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A deep understanding of Non-linear Boussinesq-Burger System of Equation using wavelets

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Abstract

This research proposal introduces an efficient mathematical scheme for solving the non-linear Boussinesq-Burger's system of equations. The scheme leverages Haar scale-3 wavelets for its development. The study also involves the exploration of numerical problems featuring various source terms. To address the challenges posed by nonlinearity, a quasi-linearization technique is employed. Additionally, analytic formulas for integrals are derived for both cases. By tackling problems with known exact solutions and considering the exact solution with parameter values, the suggested scheme's accuracy is evaluated. The findings indicate that the efficiency of Haar scale-3 surpasses that of Haar scale-2, as confirmed by a comparison with existing literature.

Keywords: Boussinesq-Burger's, Haar Wavelet method, Source term, Quasi-linearization

1. Introduction

In recent years, Haar wavelets have found extensive use in signal processing for communication systems and physics research, particularly when dealing with differential calculus and non-linear problems. Wavelets provide a means to perform algebraic modifications on mathematical models, departing from traditional approaches such as variational approximation. This approach can lead to improved criteria for the actual design process. [1].

In a study by Lepik [2], the results showed that the HW methodology is a dependable way to solve different integral equations (IE) and PDEs, including both linear and nonlinear DE. Ataie and Najafi [3] applied a finite difference approach with a higher-order scheme in time and space to solve a higher multiple-order two-dimensional (2-dim) Boussinesq wave (BW) model. They used this strategy to

successfully extract equations. In order to mimic three-dimensional surface waves, Chen and Goubet [4] investigated the long-term (L-T) behaviour of sol for a large class of dissipative Boussinesq (D-B) systems. They were able to identify the solutions' long-time asymptotes. The numerical method Chen [5] utilised to solve beginning and boundary value issues in a two-dimensional Boussinesq system (B-S) was the main topic. The suggested approach was effective and precise. Using the conventional Galerkin-finite element approach, Mitsotakis [6] constructed and quantitatively resolved a two-dimensional Boussinesq system in two spatial dimensions. Mera [7] concentrated on the creation of two-dimensional boundary circumstances for existing Boussinesq-type equations in order to improve the accuracy and applicability of the models. Using techniques like wavelets, finite difference schemes, numerical methods, and boundary conditions formulation, this research contribute to the understanding and development of methods for solving Boussinesq equations overall. Non-linear Boussinesq System has been the subject of extensive research by many researchers.

The four-parameter Boussinesq systems were developed by Bona et al. [8] using the multifaceted Euler formulae that govern free-surface movement. They also developed criteria to aid in selecting the optimal equation from this family in different modelling settings. The analysis of these systems was built on the basis of these criteria. In a related article, Bona et al. [9] focused on the numerical generalisation of the solitary wave solutions for two linked Boussinesq-type Kdv systems. They provided numerical experiments that clarified the emergence, evolution, and interaction of these waves. They also investigated the intention of generic beginning into succession representations of these vibrations and assessed the cohesion of these waves under small disruptions. Anotonopoulos et al. [10] looked at if there were any solutions and whether they were distinct for three initial-boundary-value problems pertaining to the Bona-Smith family of Boussinesq systems. Periodic boundary conditions, reflection, and nonhomogeneous Dirichlet boundary conditions that were utilised at the endpoints of a limited spatial interval contributed to these problems. They demonstrated that the initial-boundary-value problem with Dirichlet charts boundary constraints was locally well-posed in suitable contexts, in contrast to related problems with reflection and periodic boundary conditions, which were globally well-posed given particular starting data limitations. A specific type of Boussinesq systems' internal controllability and stability analysis was the focus of Micu et al.'s [11] study. They identified the relevant linear system's controllable data space for all parameter values. They created straightforward feedback controllers that resulted in exponentially stable closed-loop systems using the newly discovered perfect controllability. These works look at solitary wave solutions, initial-boundary value issues that are well-posed under various boundary scenarios, initial-boundary value (IBV) problems that are well-posed under diverse threshold conditions, internal controllability and stability qualities,

and criteria for selecting acceptable equations. This research helps us comprehend Boussinesq systems in general.

This work employs Haar wavelet techniques to numerically solve a nonlinear third-order Boussinesq system at several scales, particularly scale-2 and 3. The precise solution is contrasted with the results. The paper compares the performance of scale-2 and 3 Haar wavelet (HW) methods for the non-linear (N-L) three-dimensional (3-dim) Boussinesq-Burger (B-B) system of equations. A number of differential equations have been solved using 3-Scale Haar wavelets in the past, proving their effectiveness in mathematical models governed by these equations. Additionally, these investigations have shown that scale-3 Haar wavelets have higher convergence rates than scale-2 Haar wavelets. To the best of our knowledge, scale-3 Haar wavelets have not yet been used to examine the characteristics of solutions to the equations in the Boussinesq-Burger system. This provides as inspiration for creating a novel methodology to evaluate and contrast the consummation of 2-scale and 3-scale Haar wavelet methods in resolving Boussinesq-Burger's system of equations–governed systems.

The primary goal of the aforementioned study project is to demonstrate an entirely new numerical method for solving the recently developed Boussinesq-Burger's system of equations in the field of dispersive systems. Scale-2 and scale-3 Haar waveform bases are used in the approach that has been suggested.

The generalized Boussinesq-Burger's system of non-linear partial differential equation given as

$$\frac{\partial u}{\partial x} + pu\frac{\partial u}{\partial x} + q\frac{\partial w}{\partial x} = 0\,; x \in [a,b], t \in [0,T] \quad (1)$$

$$\frac{\partial w}{\partial t} + r\left[\frac{\partial}{\partial x}(uw)\right] + s\frac{\partial^3 u}{\partial x^3} = 0; x \in [a,b], t \in [0,T] \quad (2)$$

Subject to the boundary condition

$$u(a,t) = \eta_1(t), u(b,t) = \eta_2(t) \quad (3)$$

$$w(a,t) = \alpha_1(t), w(b,t) = \alpha_2(t)\ ¥\ t \in [0,T] \quad (4)$$

And with the constraints at the initial values

$$u(x,0) = \mu(x), w(x,0) = \lambda(x)\ ¥\ x \in [a,b] \quad (5)$$

By changing the values of these four parameters (p, q, r, s) distinct variations in the solution space can be noticed. The following is how we ordered our paper. The Haar wavelet is introduced in section 2 along with an operational matrix. In Section-3 discusses Quasi-linearization techniques. Haar wavelets

with scale-2 and scale-3 were utilised to solve a nonlinear boussinesq problem in section 4. Section 5 deals with the numerical solutions with boundary conditions with different source term. Section 6 has concluding observations.

