbb{N}$, and $\bar{A}$ is $\mathcal{F}_T$-measurable,

$$\xi^{2n+1}(\tau) := \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}}. \tag{5}$$

We note that $\xi^{2n+1}(\theta_k)$ is the pay-off at $\theta_k \wedge \tau_{2n}$ of agent 1 (up to the equality $\{\theta_k = \tau_{2n}\}$) if agent 1 plays θ_k and agent 2 plays τ_{2n}.

We also note that:

$$\xi^{2n+1}(\tau) = X^1(\tau)\mathbb{1}_{\{\tau < \tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n} \le \tau\}}.$$

Thus, $\xi^{2n+1}(\tau)$ is the pay-off at $\tau \wedge \tau_{2n}$ of agent 1 (up to the equality $\{\tau = \tau_{2n}\}$) if agent 1 plays τ and agent 2 plays τ_{2n}.

For each $S \in \Theta$, we define

$$\begin{aligned} &V^{2n+1}(S) := \text{ess sup}_{\tau\in\Theta_S} \rho^1_{S,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau)] \\ &\tilde{\tau}_{2n+1} := \text{ess inf } \tilde{\mathcal{A}}^1, \text{ where } \tilde{\mathcal{A}}^1 := \{\tau \in \Theta : V^{2n+1}(\tau) = \xi^{2n+1}(\tau)\} \\ &\tau_{2n+1} := (\tilde{\tau}_{2n+1} \wedge \tau_{2n-1})\mathbb{1}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1} < \tau_{2n}\}} + \tau_{2n-1}\mathbb{1}_{\{\tilde{\tau}_{2n+1}\wedge\tau_{2n-1} \ge \tau_{2n}\}}. \end{aligned} \tag{6}$$

Assuming that $\limsup_{k\to+\infty} X^1(\theta_k) \le X^1(T)$ (from (A4)) ensures that

$\limsup_{k\to+\infty} \xi^{2n+1}(\theta_k) \le \xi^{2n+1}(T)$. This is a technical condition on the pay-off which we use to apply Theorem 2.19, together with Prop. 2.29, in [13].

By Theorem 2.19, together with Prop. 2.29 in [13], the Bermudan stopping time $\tilde{\tau}_{2n+1}$ is optimal for the optimal stopping problem with value $V^{2n+1}(0)$, that is

$$V^{2n+1}(0) = \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}[\xi(\tilde{\tau}_{2n+1})] = \sup_{\tau\in\Theta} \rho^1_{0,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau)]. \tag{7}$$

We also recall that $V^{2n+1}(T) = \xi^{2n+1}(T)$, under the assumption of knowledge preservation on ρ.

Remark 3.1 i) It is not difficult to show, by induction, that for each $n \in \mathbb{N}$, $(\xi^{2n+1}(\tau))_{\tau\in\Theta}$ is an admissible L^p-integrable family, and τ_{2n+1} is a Bermudan stopping time (for the latter property, we use that Θ has the property of stability by concatenation of two Bermudan stopping times).

ii) For each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau \wedge \tau_{2n})$.

Indeed, we have:

$$\begin{aligned} \xi^{2n+1}(\theta_k) &= X^1(\theta_k)\mathbb{1}_{\{\theta_k < \tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n} \le \theta_k\}} \\ &= X^1(\theta_k \wedge \tau_{2n})\mathbb{1}_{\{\theta_k\wedge\tau_{2n} < \tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n} \le \theta_k\wedge\tau_{2n}\}} \\ &= \xi^{2n+1}(\theta_k \wedge \tau_{2n}). \end{aligned} \tag{8}$$

Now, let $\tau \in \Theta$ be of the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$. By definition of $\xi^{2n+1}(\tau)$, of $\xi^{2n+1}(T)$ and by Eq. (8), we have:

$$\begin{aligned}\xi^{2n+1}(\tau) &= \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}} \\ &= \sum_{k\in\mathbb{N}} \xi^{2n+1}(\theta_k \wedge \tau_{2n})\mathbb{1}_{A_k} + Y^1(\tau_{2n})\mathbb{1}_{\bar{A}} = \xi^{2n+1}(\tau \wedge \tau_{2n}).\end{aligned}$$

Proposition 3.1 *i)* $\xi^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\le\theta_k\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\le\theta_k\}}$.
ii) $V^{2n+1}(\theta_k)\mathbb{1}_{\{\tau_{2n}\le\theta_k\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\le\theta_k\}}$.
iii) $V^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\le\tau\}} = Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\le\tau\}}$.
iv) For each $n \in \mathbb{N}$, $\tilde{\tau}_{2n+1} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{2n+1}(\tau) = X^1(\tau)\} \wedge \tau_{2n}$. *In particular,* $\tilde{\tau}_{2n+1} \le \tau_{2n}$.

Proof i) On the set $\{\tau = T\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(T) = Y^1(\tau_{2n})$. On the set $\{\tau = \theta_k < T\}$, by the second statement in Remark 3.1, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\theta_k) = \xi^{2n+1}(\theta_k \wedge \tau_{2n})$. Hence, on the set $\{\tau = \theta_k < T\} \cap \{\tau_{2n} \le \theta_k\}$, we have $\xi^{2n+1}(\tau) = \xi^{2n+1}(\tau_{2n}) = Y^1(\tau_{2n})$, which proves the desired property.
ii) We have:

$$\begin{aligned}\mathbb{1}_{\{\tau_{2n}\le\theta_k\}} V^{2n+1}(\theta_k) &= \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \operatorname{ess\,sup}_{\tau\in\Theta_k} \rho^1_{\theta_k,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau \wedge \tau_{2n})] \\ &= \operatorname{ess\,sup}_{\tau\in\Theta_k} \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \rho^1_{\theta_k,\tau\wedge\tau_{2n}}[\xi^{2n+1}(\tau \wedge \tau_{2n})] \\ &= \operatorname{ess\,sup}_{\tau\in\Theta_k} \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \rho^1_{\theta_k,\tau\wedge\tau_{2n}\wedge\theta_k}[\xi^{2n+1}(\tau \wedge \tau_{2n} \wedge \theta_k)],\end{aligned}$$

where we have used property (vi) ("genrealized zero–one law") to obtain the last equality.

For any $\tau \in \Theta_k$, $\tau \wedge \tau_{2n} \wedge \theta_k = \tau_{2n} \wedge \theta_k \le \theta_k$. Hence,

$$\begin{aligned}&\mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \rho^1_{\theta_k,\tau\wedge\tau_{2n}\wedge\theta_k}[\xi^{2n+1}(\tau \wedge \tau_{2n} \wedge \theta_k)] \\ &= \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \rho^1_{\theta_k,\tau_{2n}\wedge\theta_k}[\xi^{2n+1}(\tau_{2n} \wedge \theta_k)] \\ &= \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \xi^{2n+1}(\tau_{2n} \wedge \theta_k),\end{aligned}$$

where we have used property (iii) (knowledge preserving property) of ρ to obtain the last equality.

Finally, we get

$$\mathbb{1}_{\{\tau_{2n}\le\theta_k\}} V^{2n+1}(\theta_k) = \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} \xi^{2n+1}(\tau_{2n}) = \mathbb{1}_{\{\tau_{2n}\le\theta_k\}} Y^1(\tau_{2n}).$$

iii) Let $\tau \in \Theta$ be the form $\tau = \sum_{k\in\mathbb{N}} \theta_k \mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$. Then, by property (ii) (admissibility) of ρ, we have

$$V^{2n+1}(\tau) = \sum_{k\in\mathbb{N}} V^{2n+1}(\theta_k)\mathbb{1}_{A_k} + V^{2n+1}(T)\mathbb{1}_{\bar{A}}.$$

Hence,

$$\begin{aligned} V^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\le\tau\}} &= \sum_{k\in\mathbb{N}} V^{2n+1}(\theta_k)\mathbb{1}_{A_k\cap\{\tau_{2n}\le\tau\}} + V^{2n+1}(T)\mathbb{1}_{\bar{A}\cap\{\tau_{2n}\le\tau\}} \\ &= \sum_{k\in\mathbb{N}} V^{2n+1}(\theta_k)\mathbb{1}_{A_k\cap\{\tau_{2n}\le\theta_k\}} + V^{2n+1}(T)\mathbb{1}_{\bar{A}\cap\{\tau_{2n}\le T\}} \\ &= \sum_{k\in\mathbb{N}} Y^1(\tau_{2n})\mathbb{1}_{A_k\cap\{\tau_{2n}\le\theta_k\}} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}\cap\{\tau_{2n}\le T\}}, \end{aligned}$$

where we have used the previous property ii) to obtain the last equality. Hence, we get

$$\begin{aligned} V^{2n+1}(\tau)\mathbb{1}_{\{\tau_{2n}\le\tau\}} &= \sum_{k\in\mathbb{N}} Y^1(\tau_{2n})\mathbb{1}_{A_k\cap\{\tau_{2n}\le\theta_k\}} + \xi^{2n+1}(T)\mathbb{1}_{\bar{A}\cap\{\tau_{2n}\le T\}} \\ &= \sum_{k\in\mathbb{N}} Y^1(\tau_{2n})\mathbb{1}_{A_k\cap\{\tau_{2n}\le\theta_k\}} + Y^1(\tau_{2n})\mathbb{1}_{\bar{A}\cap\{\tau_{2n}\le T\}} \\ &= Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}\le\tau\}}. \end{aligned}$$

iv) By the previous property iii), we have, $V^{2n+1}(\tau) = \xi^{2n+1}(\tau)$ if and only if $V^{2n+1}(\tau)\mathbb{1}_{\{\tau<\tau_{2n}\}} = \xi^{2n+1}(\tau)\mathbb{1}_{\{\tau<\tau_{2n}\}}$. Hence,

$$\tilde{\tau}_{2n+1} = \operatorname{ess\,inf}\{\tau\in\Theta : V^{2n+1}(\tau) = X^1(\tau)\}\wedge\tau_{2n}. \qquad \square$$

Similarly to (4), (5) and (6), we define:

$$\begin{aligned} \xi^{2n+2}(\theta_k) &:= X^2(\theta_k)\mathbb{1}_{\{\theta_k<\tau_{2n+1}\}} + Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\le\theta_k\}}, \text{ and} \\ \xi^{2n+2}(T) &:= Y^2(\tau_{2n+1}). \end{aligned}$$

For $\tau\in\Theta$ of the form $\tau = \sum_{k\in\mathbb{N}}\theta_k\mathbb{1}_{A_k} + T\mathbb{1}_{\bar{A}}$, we define

$$\begin{aligned} &\xi^{2n+2}(\tau) := \sum_{k\in\mathbb{N}} \xi^{2n+2}(\theta_k)\mathbb{1}_{A_k} + \xi^{2n+2}(T)\mathbb{1}_{\bar{A}} \\ &V^{2n+2}(S) := \operatorname{ess\,sup}_{\tau\in\Theta_S}\rho^2_{S,\tau\wedge\tau_{2n+1}}[\xi^{2n+2}(\tau)] \\ &\tilde{\tau}_{2n+2} := \operatorname{ess\,inf}\tilde{\mathcal{A}}^2, \text{ where } \tilde{\mathcal{A}}^2 := \{\tau\in\Theta : V^{2n+2}(\tau) = \xi^{2n+2}(\tau)\} \\ &\tau_{2n+2} := (\tilde{\tau}_{2n+2}\wedge\tau_{2n})\mathbb{1}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}<\tau_{2n+1}\}} + \tau_{2n}\mathbb{1}_{\{\tilde{\tau}_{2n+2}\wedge\tau_{2n}\ge\tau_{2n+1}\}}. \end{aligned} \tag{9}$$