2. Haar Wavelet along with its Operational matrix

Wavelet approach has been widely employed in image digital processing, general relativity, mathematical methods, and many other domains in recent years as a powerful mathematical weapon.

Equations 6, 7 and 8 give the mathematical equations for the father wavelet (Scale 3 Haar function) and mother wavelets in the scale 3 Haar wavelet family with dilation factor three.

The scale-3 Haar wavelet integral methodology is used to solve a non-linear system of partial differential equation which are of third order. In which the differential equations maximum derivative is expanded into scale-3 Haar wavelets and the derivatives of low order are graded by integrating the differential equations. The Haar scale-3 wavelet is more accurate and converges faster than the Haar scale-2 wavelet. Using the orthogonality condition in wavelets, every square integrable function defined on the interval [0, 1] may be simply expressed in terms of the infinite sum of Haar wavelet series.

$$f(x) \approx c_1\varphi_1(x) + \sum_{even\ index\ i\geq 2}^{\infty} c_i {\Psi_i}^1(x) + \sum_{odd\ index\ i\geq 3}^{\infty} c_i {\Psi_i}^2(x)$$

Haar Scaling Function $h_1(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & elsewhere \end{cases}$ (6)

$$h_i(t) = \Psi^1\left(3^j t - k\right) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \alpha_1(i) \leq t \leq \alpha_2(i) \\ 2 & \alpha_2(i) \leq t \leq \alpha_3(i) \\ -1 & \alpha_3(i) \leq t \leq \alpha_4(i) \\ 0 & elsewhere \end{cases} \quad \text{for i} = 2, 4 \ldots 3p - 1 \tag{7}$$

$$h_i(t) = \Psi^2\left(3^j t - k\right) = \sqrt{\frac{3}{2}} \begin{cases} 1 & \alpha_1(i) \leq t \leq \alpha_2(i) \\ 0 & \alpha_2(i) \leq t \leq \alpha_3(i) \\ -1 & \alpha_3(i) \leq t \leq \alpha_4(i) \\ 0 & elsewhere \end{cases} \quad \text{for i} = 3, 6 \ldots \ldots 3p \tag{8}$$

Where $\alpha_1(i) = \frac{k}{p}, \alpha_2(i) = \frac{3k+1}{3p}, \alpha_3(i) = \frac{3k+2}{3p}, \alpha_4(i) = \frac{k+1}{p}$

$$p = 3^j; j = 0, 1, 2 \ldots k = 0, 1, 2..p - 1$$

The wavelet number, level of resolution (dilation), and translation parameters of the wavelet family are represented by i, j and k respectively. With the help of j, k the values of i may be computed using the relationships $i - 1 = 3j + 2k$ for even values of i and $i - 2 = 3j + 2k$ for odd values of i. Using this

connection for various dilations and translations of $h_2(t), h_3(t)$ we get the wavelet family $h_1(t), h_2(t), h_3(t), h_4(t), h_5(t), h_6(t)$ ... where $h_2(t)$ and $h_3(t)$ are also known as mother wavelets, while the rest of the wavelets are known as daughter wavelets.

Let us define the collocation points $x_l = (l - 0.5)/3M,\ l = 1, 2, 3 \ldots .3M$ and Discredit the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i, l) = (h_i(x_j))$ which has the dimension 3M*3M.

The operational matrix of the integration P, which is 3M square matrix is defined. If the solution process of differential equation of any order. We need to integrate Haar scale-3 wavelets i.e., we employ the integral

$$\varphi_{1,1}(\mathrm{t}) = \int_0^x \varphi_1(\mathrm{t})\,\mathrm{dt} = \begin{cases} t\,;\ [A_1, B_1) \\ 0;\ elsewhere \end{cases} \tag{9}$$

$$\Psi_{i,1}{}^{(1)}(t) = \int_0^x \Psi_i{}^{(1)}\,dt = \frac{1}{\sqrt{2}} \begin{cases} p(i) - t & ;\ p(i) \le t < q(i) \\ 2t - 3\,q(i) + p(i) & ;\ q(i) \le t < r(i) \\ p(i) + 3\,r(i) - 3\,q(i) - t & ;\ r(i) \le t < s(i) \end{cases} \tag{10}$$

$$\Psi_{i,1}{}^{(2)}(t) = \int_0^x \Psi_i{}^{(2)}\,dt = \sqrt{\frac{3}{2}} \begin{cases} t - p(i) & ;\ p(i) \le t < q(i) \\ q(i) - p(i) & ;\ q(i) \le t < r(i) \\ r(i) + q(i) - p(i) - t & ;\ r(i) \le t < s(i) \end{cases} \tag{11}$$

Moreover, we introduce

$$\varphi_{1,s+1}(t) = \int_0^x \varphi_{1,s}(t)\,dt \tag{12}$$

$$\Psi_{1,s+1}{}^{(1)}(t) = \int_0^x \Psi_{1,s}{}^{(1)}(t)\,dt \tag{13}$$

$$\Psi_{1,s+1}{}^{(2)}(t) = \int_0^x \Psi_{1,s}{}^{(2)}(t)\,dt \tag{14}$$

Which can explicitly be written as

$$\varphi_{i,s+1}(x) = \begin{cases} \frac{t^{s+1}}{(s+1)!} & ;\ [A_1, B_1) \\ 0 & ;\ Elsewhere \end{cases} \tag{15}$$

$$\Psi_{i,s+1}{}^{(1)}(t) = \frac{1}{\sqrt{2}} \begin{cases} 0 & ;\ 0 \le t < p_1(i) \\ \frac{-[t - p_1(i)]^{s+1}}{(s+1)!} & ;\ p_1(i) \le t < p_2(i) \\ \frac{3\,[t - p_2(i)]^{s+1} - [t - p_1(i)]^{s+1}}{(s+1)!} & ;\ p_2(i) \le t < p_3(i) \\ \frac{3\,[t - p_2(i)]^{s+1} - 3\,[t - p_3(i)]^{s+1} - [t - p_1(i)]^{s+1}}{(s+1)!} & ;\ p_3(i) \le t < p_4(i) \\ \frac{3\,[t - p_2(i)]^{s+1} - 3[t - p_3(i)]^{s+1} - [t - p_1(i)]^{s+1} + [t - p_4(i)]^{s+1}}{(s+1)!} & ;\ p_4(i) \le t < 1 \end{cases} \tag{16}$$

$$\Psi_{1,s+1}{}^{(2)}(t) =$$

$$\sqrt{\frac{3}{2}}\begin{cases} 0 & ;\ 0 \le t < p_1(i) \\ \frac{[t-p_1(i)]^{s+1}}{(s+1)!} & ;\ p_1(i) \le t < p_2(i) \\ \frac{[t-p_1(i)]^{s+1} - [t-p_2(i)]^{s+1}}{(s+1)!} & ;\ p_2(i) \le t < p_3(i) \\ \frac{[t-p_1(i)]^{s+1} - [t-p_2(i)]^{s+1} - [t-p_3(i)]^{s+1}}{(s+1)!} & ;\ p_3(i) \le t < p_4(i) \\ \frac{[t-p_1(i)]^{s+1} - [t-p_2(i)]^{s+1} - [t-p_3(i)]^{s+1} + [t-p_4(i)]^{s+1}}{(s+1)!} & ; p_4(i) \le t < 1 \end{cases}$$

(17)

3. Quasilinearization Technique:

The quasi-linearization approach [14]-[15] is a generalised form of the Newton–u Raphson method for linearizing nonlinear differential equations. It quadratically converges to the exact value. If there is any convergence at all, and it is monotonic. The main idea to use this technique is based upon the fact there is no analytic method to solve many non-linear equations on the society demand we have the need to find the solution of these equation. In equation (3.1) φ is a non-linear term then we have to use the below recurrence relation

$$\varphi''_{r+1} = \psi(\varphi'_r, \varphi_r) + (\varphi_{r+1} - \varphi_r)\psi_{\varphi}(\varphi'_r, \varphi_r) + (\varphi'_{r+1} - \varphi'_r)\psi'_{\varphi}(\varphi'_r, \varphi_r) \tag{19}$$

Where ψ is a non-linear function of $\varphi^{n-1}{}_r$, $\varphi^{n-2}{}_r$, $\varphi^{n-3}{}_r$ ... φ'_r , φ_r and φ_r will be known value at each step which will be used to calculate φ_{r+1}.

By applying the quasi-linearization technique to linearize the non-linear term of the equation (1) and (2)

For First Non-linear term-

$$[uu_x]_{r+1} = [uu_x]_r + [u_{r+1} - u_r][u_x]_r + [[u_x]_{r+1} - [u_x]_r][u]_r$$

$$[uu_x]_{r+1} = [u]_{r+1}[u_x]_r + [u]_r[u_x]_{r+1} - u_r[u_x]_r \tag{20}$$

For Second Non-linear term-

$$[[uw]_x]_{r+1} = [u_x w]_{r+1} + [w_x u]_{r+1}$$

$$[[uw]_x]_{r+1} = [u_x]_{r+1}w_r - [u_x]_r w_r + [u_x]_r w_{r+1} + [w_x]_{r+1}u_r - [w_x]_r u_r + [w_x]_r u_{r+1} \tag{21}$$

4. Method of Solution:

After applying the quasi-linearization, we have equations (20), (21) then using these, non-linear system of partial differential equation (1), (2), (3), (4), (5) transformed in to the series of linear differential equation (22), (23) we have

$$[\frac{\partial u}{\partial t}]_{r+1} + p[u_{r+1}[u_x]_r + u_r[u_x]_{r+1} - u_r[u_x]_r] + q[\frac{\partial w}{\partial x}]_{r+1} = 0 \quad (22)$$

$$[\frac{\partial w}{\partial t}]_{r+1} + r[[u_x]_{r+1}w_r - [[u_x]_r w_r + [u_x]_r w_r + [u_x]_r w_{r+1} + [w_x]_{r+1}u_r - [w_x]_r u_r + [w_x]_r u_{r+1}] + s[\frac{\partial^3 u}{\partial x^3}]_{r+1} = 0 \quad (23)$$

Subjected to the boundary conditions given by

$$u(a, t_{r+1}) = \eta_1(t_{r+1}),\ u(b, t_{r+1}) = \eta_2(t_{r+1}) \quad (24)$$

$$w(a, t_{r+1}) = \xi_1(t_{r+1}), w(b, t_{r+1}) = \xi_2(t_{r+1}) \quad (25)$$

With the constraints on initial values

$$u(x, 0) = \mu(x), w(x, 0) = \lambda(x)\ \forall\ x \in [a, b], t_{r+1} \in [0, t] \quad (26)$$

Here t_{r+1} represent $(r+1)^{th}$ approximation for t in the process of quasilinearization

$$u_{xxxt}(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_i h_i(x) h_l(t) \quad (27)$$

$$w_{xt}(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} b_i h_i(x) h_l(t) \quad (28)$$

Integrating the equation (27) with respect to t with limits of integration we have

$$u_{xxx}(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il} h_i(x) Q_{1,l}(t) + u_{xxx}(x, 0) \quad (29)$$

Integrating equation (29) with respect to x three times them we have

$$u_{xx}(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il} Q_{1,i}(x) Q_{1,l}(t) + [u_{xx}(x, 0) - u_{xx}(0, 0)] + u_{xx}(0, t) \quad (30)$$

$$u_x(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il} Q_{2,i}(x) Q_{1,l}(t) + x[u_{xx}(0, t) - u_{xx}(0, 0)] + u_x(0, t) + [u_x(x, 0) - u_x(0,0)] \quad (31)$$

$$u(x, t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il} Q_{3,i}(x) Q_{1,l}(t) + \frac{x^2}{2}[u_{xx}(0, t) - u_{xx}(0, 0)] + x[u_x(0, t) - u_x(0,0)] + u(0, t) + [u(x, 0) - u(0,0)] \quad (32)$$