The random variable $\xi^{2n+2}(\tau)$ is exactly the pay-off at $\tau\wedge\tau_{2n+1}$ of agent 2, if agent 1 plays τ_{2n+1} and agent 2 plays τ. Hence, $V^{2n+2}(S)$ is the optimal value at time S for agent 2, when agent 1's strategy is fixed to τ_{2n+1}. Assuming that $\limsup_{k\to+\infty} X^2(\theta_k)\le X^2(T)$ leads to $\limsup_{k\to+\infty}\xi^{2n+2}(\theta_k)\le\xi^{2n+2}(T)$, which we use again in applying Proposition 2.29 and Theorem 2.19 in [13]. By [13], the Bermudan stopping time $\tilde{\tau}_{2n+2}$ is optimal for the problem with value $V^{2n+2}(0)$, that

is,

$$V^{2n+2}(0) = \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}\big[\xi^{2n+2}(\tilde{\tau}_{2n+2})\big] = \sup_{\tau\in\Theta}\rho^2_{0,\tau\wedge\tau_{2n+1}}\big[\xi^{2n+2}(\tau)\big]. \quad (10)$$

Remark 3.2 Let us recall that (cf. [13]) $\tilde{\tau}_n = \operatorname{ess\,inf}\{\tau \in \Theta : V^n(\tau) = \xi^n(\tau)\}$ satisfies the property: $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n)$. (This is due to the property of stability of Θ by monotone limit and to the right-continuity-along Bermudan stopping strategies of the families $(V^n(\tau))$ and $(\xi^n(\tau))$).

Remark 3.3 By analogy with Remark 3.1, we have:
i) $(\xi^{2n+2}(\tau))$ is an admissible L^p-integrable family;
ii) for each $n \in \mathbb{N}$, for each $\tau \in \Theta$, $\xi^{2n+2}(\tau) = \xi^{2n+2}(\tau \wedge \tau_{2n+1})$.

By analogy with Proposition 3.1 and using Remark 3.3, the following holds true:

Proposition 3.2 *i)* $\xi^{2n+2}(\tau)\mathbb{1}_{\{\tau_{2n+1}\le\theta_k\}} = Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\le\theta_k\}}$
ii) $V^{2n+2}(\theta_k)\mathbb{1}_{\{\tau_{2n+1}\le\theta_k\}} = Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\le\theta_k\}}$.
iii) $V^{2n+2}(\tau)\mathbb{1}_{\{\tau_{2n+1}\le\tau\}} = Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\le\tau\}}$
iv) For each $n \in \mathbb{N}$, $\tilde{\tau}_{2n+2} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{2n+2}(\tau) = X^2(\tau)\} \wedge \tau_{2n+1}$. *In particular,* $\tilde{\tau}_{2n+2} \le \tau_{2n+1}$.

Proof The proof follows the same steps as the proof of Proposition 3.1 and is left to the reader. □

Proposition 3.3 *We assume that* ρ *satisfies property (vi bis) (usual "zero–one law"). Then, for all* $m \ge 1$, $\tilde{\tau}_{m+2} \le \tau_m$.

Proof We suppose, by way of contradiction, that there exists $m \ge 1$ such that $P(\tilde{\tau}_{m+2} > \tau_m) > 0$, and we set $n := \min\{m \ge 1 : P(\tilde{\tau}_{m+2} > \tau_m) > 0\}$. We have $\tilde{\tau}_{n+1} \le \tau_{n-1}$, by definition of n. This observation, together with the definition of τ_{n+1} and with the inequality of part (iv) of Proposition 3.1 gives:

$$\begin{aligned}\tau_{n+1} &= (\tilde{\tau}_{n+1} \wedge \tau_{n-1})\mathbb{1}_{\{\tilde{\tau}_{n+1}\wedge\tau_{n-1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}\wedge\tau_{n-1}\ge\tau_n\}} \\ &= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}\ge\tau_n\}} \\ &= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}}\end{aligned} \quad (11)$$

For similar reasons, we have

$$\tau_n = \tilde{\tau}_n\mathbb{1}_{\{\tilde{\tau}_n<\tau_{n-1}\}} + \tau_{n-2}\mathbb{1}_{\{\tilde{\tau}_n=\tau_{n-1}\}}. \quad (12)$$

For the easing of the presentation, we set $\Gamma := \{\tau_n < \tilde{\tau}_{n+2}\}$.

On the set Γ, we have:

1) $\tau_n < \tilde{\tau}_{n+2} \le \tau_{n+1}$, the last inequality being due to property iv) of Proposition 3.1.

2) $\tau_{n+1} = \tau_{n-1}$. This is due to 1), together with Eq. (11). Indeed, by Eq. (11), $\tau_{n+1} = \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}}$. By 1), we have: $\tau_{n+1} > \tau_n$. Hence, $\tau_{n+1} =$

$\tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n<\tau_{n+1}\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}}$. Hence, as we cannot have: $\tau_{n+1} = \tilde{\tau}_{n+1}$ and $\tilde{\tau}_{n+1} < \tau_{n+1}$ simultaneously, we get $\tau_{n+1} = \tau_{n-1}$.

3) $\xi^{n+2} = \xi^n$. This is a consequence of 2) and the definitions of ξ^{n+2} and ξ^n.

4) $\tau_n = \tilde{\tau}_n$.

We prove that $\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \emptyset$, which together with Eq. (12), gives the desired statement.

Due to Eq. (12), we have $\{\tilde{\tau}_n = \tau_{n-1}\} = \{\tilde{\tau}_n = \tau_{n-1}\} \cap \{\tau_n = \tau_{n-2}\}$. Thus, we have

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1}, \tau_n = \tau_{n-2} < \tilde{\tau}_{n+2}\}.$$

Now, we have $\tilde{\tau}_n \leq \tau_{n-2}$ (due to the definition of n). Hence,

$$\{\tilde{\tau}_n = \tau_{n-1}\} \cap \Gamma = \{\tilde{\tau}_n = \tau_{n-1} \leq \tau_{n-2} = \tau_n < \tilde{\tau}_{n+2}\} = \emptyset,$$

where the equality with $\emptyset$ is due to $\tilde{\tau}_{n+2} \leq \tau_{n-1}$.