Integrating the equation (28) with respect to t with limits of integration we have

$$w_x(x,t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} b_{il}h_i(x)Q_{1,l}(t) + [w_x(x,0) - w_x(0,0)] \quad (33)$$

Now integrating with respect to x we have

$$w(x,t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} b_{il}Q_{1,i}(x)Q_{1,l}(t) + w(0,t) + [w(x,0) - w(0,0)] \quad (34)$$

Differentiating equation (32) with respect to t we get

$$u_t(x,t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il}Q_{3,i}(x)h_l(t) + \frac{x^2}{2}[u_{xx}(0,t) - u_{xx}(0,0)]_t + x[u_x(0,t)]_t + [u(0,t)]_t - x[u_x(0,0)]_t \quad (35)$$

Differentiating equation (34) with respect to t we get

$$w_t(x,t) = \sum_{i=1}^{3p}\sum_{l=1}^{3p} b_{il}Q_{1,i}(x)h_l(t) + [w(0,t)]_t \quad (36)$$

Using all these values equation (1), (2) becomes

$$\sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il}[Q_{3,i}(x)h_l(t) + p\big(Exact\ sol(u)\big)_x Q_{3,i}(x)\,Q_{1,l}(t) + p\big(Exact\ sol(u)\big)Q_{2,i}(x)Q_{1,l}(t) +$$

$$q\sum_{i=1}^{3p}\sum_{l=1}^{3p} b_{il}h_i(x)Q_{1,l}(t) = [x[u_x(0,0)]_t - [u(0,t)]_t - x[u_x(0,t)]_t - \frac{x^2}{2}[u_{xx}(0,t)]_t + \frac{x^2}{2}$$

$$[u_{xx}(0,0)]_t] - p\big(Exact\ sol(u)\big)_x[\frac{x^2}{2}\big(u_{xx}(0,t) - u_{xx}(0,0)\big) + x[u_x(0,t)] - x[u_x(0,0)] + u(0,t)$$

$$+[\mathrm{u(x,0) - u(0,0)}] - \mathrm{xp\big(Exact\ sol(u)\big)}u_{xx}(0,t) + p\big(Exact\ sol(u)\big)u_{xx}(0,0) -$$

$$p(Exact\ sol(u))u_x(x,0) +$$

$$p(Exact\ sol)u_x(0,0) - p(Exact\ sol)u_x(0,t) + p(Exact\ sol)u_x(0,0) + (Exact\ sol)(Exact\ sol)_x$$

$$-[w_x(x,0) - w_x(0,0)] \quad (37)$$

Now equation (2) becomes

$$\sum_{i=1}^{3p}\sum_{l=1}^{3p} b_{il}\left[h_l(t)Q_{1,i}(x) + r\big(Exact\ sol(u)\big)_x Q_{1,i}(x)Q_{1,l}(t) + r\big(Exact\ sol(u)\big)h_i(x)Q_{1,l}(t)\right] +$$

$$\sum_{i=1}^{3p}\sum_{l=1}^{3p} a_{il}[r\left(Exact\ sol(w)\big)Q_{2,i}(x)Q_{1,l}(t) + r\big(Exact\ sol(w)\big)_x Q_{3,i}(x)Q_{1,l}(t) + sh_i(x)Q_{1,l}(t)\right.$$

$$= -[w(0,t)]_t - [r\big(Exact\ sol(w)\big)\big(x\big(u_{xx}(0,t)\big) - (u_{xx}(0,0))\big) + \Big(\big(u_x(x,0)\big) - \big(u_x(0,0)\big)\Big) +$$

$$\Big(\big(u_x(0,t)\big) - \big(u_x(0,0)\big)\Big) + \Big[r\big(Exact\ sol(u)\big)_x\big(Exact\ sol(w)\big)\Big] - r\big(Exact\ sol(u)\big)_x[w(x,0)$$

$$-w(0,0)+w(0,t)]-r(Exact\ sol(u))[w_x(x,0)-w_x(0,0)]$$
$$+r(Exact\ sol(w))(Exact\ sol(u))$$

$$-r\big(Exact\ sol(w)\big)_x[\tfrac{x^2}{2}\Big(\big(u_{xx}(0,t)\big)-\big(u_{xx}(0,0)\big)\Big)+x(u_x(0,t)-u_x(0,0))+u(0,t)+u(x,0)-$$
$$u(0,0) \quad (38)$$

Now using the Boundary conditions and discretizing the space variable as x tend to x_l where $x_l = {2l-1}/{6p}$, $l = 0, 1, 2, 3, \ldots 2p$ in the equation (32), (34) and substituting the value obtained in the given system of equation are obtained for different value of r

$$a_{1\times 3p}A_{3p\times 3p}+b_{1\times 3p}B_{3p\times 3p}=C_{1\times 3p} \quad (39)$$

$$b_{1\times 3p}D_{3p\times 3p}+a_{1\times 3p}E_{3p\times 3p}=F_{1\times 3p} \quad (40)$$

Where the equations (40), (41), (42), (43), (44), (45) and (46) respectively represents the value of