We note that combining properties 1) and 4) gives $\tilde{\tau}_n < \tilde{\tau}_{n+2}$ on Γ. We will obtain a contradiction with this property. To this end, we will show that:

$$\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma \xi^{n+2}(\tilde{\tau}_n). \tag{13}$$

By definition of $\tilde{\tau}_n$ and by Remark 3.2, we have: $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n)$. This property, together with property 3) on Γ, gives $V^n(\tilde{\tau}_n) = \xi^n(\tilde{\tau}_n) = \xi^{n+2}(\tilde{\tau}_n)$ on Γ. In order to show Eq. (13), it suffices to show

$$\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n).$$

By property 4) on Γ and Proposition A.1 (applied with $A = \Gamma \in \mathcal{F}_{\tau_n}$ and $\tau = \tau_n$), we have $\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^{n+2}(\tau_n) = V_\Gamma^{n+2}(\tau_n)$. Due to property 3) on Γ, V_Γ^{n+2} and V_Γ^n have the same pay-off, and by applying again Proposition A.1 and property 4) on Γ, we have

$$V_\Gamma^{n+2}(\tau_n) = V_\Gamma^n(\tau_n) = \mathbb{1}_\Gamma V^n(\tau_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n).$$

We have thus shown that $\mathbb{1}_\Gamma V^{n+2}(\tilde{\tau}_n) = \mathbb{1}_\Gamma V^n(\tilde{\tau}_n)$, which is the desired equality. Hence, we get $\tilde{\tau}_{n+2} \leq \tilde{\tau}_n$ on Γ (as by definition $\tilde{\tau}_{n+2} = \operatorname{ess\,inf}\{\tau \in \Theta : V^{n+2}(\tau) = \xi^{n+2}(\tau)\}$). However, this is in contradiction with the property $\tilde{\tau}_{n+2} > \tilde{\tau}_n$ on Γ. The proof is complete. □

Lemma 3.1 *a) For all* $n \geq 2$, $\tau_{n+1} = \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tau_{n-1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}}$.

b) For all $n \geq 2$, $\tilde{\tau}_{n+1} = \tau_{n+1} \wedge \tau_n$.

c) On $\{\tau_n = \tau_{n-1}\}$, $\tau_m = T$, *for all* $m \in \{1, \ldots, n\}$.

Proof a) This property follows from the definition of τ_{n+1}, together with Proposition 3.3, and with property iv) of Proposition 3.1.

b) By using a), we get

$$\begin{aligned}\tau_{n+1} \wedge \tau_n &= (\tilde{\tau}_{n+1} \wedge \tau_n)\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + (\tau_{n-1} \wedge \tau_n)\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}} \\ &= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + (\tau_{n-1} \wedge \tilde{\tau}_{n+1})\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}} \\ &= \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}<\tau_n\}} + \tilde{\tau}_{n+1}\mathbb{1}_{\{\tilde{\tau}_{n+1}=\tau_n\}},\end{aligned}$$

where we have used Proposition 3.3 for the last equality.

Finally, by using property iv) of Proposition 3.1, we get $\tau_{n+1} \wedge \tau_n = \tilde{\tau}_{n+1}$.

c) To prove this property, we proceed by induction. The property is true for $n = 2$. We suppose that the property is true at rank $n-1$ (where $n \geq 3$), that is on $\{\tau_{n-1} = \tau_{n-2}\}$, $\tau_m = T$, for all $m \in \{1, \ldots, n-1\}$.

From the expression for τ_n from statement a), we get

$$\tau_n = \tilde{\tau}_n \mathbb{1}_{\{\tilde{\tau}_n<\tau_{n-1}\}} + \tau_{n-2}\mathbb{1}_{\{\tilde{\tau}_n=\tau_{n-1}\}}.$$

Hence, $\tau_n = \tau_{n-2}$ on the set $\{\tau_n = \tau_{n-1}\}$. We conclude by the induction hypothesis. □

Lemma 3.2 *The following inequalities hold true:*

i) $J_1(\tau, \tau_{2n}) \leq J_1(\tau_{2n+1}, \tau_{2n})$, *for all* $\tau \in \Theta$.

ii) $J_2(\tau_{2n+1}, \tau) \leq J_2(\tau_{2n+1}, \tau_{2n+2})$, *for all* $\tau \in \Theta$.

Proof Let us first prove statement i):

By (A1), we have $X^1 \leq Y^1$; it follows

$$X^1(\tau)\mathbb{1}_{\{\tau\leq\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}<\tau\}} \leq \xi^{2n+1}(\tau).$$

Hence, by monotonicity, and by definition of $V^{2n+1}(0)$, we have

$$J_1(\tau, \tau_{2n}) \leq V^{2n+1}(0) \tag{14}$$

We will now show that $V^{2n+1}(0) = J_1(\tau_{2n+1}, \tau_{2n})$, which will complete the proof of statement i).

We have

$$\begin{aligned}J_1(\tau_{2n+1}, \tau_{2n}) &= \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}\left[X^1(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\leq\tau_{2n}\}} + Y^1(\tau_{2n})\mathbb{1}_{\{\tau_{2n}<\tau_{2n+1}\}}\right] \\ &= \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tau_{2n+1})\right],\end{aligned}$$

where we have used c) from Lemma 3.1, and $X^1(T) = Y^1(T)$ from (A2) to show the last equality.

On the other hand, by ii) from Remark 3.1 and b) from Lemma 3.1,

$$\begin{aligned}\rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tau_{2n+1})\right] &= \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tau_{2n+1} \wedge \tau_{2n})\right] \\ &= \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tilde{\tau}_{2n+1})\right].\end{aligned}$$

By optimality of $\tilde{\tau}_{2n+1}$ for $V^{2n+1}(0)$ (cf. Eq. 7), we get

$$\rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tilde{\tau}_{2n+1})\right] = V^{2n+1}(0).$$

Hence, we conclude

$$\begin{aligned} J_1(\tau_{2n+1},\tau_{2n}) = \rho^1_{0,\tau_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tau_{2n+1})\right] &= \rho^1_{0,\tilde{\tau}_{2n+1}\wedge\tau_{2n}}\left[\xi^{2n+1}(\tilde{\tau}_{2n+1})\right] \\ &= V^{2n+1}(0). \end{aligned} \tag{15}$$

From Eqs. (15) and (14), we get $J_1(\tau,\tau_{2n}) \le J_1(\tau_{2n+1},\tau_{2n})$.

Let us now prove statement ii):

We have

$$J_2(\tau_{2n+1},\tau) \le V^{2n+2}(0), \tag{16}$$

by definition of $V^{2n+2}(0)$ (cf. Eq. 9).

We will now show that $J_2(\tau_{2n+1},\tau_{2n+2}) = V^{2n+2}(0)$, which will complete the proof.

By definition of $\xi^{2n+2}(\tau_{2n})$, by ii) from Remark 3.3, and by b) from Lemma 3.1, we have

$$\begin{aligned} &J_2(\tau_{2n+1},\tau_{2n+2}) \\ &= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}\left[X^2(\tau_{2n+2})\mathbb{1}_{\{\tau_{2n+2}<\tau_{2n+1}\}} + Y^2(\tau_{2n+1})\mathbb{1}_{\{\tau_{2n+1}\le\tau_{2n+2}\}}\right] \\ &= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}\left[\xi^{2n+2}(\tau_{2n+2})\right] \\ &= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}\left[\xi^{2n+2}(\tau_{2n+2}\wedge\tau_{2n+1})\right] \\ &= \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}\left[\xi^{2n+2}(\tilde{\tau}_{2n+2})\right]. \end{aligned}$$

By Eq. (10), we have $\rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}\left(\xi^{2n+2}(\tilde{\tau}_{2n+2})\right) = V^{2n+2}(0)$. Hence, we conclude

$$\begin{aligned} J_2(\tau_{2n+1},\tau_{2n+2}) &= \rho^2_{0,\tau_{2n+1}\wedge\tau_{2n+2}}\left(\xi^{2n+2}(\tau_{2n+2})\right) \\ &= \rho^2_{0,\tilde{\tau}_{2n+2}\wedge\tau_{2n+1}}\left(\xi^{2n+2}(\tilde{\tau}_{2n+2})\right) = V^{2n+2}(0). \end{aligned} \tag{17}$$

From Eqs. (17) and (16), we get $J_2(\tau_{2n+1},\tau) \le J_2(\tau_{2n+1},\tau_{2n+2})$ □

Remark 3.4 As a by-product of the previous proof, we find that τ_{2n+1} is optimal for the problem with value $V^{2n+1}(0)$, and τ_{2n+2} is optimal for the problem with value $V^{2n+2}(0)$.

Definition 3.2 We define $\tau_1^* = \lim_{n\to+\infty}\tau_{2n+1}$, and $\tau_2^* = \lim_{n\to+\infty}\tau_{2n}$.

Proposition 3.4 *We assume that ρ^1 and ρ^2 satisfy properties $(i)-(x)$. Then:*

i) For all $\tau\in\Theta$, $\lim_{n\to+\infty} J_1(\tau,\tau_{2n}) = J_1(\tau,\tau_2^*)$.

ii) For all $\tau\in\Theta$, $\lim_{n\to+\infty} J_2(\tau_{2n+1},\tau) = J_2(\tau_1^*,\tau)$.

iii) For all $\tau\in\Theta$, $\lim_{n\to+\infty} J_1(\tau_{2n+1},\tau_{2n+2}) = J_1(\tau_1^*,\tau_2^*)$.

iv) For all $\tau\in\Theta$, $\lim_{n\to+\infty} J_2(\tau_{2n+1},\tau_{2n+2}) = J_2(\tau_1^*,\tau_2^*)$.

Proof Let us first show statement i).

Let us recall the following notation: for a fixed $\tau \in \Theta$,

$$I^1(\tau, \nu) := X^1(\tau)\mathbb{1}_{\{\tau \leq \nu\}} + Y^1(\nu)\mathbb{1}_{\{\nu < \tau\}},$$
$$I^1(\tau, \tau_2^*) := X^1(\tau)\mathbb{1}_{\{\tau \leq \tau_2^*\}} + Y^1(\tau_2^*)\mathbb{1}_{\{\tau_2^* < \tau\}}.$$

With this notation, we have

$$J_1(\tau, \tau_{2n}) = \rho^1_{0,\tau\wedge\tau_{2n}}\big[I^1(\tau, \tau_{2n})\big], \quad \text{and} \quad J_1(\tau, \tau_2^*) = \rho^1_{0,\tau\wedge\tau_2^*}\big[I^1(\tau, \tau_2^*)\big].$$

We note that the sequence (τ_{2n}) and $(\tau \wedge \tau_{2n})$ converges from above to τ_2^* and $\tau \wedge \tau_2^*$, respectively. Moreover, for each $\tau \in \Theta$, the family $(I^1(\tau, \nu))_{\nu\in\Theta}$ is admissible. Indeed, for each $\nu \in \Theta$, $I^1(\tau, \nu)$ is $\mathcal{F}_\nu$-measurable. Moreover, if $\{\nu = \nu'\}$, $I^1(\tau, \nu) = I^1(\tau, \nu')$ a.s. Hence, as any admissible family in our framework is right-continuous along Bermudan stopping strategies (cf. Remark 2.27 in [13]), we get

$$\lim_{n\to+\infty} I^1(\tau, \tau_{2n}) = I^1(\tau, \tau_2^*).$$

Hence, by property (x) (cf. Eq. 3) on ρ^1, we get

$$\limsup_{n\to+\infty} \rho^1_{0,\tau\wedge\tau_{2n}}\big[I^1(\tau, \tau_{2n})\big] = \rho^1_{0,\tau\wedge\tau_2^*}\big[I^1(\tau, \tau_2^*)\big].$$

Now, let us prove statement ii).