A, B, C, D, E, F

$$A=[Q_{3,i}(x)h_l(t)+p\big(Exact\ sol(u)\big)_xQ_{3,i}(x)Q_{1,l}(t)+p\big(Exact\ sol(u)\big)Q_{2,i}(x)Q_{1,l}(t)$$

$$B=h_i(x)Q_{1,l}(t)$$

$$C=[x[u_x(0,0)]_t-[u(0,t)]_t-x[u_x(0,t)]_t-\tfrac{x^2}{2}[u_{xx}(0,t)]_t+\tfrac{x^2}{2}$$

$$[u_{xx}(0,0)]_t]-p\big(Exact\ sol(u)\big)_x[\frac{x^2}{2}\big(u_{xx}(0,t)-u_{xx}(0,0)\big)+x[u_x(0,t)]-x[u_x(0,0)]+u(0,t)$$

$$+[\mathrm{u(x,0)-u(0,0)}]-\mathrm{xp(Exact\ sol)}u_{xx}(0,t)+p(Exact\ sol)u_{xx}(0,0)-p(Exact\ sol)u_x(x,0)+$$

$$p(Exact\ sol)u_x(0,0)-p(Exact\ sol)u_x(0,t)+p(Exact\ sol)u_x(0,0)+(Exact\ sol)(Exact\ sol)_x$$

$$-[w_x(x,0)-w_x(0,0)]$$

$$D=\Big[h_l(t)Q_{1,i}(x)+r\big(Exact\ sol(u)\big)_xQ_{1,i}(x)Q_{1,l}(t)+r\big(Exact\ sol(u)\big)h_i(x)Q_{1,l}(t)\Big]$$

$$E=r\big(Exact\ sol(w)\big)Q_{2,i}(x)Q_{1,l}(t)+r\big(Exact\ sol(w)\big)_xQ_{3,i}(x)Q_{1,l}(t)+sh_i(x)Q_{1,l}(t)$$

$$F=-[w(0,t)]_t-[r\big(Exact\ sol(w)\big)\big(x\big(u_{xx}(0,t)\big)-(u_{xx}(0,0))\big)+\Big(\big(u_x(x,0)\big)-\big(u_x(0,0)\big)\Big)+$$

$$\Big(\big(u_x(0,t)\big)-\big(u_x(0,0)\big)\Big)+\Big[r\big(Exact\ sol(u)\big)_x\big(Exact\ sol(w)\big)\Big]-r\big(Exact\ sol(u)\big)_x[w(x,0)$$

$$-w(0,0)+w(0,t)]-r(Exact\ sol(u))[w_x(x,0)-w_x(0,0)]$$
$$+r(Exact\ sol(w))(Exact\ sol(u))$$

$$-r\big(Exact\ sol(w)\big)_x[\frac{x^2}{2}\Big(\big(u_{xx}(0,t)\big)-\big(u_{xx}(0,0)\big)\Big)+x(u_x(0,t)-u_x(0,0))+u(0,t)+u(x,0)-u(0,0)$$

The process of solution starts by taking $r=0$ and $t_0=0$ and the boundary conditions is given by

$$u(x_l,t_r)=u(x_l,0)=\mu(x_l) \qquad w(x_l,t_r)=w(x_l,0)=\lambda(x_l)$$

$$u_x(x_l,t_r)=u_x(x_l,0)=\mu_x x_l \quad \text{and} \quad w_x(x_l,t_r)=w_x(x_l,0)=\lambda_x x_l$$

$$u_{xx}(x_l,t_r)=u_{xx}(x_l,0)=\mu_{xx}x_l \qquad w_{xx}(x_l,t_r)=w_{xx}(x_l,0)=\lambda_{xx}x_l \quad (47)$$

The values of coefficient can be calculated successively for different values of r by using these equation (48), (49)

$$a=(C-FD^{-1}B)*(A-ED^{-1}B) \quad (48)$$

$$b=(C-FE^{-1}A)*(B-DE^{-1}A) \quad (49)$$

Then by putting the values of the wavelet coefficient in the given equation and can obtain numerically approximately solution successively for $u(x,t)$ and $w(x,t)$ for values of r then we have

$$u(x,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}a_{il}Q_{3,i}(x)Q_{1,l}(t)+\frac{x^2}{2}[u_{xx}(0,t)-u_{xx}(0,0)]+x[u_x(0,t)-u_x(0,0)]+u(0,t)+[u(x,0)-u(0,0)]$$

$$w(x,t)=\sum_{i=1}^{3p}\sum_{l=1}^{3p}b_{il}Q_{1,i}(x)Q_{1,l}(t)+w(0,t)+[w(x,0)-w(0,0)] \quad (50)$$

5. Numerical Examples:

The MATLAB computer language was used to do numerical calculations and to generate graphical outputs. A discrete form of the Haar scale-2 and Haar scale-3 wavelet series is required to determine the numerical solution of a third order system of PDE using Haar-scale-3 wavelet. As a result, at the initial level of resolution J = 1, the collocation points at the point of discontinuity approach given in equations which is used to pick collocation points. The efficacy of the present scheme was tested by analysing the solutions of three issues acquired by the present scheme and calculating absolute errors to explain the applicability of the present scheme for the third order system of PDE.

Numerical Problem-1

Consider the following Non-linear Boussinesq Burger's System of equations

$$\begin{cases} \frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial w}{\partial x} = 0 \; ; x \in [a,b], t \in [0,T] \\ \frac{\partial w}{\partial t} + 2\left[\frac{\partial}{\partial x}(uw)\right] - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} = 0; x \in [a,b], t \in [0,T] \end{cases} \tag{52}$$

Subjected to the boundary conditions are given in equation (53)

$$u(0,t) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{\frac{t}{2} - log2}{2}\right), \qquad w(0,t) = -\frac{1}{8}sech^2\left(\frac{-\frac{t}{2} + log2}{2}\right)$$