For $\tau \in \Theta$, we recall the following notation:

$$I^2(\nu, \tau) := X^2(\tau)\mathbb{1}_{\{\tau < \nu\}} + Y^2(\nu)\mathbb{1}_{\{\nu \leq \tau\}},$$

The family $(I^2(\nu, \tau)_{\nu\in\Theta})$ is admissible. Indeed, for each $\nu \in \Theta$, $I^2(\nu, \tau)$ is $\mathcal{F}_\nu$-measurable. Moreover, on $\{\nu_1 = \nu_2\}$, $I^2(\nu_1, \tau) = I^2(\nu_2, \tau)$ a.s.

As (τ_{2n+1}) converges from above to τ_1^*, and as $(I^2(\nu, \tau)_{\nu\in\Theta})$ is right-continuous along Bermudan stopping strategies (cf. Remark 2.27 in [13]), we get

$$\lim_{n\to+\infty} I^2(\tau_{2n+1}, \tau) = I^2(\tau_1^*, \tau).$$

By property (x) (cf. Eq. 3) on ρ^2, we get

$$\limsup_{n\to+\infty} \rho^2_{0,\tau_{2n+1}\wedge\tau}\big[I^2(\tau_{2n+1}, \tau)\big] = \rho^2_{0,\tau_1^*\wedge\tau}\big[I^2(\tau_1^*, \tau)\big].$$

We now prove statement iii).

The proof relies again on the Bermudan structure on Θ. For any sequence $(\tau_n) \in \Theta^{\mathbb{N}}$ converging from above to $\tau \in \Theta$, we have: for almost each $\omega \in \Omega$, there exists $n_0 = n_0(\omega)$ such that for all $n \geq n_0$, $\tau_n(\omega) = \tau(\omega)$ (cf. Remark 2.27 in [13]).

Hence, for almost each $\omega \in \Omega$, there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$, $\tau_{2n+1}(\omega) = \tau_1^*(\omega)$, $\tau_{2n+2}(\omega) = \tau_2^*(\omega)$ and

$$I(\tau_{2n+1}, \tau_{2n+2})(\omega) = X^1(\tau_1^*)(\omega)\mathbb{1}_{\{\tau_1^* \leq \tau_2^*\}}(\omega) + Y^1(\tau_2^*)(\omega)\mathbb{1}_{\{\tau_2^* < \tau_1^*\}}(\omega).$$

By property (x) (cf. Eq. 3) on ρ^1, we get

$$\lim_{n \to +\infty} J_1(\tau_{2n+1}, \tau_{2n+2}) = J_1(\tau_1^*, \tau_2^*).$$

The proof of iv) is based on the same type of arguments. □

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1 By combining Lemma 3.2 and Proposition 3.4, we get:

$$J_1(\tau, \tau_2^*) \leq J_1(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$
$$J_2(\tau_1^*, \tau) \leq J_2(\tau_1^*, \tau_2^*), \text{ for all } \tau \in \Theta.$$

Hence, (τ_1^*, τ_2^*) is a Nash equilibrium point. □

We have thus shown that the non-linear non-zero-sum Dynkin game with Bermudan strategies admits a Nash equilibrium point.

4 Conclusions

We have established the existence of a Nash equilibrium point for a non-zero-sum game between two *Bermudan stoppers*, where each stopper assesses their pay-off with their own general non-linear assessment functional. The Bermudan stopping strategies of the agents allow for a countable number of stopping opportunities, where the number of stopping opportunities and the stopping opportunities themselves may depend on the scenario ω (in a non-anticipative way). The non-linear evaluation functionals include those induced by BSDEs with Lipschitz driver, as well as the dynamic concave utilities.

A Localisation Property

Proposition A.1 (Localisation property) *Let $(\xi(\tau))_{\tau \in \Theta}$ be a given admissible p-integrable family. Let $(V(\tau))_{\tau \in \Theta}$ be the value family of the following optimal stopping problem: for $S \in \Theta$,*

$$V(S) = \operatorname{ess\,sup}_{\tau \in \Theta_S} \rho_{S,\tau}[\xi(\tau)].$$

Let $S \in \Theta$, and let A be in $\mathcal{F}_S$. We consider the pay-off family $(\xi(\tau)\mathbb{1}_A)_{\tau \in \Theta_S}$, and we denote by $(V_A(\tau))_{\tau \in \Theta_S}$ the corresponding value family, defined by:

$$V_A(\tau) = \operatorname{ess\,sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[\xi(\nu)\mathbb{1}_A].$$

If ρ satisfies the usual "zero–one law" (that is $\mathbb{1}_A \rho_{S,\tau}[\eta] = \mathbb{1}_A \rho_{S,\tau}[\mathbb{1}_A \eta]$ for all $A \in \mathcal{F}_S$, for all $\eta \in L^p(\mathcal{F}_\tau)$), then for each $\tau \in \Theta_S$,

$$\mathbb{1}_A V_A(\tau) = \mathbb{1}_A V(\tau).$$

Proof By the definition of $V(\tau)$ and the usual "zero–one law", we have

$$\begin{aligned}\mathbb{1}_A V(\tau) &= \mathbb{1}_A \text{ess sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[\xi(\nu)] = \text{ess sup}_{\nu \in \Theta_\tau} \mathbb{1}_A \rho_{\tau,\nu}[\xi(\nu)] \\ &= \text{ess sup}_{\nu \in \Theta_\tau} \mathbb{1}_A \rho_{\tau,\nu}[\mathbb{1}_A \xi(\nu)] = \mathbb{1}_A \text{ess sup}_{\nu \in \Theta_\tau} \rho_{\tau,\nu}[\mathbb{1}_A \xi(\nu)] = \mathbb{1}_A V_A(\tau).\end{aligned}$$

□

B Examples of Non-linear Operators

B.1 Non-linear Evaluations Induced by BSDEs

Let $p = 2$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented natural filtration of a one-dimensional Brownian motion $(W_t)_{t\in[0,T]}$. Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a standard Lipschitz driver, that is: $g(\cdot, \cdot, y, z)$ is progressively measurable, such that g is Lipschitz in (y, z), uniformly in (ω, t), and $\mathbb{E}[\int_0^T (g(t, 0, 0))^2 dt] < +\infty$. For each $0 \le s \le t \le T$, and $\eta \in L^2(\mathcal{F}_t)$, we define: $\mathcal{E}^g_{s,t}(\eta) := Y_s$, where (Y_u, Z_u) is the adapted pair of processes solving the BSDE with terminal condition η and terminal time t:

$$Y_s = \eta + \int_s^t g(u, Y_u, Z_u)du - \int_s^t Z_u dW_u, \quad \text{for } 0 \le s \le t.$$

The operator $\mathcal{E}^g_{s,t}(\cdot)$ is sometimes referred to as g-(conditional) evaluation in the mathematical finance and risk measurement literature.