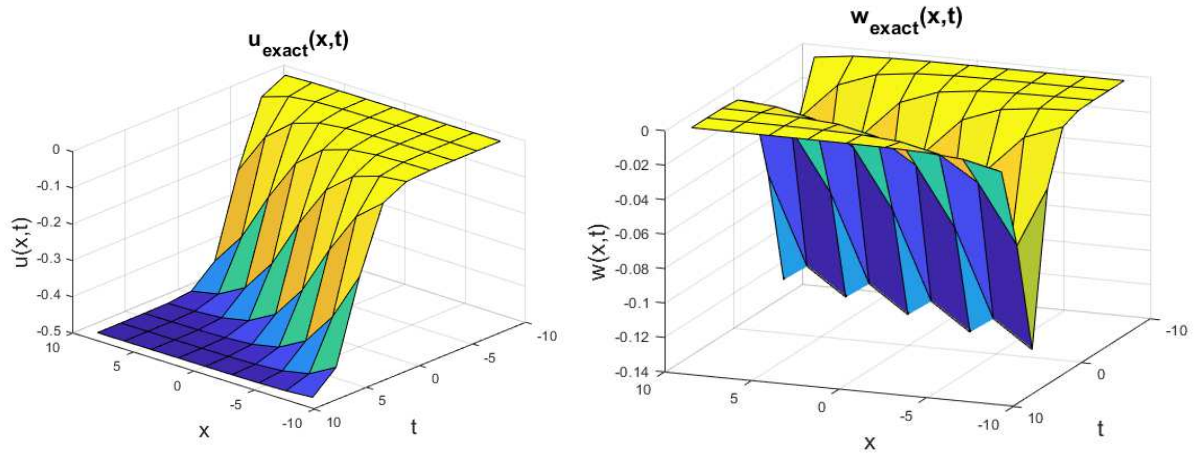
$$u(1,t) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{1 + \frac{t}{2} - log2}{2}\right), \qquad w(1,t) = -\frac{1}{8}sech^2\left(\frac{-1 - \frac{t}{2} + log2}{2}\right) \tag{53}$$

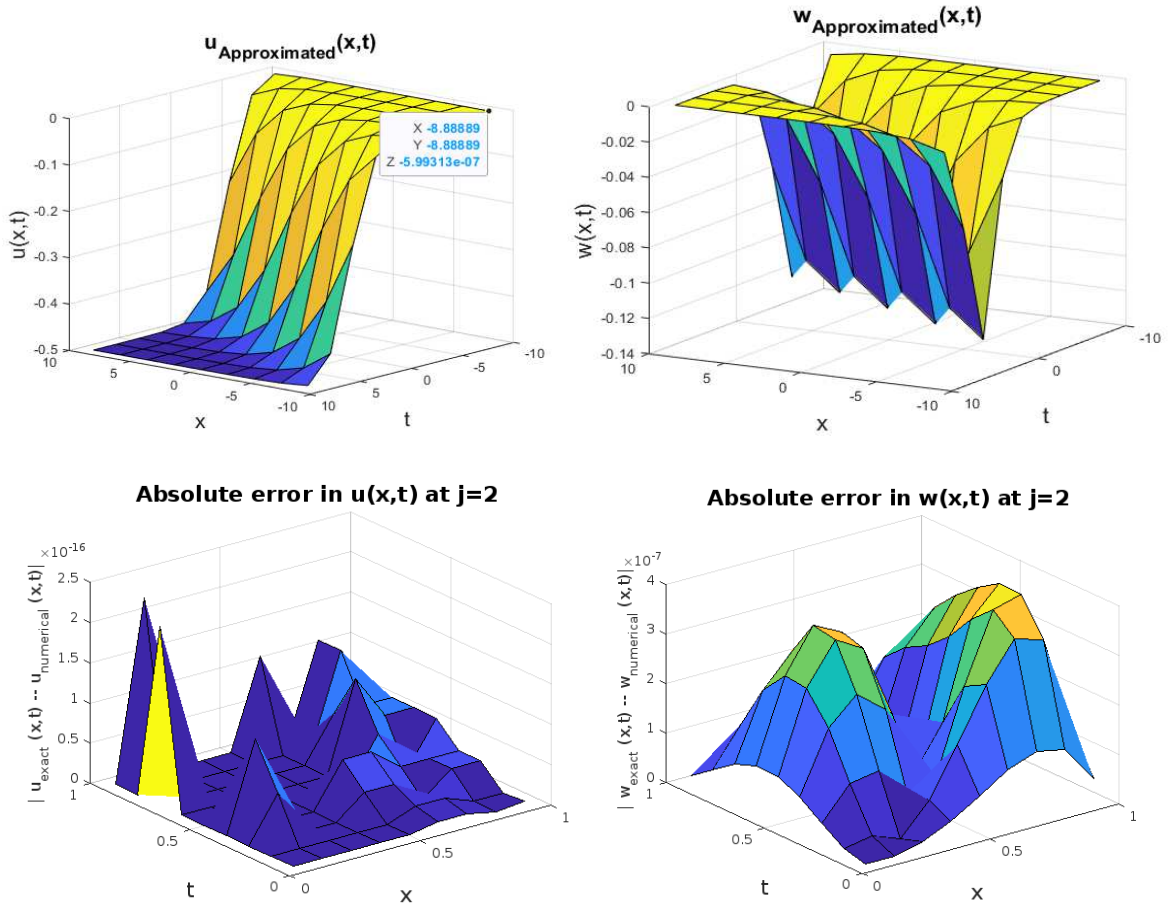
And with the initial conditions given in equation (54)

$$u(x,0) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{x - log2}{2}\right), \qquad v(x,0) = -\frac{1}{8}sech^2\left(\frac{-x + log2}{2}\right) \tag{54}$$

The exact solution of the Equation (52) subjected to the conditions given in Equation (53) and (54) for $p = 2, q = -\frac{1}{2}, r = 2, s = -\frac{1}{2}$ is

$$u(x,t) = -\frac{1}{4} - \frac{1}{4}\tanh\left(\frac{x + \frac{t}{2} - log2}{2}\right), \qquad w(x,t) = -\frac{1}{8}sech^2\left(\frac{-x - \frac{t}{2} + log2}{2}\right) \tag{55}$$





Graphical representation of Scale-3 Haar Wavelets for u(x, t), w(x, t) for numerical problem-1

Level of Resolution	J = 1	J = 2	J =3
$L_2 - error(HS2WM)$	2.089608e-15	2.329367e-16	2.483600e-18
$L_2 - error(HS3WM)$	2.426703e-21	2.467190e-23	2.422743e-23
$L_\infty - error(HS2WM)$	1.297012e-17	1.198329e-17	1.227092e-19
$L_\infty - error(HS3WM)$	1.923255e-22	1.929064e-22	2.098233e-24

Table-1 Comparison result achieved from Haar Scale-2 and Haar Scale-3 Method for numerical Problem-1

The graphical representation of approximated and exact solutions of Scale-3 Haar wavelet for Problem-1 as shown in above Figures. Figures illustrates that the exact and numerical results for J=1 are compatible. Table shows the results about errors like $L_2 and\ L_\infty\ for\ J = 1, 2, 3$ in case of both scale-2 and Scale-3 Haar wavelets.

Numerical Problem-2

Consider the following Non-linear Boussinesq Burger's System of equations

$$\begin{cases} \frac{\partial u}{\partial x} + 2u\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial w}{\partial x} = 0 \; ; x \in [a,b], t \in [0,T] \\ \frac{\partial w}{\partial t} + 2\left[\frac{\partial}{\partial x}(uw)\right] - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} = 0; x \in [a,b], t \in [0,T] \end{cases} \quad (56)$$

Subjected to the boundary conditions are given in equation (57)

$$u(0,t) = 0, \qquad w(0,t) = 0$$

$$u(1,t) = \sin(2\pi)cost, \qquad w(1,t) = \sin(2\pi)cost \quad (57)$$

And with the initial conditions given in equation (54)

$$u(x,0) = \sin(2\pi x), \qquad w(x,0) = \sin(2\pi x) \quad (58)$$

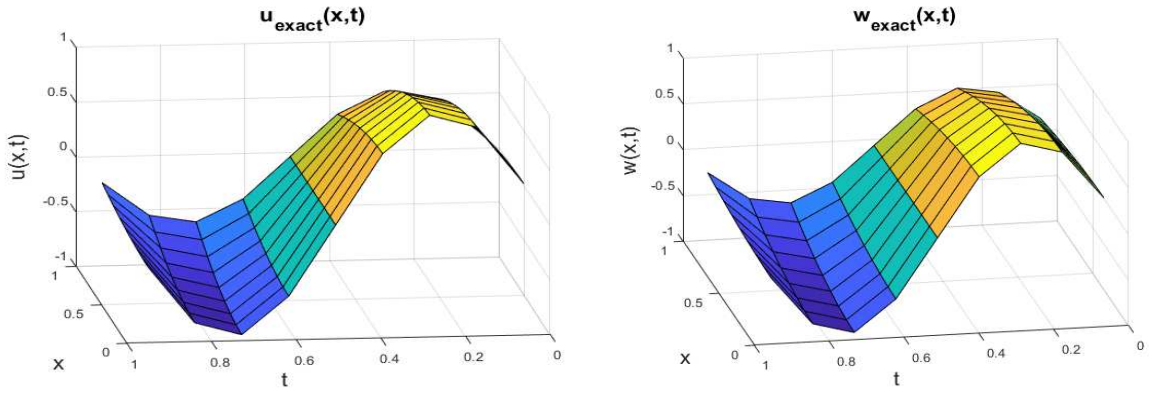
The exact solution of the Equation (56) subjected to the conditions given in Equation (57) and (58) for $p = 2, q = -\frac{1}{2}, r = 2, s = -\frac{1}{2}$ is

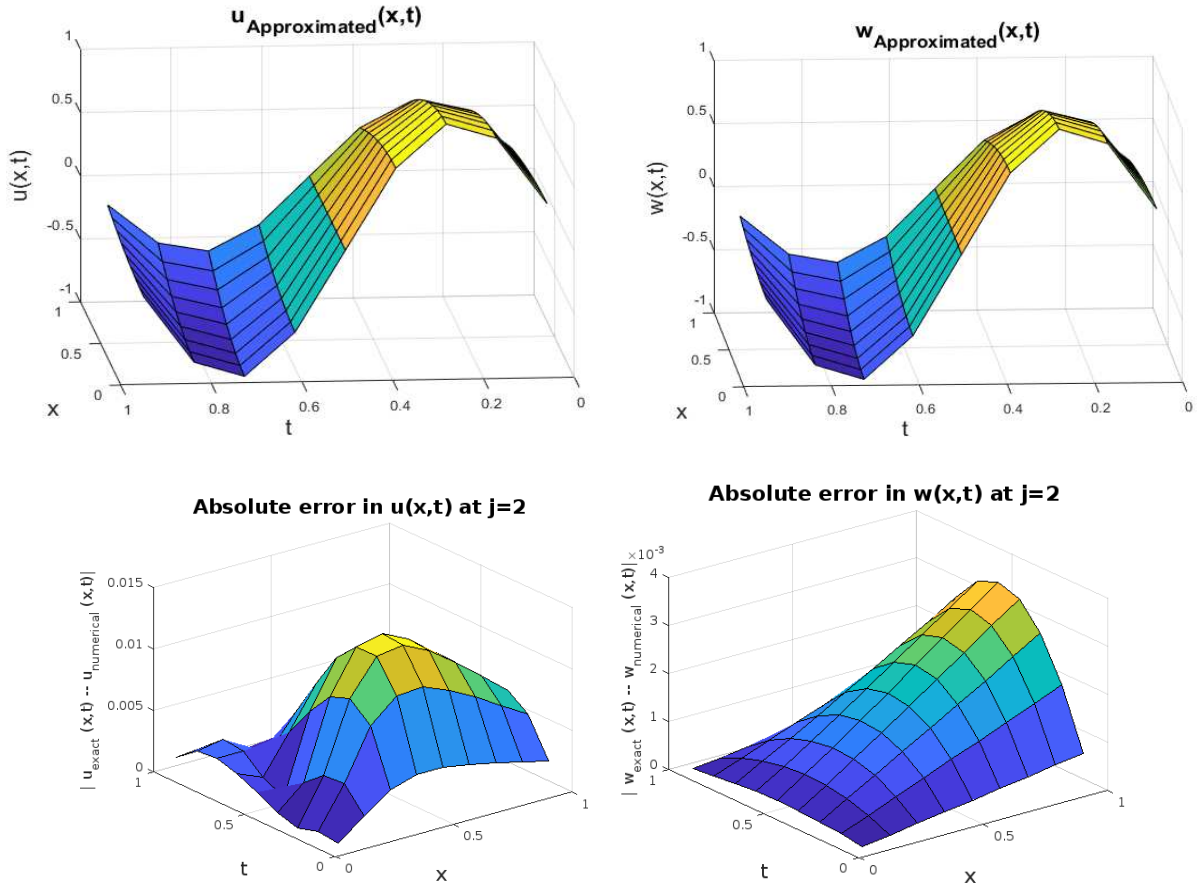
$$u(x,t) = \sin(2\pi x)\, cost, \qquad w(x,t) = \sin(2\pi x)cost \quad (59)$$