It is well-known (cf. [11]) that $(\mathcal{E}^g_{s,t}(\cdot))$ can be extended to the case where the terminal time is a stopping time τ (valued in [0, T]) and the terminal condition η is $\mathcal{F}_\tau$-measurable ($\eta \in L^2(\mathcal{F}_\tau)$) by considering the BSDE with Lipschitz driver $g^\tau(t, y, z) := g(t, y, z)\mathbb{1}_{\{t\le\tau\}}$, terminal time T, and terminal condition η.

We recall that $\rho_{S,\tau}(\cdot) := \mathcal{E}^g_{S,\tau}(\cdot)$ satisfies the properties of admissibility (ii), knowledge preservation (iii), monotonicity (iv), consistency (v), and "generalized zero–one law" (vi). For the proof of the "generalized zero–one law", we refer for instance, to [14]. We note that if, moreover, the driver is "grounded", in the sense that $g(t, 0, 0) = 0$, then $\mathcal{E}^g_{S,\tau}(\cdot)$ satisfies the usual zero–one law (property (vi bis)). In mathematical finance and risk measurement, we often have $g(t, 0, 0) = 0$: this ensures, in particular, that $\mathcal{E}^g_{S,\tau}(\eta) \ge 0$ a.s. for $\eta \ge 0$ a.s.

Remark B.1 If, moreover, $g(t, y, 0) = 0$, for all $y \in \mathbb{R}$, then the operator $\mathcal{E}^g_{S,\tau}(\cdot)$ does not depend on the second index τ (but on the first index only) and is known as Peng's g-expectation (cf. [34]).

It is also well-known (cf. e.g. [35]) that $\mathcal{E}^g_{S,\tau}(\cdot)$ satisfies the property of continuity with respect to the terminal time and the terminal condition:

(Continuity)($\star$) Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times, let τ be a stopping time such that $\lim_{n\to\infty}\tau_n = \tau$ a.s. Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of random variables, such that $\eta_n \in L^2(\mathcal{F}_{\tau_n})$, $\sup_n \eta_n \in L^2$ and $\lim_{n\to\infty}\eta_n = \eta$ a.s. Then, we have $\lim_{n\to\infty}\mathcal{E}^g_{S,\tau_n}(\eta_n) = \mathcal{E}^g_{S,\tau}(\eta)$ a.s.

Based on the continuity property ($\star$), $\mathcal{E}^g_{S,\tau}(\cdot)$ satisfies the monotone Fatou property (vii) and property (viii) (for the proof of property (viii), cf. e.g., Lemma A.5 in [8]).

Moreover, property (ix) is also satisfied as, in the Brownian framework, the first component (Y_t) of the solution of the BSDE with Lipschitz driver g, terminal time T, and terminal condition $\eta \in L^2(\mathcal{F}_T)$, has continuous trajectories (in t). Hence, for any non-decreasing sequence (τ_n), such that $\lim_{n\to+\infty}\tau_n = T$,

$$\lim_{n\to+\infty}\mathcal{E}^g_{\tau_n,T}(\eta) = \lim_{n\to+\infty} Y_{\tau_n} = Y_T = \mathcal{E}^g_{T,T}(\eta) = \eta.$$

Finally, we can show that $\mathcal{E}^g_{S,\tau}(\cdot)$ satisfies property (x) also as a consequence of the continuity with respect to terminal time and terminal condition (property ($\star$)). Indeed, let $(\nu_n) \in \Theta^{\mathbb{N}}$ and $\nu \in \Theta$ be such that $\nu_n \downarrow \nu$. As ϕ is right-continuous along Bermudan stopping times (cf. Remark 2.27 in [13]), we get: $\lim_{n\to+\infty}\phi(\nu_n) = \phi(\nu)$ a.s. Then, by property ($\star$), we conclude: $\lim_{n\to+\infty}\mathcal{E}^g_{S,\nu_n}(\phi(\nu_n)) = \mathcal{E}^g_{S,\nu}(\phi(\nu))$.

Thus, all the required properties (i) to (x) hold true for $(\mathcal{E}^g_{S,\tau}(\cdot))$, when g is a standard Lipschitz driver such that $g(t,0,0) = 0$.

B.2 Dynamic Concave Utilities

Let $p = +\infty$. We recall the definition of the Dynamic concave utilities (abridged DCUs) from [6] (cf. also [2]).

Definition B.1 (Dynamic concave utility) For S, τ stopping times valued in [0, T], such that $S \le \tau$, let

$$\{u_{S,\tau}(\cdot) : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_S)\}$$

be a family of operators. This family is called a dynamic concave utility, if it satisfies the following properties:

(D1) (Monotonicity) $u_{S,\tau}(\eta) \le u_{S,\tau}(\eta')$, if $\eta \le \eta'$;

(D2) (Translation invariance) $u_{S,\tau}(\eta + X) = u_{S,\tau}(\eta) + X$, if $\eta \in L^\infty(\mathcal{F}_\tau)$ and $X \in L^\infty(\mathcal{F}_S)$;

(D3) (Concavity) $u_{S,\tau}(\lambda\eta + (1-\lambda)\eta') \ge \lambda u_{S,\tau}(\eta) + (1-\lambda)u_{S,\tau}(\eta')$, for any $\lambda \in [0,1]$ and $\eta, \eta' \in L^\infty(\mathcal{F}_\tau)$;

(D4) (Normalisation) $u_{S,\tau}(0) = 0$.

Moreover, in [2, 6] the following properties on the dynamic concave utilities are assumed: (D5) (Time consistency) for any stopping time $\sigma \in \mathcal{T}_{S,\tau}$, we have $u_{S,\sigma}(u_{\sigma,\tau}(\eta)) = u_{S,\tau}(\eta)$;

(D6) (Continuity from above) for any non-increasing sequence $(\eta_n) \subset L^\infty(\mathcal{F}_\tau)$ with $\eta = \lim_{n\to\infty} \downarrow \eta_n \in L^\infty(\mathcal{F}_\tau)$, we have $\lim_{n\to\infty} \downarrow u_{S,\tau}(\eta_n) = u_{S,\tau}(\eta)$

(D7) (Local property) $u_{S,\tau}(\eta \mathbb{1}_A + \xi \mathbb{1}_{A^c}) = u_{S,\tau}(\eta)\mathbb{1}_A + u_{S,\tau}(\xi)\mathbb{1}_{A^c}$, for any $A \in \mathcal{F}_S$ and for any $\eta, \xi \in L^\infty(\mathcal{F}_\tau)$;

(D8) $E_P[\eta|\mathcal{F}_t] \geq 0$ for any $\eta \in L^\infty(\mathcal{F}_T)$, such that $u_{t,T}(\eta) \geq 0$.

By the results of [6], any functional ρ satisfying properties (D1)–(D8), has the following representation:

$$\begin{aligned} u_{S,\tau}(\eta) &= \operatorname*{ess\,inf}_{Q:Q\sim P, Q=P \text{ on } \mathcal{F}_S} \left\{ E_Q[\eta|\mathcal{F}_S] + c_{S,\tau}(Q) \right\} \\ &= \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} E_Q[\eta + \int_S^\tau f(u, \psi_u^Q)du|\mathcal{F}_S], \end{aligned} \tag{18}$$

where the function f is such that $f(\cdot, \cdot, x)$ is predictable for any x; f is a proper, convex function in the space variable x, and valued in $[0, +\infty]$, and the process (ψ_t^Q) is the process from the Doleans-Dade exponential representation for the density process (Z_t^Q), where $Z_t^Q = \frac{dQ}{dP}|_{\mathcal{F}_t}$, and

$$\mathcal{Q}_S = \{Q : Q \sim P, \psi_t^Q(\omega) = 0, dt \otimes dP \text{ a.e. on } [[0, S[[, \ E_Q[\int_S^T f(s, \psi_s^Q)ds] < +\infty\}.$$

The dynamic concave utilities $u_{S,\tau}$ do not enter directly into the framework of the present paper, as they are defined only for S, τ such that $S \leq \tau$ a.s. (cf. [2, 6]). There is, however, a "natural" extension of $u_{S,\tau}$ in view of the representation property (18). This extension is as follows:

For S and τ stopping times, and $\eta \in L^\infty(\mathcal{F}_\tau)$, we define

$$\mathbb{1}_{\{S>\tau\}} u_{S,\tau}(\eta) = \mathbb{1}_{\{S>\tau\}} \times \eta. \tag{19}$$

Thus, by assumptions, the DCUs $(u_{S,\tau}(\cdot))$ satisfy properties (i), admissibility (ii), monotonicity (iv), consistency (v), "generalized zero–one law"(vi) (cf. the local property of DCUs). Moreover, by property (D4) and (D7), the DCUs satisfy the "generalized zero–one law" (cf. e.g. Remark 3.9 in [13]). By property (D2) and (D4), the DCUs satisfy the knowledge preserving property(iii). Using the representation (18), one can show that properties (viii) and (ix) are also satisfied (cf. Remark 3.11 and Remark 3.12 in [13]).

Finally, we check that the DCUs satisfy property (x) of ρ. Let (ν_n) be a non-increasing sequence in $\Theta^{\mathbb{N}}$ such that $\nu_n \downarrow \nu$, where $\nu \in \Theta$. Let ϕ be a p-integrable admissible family such that $\sup_n |\phi(\nu_n)| \in L^\infty$. By right-continuity of ϕ along Bermudan stopping strategies and by continuity of the integral $\int_S^\cdot du$, we get $\lim_{n\to+\infty} \phi(\nu_n) + \int_S^{\nu_n} f(u, \psi_u^Q)du = \phi(\nu) + \int_S^\nu f(u, \psi_u^Q)du =: \eta$. As, for each $Q \in \mathcal{Q}_S$,

$$E_Q(\sup_n |\phi(\nu_n)|) + \int_S^{\nu_n} f(u, \psi_u^Q)du \leq E_Q(\sup_n |\phi(\nu_n)|) + \int_S^T f(u, \psi_u^Q)du < +\infty,$$

we get (by dominated convergence):

$$\lim_{n\to+\infty} E_Q(\phi(\nu_n) + \int_S^{\nu_n} f(u, \psi_u^Q)du|\mathcal{F}_S) = E_Q(\eta|\mathcal{F}_S).$$

Hence,

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} \lim_{n\to+\infty} E_Q(\phi(\nu_n) + \int_S^{\nu_n} f(u, \psi_u^Q)du|\mathcal{F}_S) = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} E_Q(\eta|\mathcal{F}_S) = u_{S,\nu}(\phi(\nu)).$$

Moreover, we have:

$$\limsup_{n\to+\infty} \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} E_Q(\phi(\nu_n) + \int_S^{\nu_n} f(u, \psi_u^Q)du|\mathcal{F}_S) \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_S} \limsup_{n\to+\infty} E_Q(\phi(\nu_n) + \int_S^{\nu_n} f(u, \psi_u^Q)du|\mathcal{F}_S),$$

Hence, we get the result.

We have thus checked that the DCUs $(u_{S,\tau}(\cdot))$ satisfy all the properties (i)–(x).

Acknowledgements We are deeply grateful to the referees and to the associate editor for their helpful comments and remarks.

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