With the defined source term

$$f(x,t) = 2\sin(2\pi x)\cos(2\pi x)\, cos^2 t - \left(1 + \frac{7}{15}\right)\sin(2\pi x)\, sint + \left(1 + \frac{7}{30}\right)\cos(2\pi x)\, cost$$

$$g(x,t) = \sin(2\pi x)\cos(2\pi x)\, cos^2 t - \left(1 + \frac{1}{2}\right)\sin(2\pi x)\, sint + \left(1 + \frac{2}{5}\right)\cos(2\pi x)\, cost \quad (60)$$





Graphical representation of Scale-3 Haar Wavelets for u(x, t), w(x, t) for numerical problem-2

Level of Resolution	J = 1	J = 2	J =3
$L_2 - error(HS2WM)$	3.147904e-11	3.810721e-12	3.377801e-14
$L_2 - error(HS3WM)$	3.348911e-13	3.438901e-13	3.234010e-15
$L_\infty - error(HS2WM)$	1.368012e-14	1.111456e-14	1.156723e-16
$L_\infty - error(HS3WM)$	1.100098e-15	1.678902e-17	2.230089e-22

Table-2 Comparison result achieved from Haar Scale-2 and Haar Scale-3 Method for numerical Problem-2

The graphical representation of approximated and exact solutions of Scale-3 Haar wavelet for Problem-2 as shown in above Figures. Figures illustrates that the exact and numerical results for J=1 are compatible. Table shows the results about errors like $L_2 and\ L_\infty\ for\ J = 1, 2, 3$ in case of both scale-2 and Scale-3 Haar wavelets.

In above two examples help us to enhance the accuracy of Haar scale-3 wavelet method as compare to Haar scale-2 wavelet method. The higher the resolution J, the more exact the solution approximation. For solving Boussinesq-Burger system of equations, the numerical techniques are reliable and convenient. The simplicity, adaptability, and lack of computational errors are the key benefits of these

techniques. The outcomes we obtained from the results of numerical experiments performed on three test problems with the proposed technique, we conclude that the 2D Dispersive equations of third order containing non-linearity can easily be solved with high accuracy and less computational cost by the discussed scheme. We have solved various types of Dispersive equations the use of common MATLAB subprograms makes it more computer-friendly. The proposed scheme for a very small number of collocation points becomes a strong solver for these kinds of partial differential equations and good accuracy is obtained. In comparison to the classical Scale-2 Haar wavelet approach, numerical findings indicate that quasi-linearization using Scale-3 Haar wavelet converges rapidly even for small numbers of grid points.

Conclusion

The Haar wavelet method was used to solve the nonlinear third-order Boussinesq-Burger system in this article. The basic idea behind the Haar wavelet approach is to turn a third-order Boussinesq system into a set of algebra equations with a finite number of variables. By comparing the Haar wavelet to the exact solution of the Boussinesq system at the same time, we discovered that it had a decent approximation effect. The higher the resolution J, the more exact the solution approximation. For solving Boussinesq-Burger system of equations, the numerical techniques are reliable and convenient. The simplicity, adaptability, and lack of computational errors are the key benefits of these techniques. Furthermore, if we improve the level of resolution, which results in a bigger number of collocation points, the mistakes may be greatly decreased.

MATLAB language is used in finding the results and figure draw, it's a characteristic at high accuracy and large speed.

References:

Ataie-Ashtiani, B., & Najafi-Jilani, A. (2007). A higher-order two-dimensional Boussinesq wave model. Journal of Coastal Research, 50, 1183–1187.

Bona, J. L., Chen, M., & Saut, J. C. (2002). Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media I: Derivation and linear theory. Journal of Nonlinear Science, 12, 283–318.

Bona, J. L., Dougalis, V. A., & Mitsotakis, D. E. (2005). Numerical solution of coupled-KdV systems of Boussinesq equations: II. Generation, interactions and stability of generalized solitary waves. Preprint submitted to Elsevier Science.

Chen, M. (2009). Numerical investigation of a two-dimensional Boussinesq system. Discrete and Continuous Dynamical Systems, 23(4), 1169–1190.

Chen, M., & Goubet, O. (2009). Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems. Discrete and Continuous Dynamical Systems, 2, 37–53.

Debnath, L. (2002). Wavelet transforms and their applications. Birkhäuser.

Jiwari, R. (2012). A Haar wavelet quasilinearization approach for numerical simulation of Burgers' equation. Computer Physics Communications, 183, 2413–2423.

Kaur, H., Mittal, R. C., & Mishra, V. (2013). Haar wavelet approximate solutions for the generalized Lane-Emden equations arising in astrophysics. Computer Physics Communications, 184, 2169–2177.

Lepik, U. (2007). Application of the Haar wavelet transform to solving integral and differential equations. Proceedings of the Estonian Academy of Sciences: Physics and Mathematics, 56(1), 28–46.

Lepik, U. (2007). Numerical solution of evolution equations by the Haar Wavelet method. Applied Mathematics and Computation, 185, 695–704.

Mera, M. (2011). Boundary conditions for 2D Boussinesq-type wave-current interaction equations. Civil Engineering Dimension, 13(1), 37–41.

Micu, S., Ortega, J. H., Rosier, L., & Zhang, B. Y. (2009). Control and stabilization of a family of Boussinesq systems. Discrete and Continuous Dynamical Systems, 24(2), 273–313.

Mitsotakis, D. E. (2009). Boussinesq systems in two space dimensions over a variable bottom for the generation and propagation of tsunami waves. Mathematics and Computers in Simulation, 80, 860–873. https://arxiv.org/abs/0906.2491

Zhi, S., Li-Yuan, D., & Qing, J. C. (2007). Numerical solution of differential equations by using Haar wavelets. In Proceedings of a Conference (pp. 1039–1044). Beijing, China.