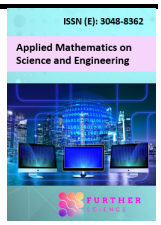
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A Brief Overview of Applications of Tree-Width and Other Graph Width Parameters

Takaaki Fujita [1*]

[1] *Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.*
[*] *Corresponding author: takaaki.fujita060@gmail.com*

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Abstract

Graph theory, a fundamental branch of mathematics, centers on the study of networks composed of vertices (nodes) and edges, examining their paths, structures, and properties. One essential metric in this field is the "graph width parameter," which quantifies the maximum width across all cuts or layers within a hierarchical decomposition of the graph. Tree-width, in particular, has garnered significant attention due to its broad range of applications. In this work, we revisit these parameters and their applications, focusing specifically on tree-width and related graph width parameters in the contexts of Bond Graphs, Factor Graphs, and Graph Entropy.

1. Introduction

1.1. Graph Structures and Graph Classes

Graph theory, a fundamental area of mathematics, focuses on the study of structures consisting of vertices (nodes) and edges that represent relationships or connections [1]. Due to its flexibility and strong theoretical foundation, graph theory has been extensively applied across a wide range of disciplines, including computer science, biology, social sciences, and engineering [2–4].

The study of graph theory often involves the investigation of specific graph structures such as paths [5], trees [6], cycles [7], complete graphs [8], cliques [9], wheels [10], and forests [11] (cf. [12]). For example, a tree is defined as a connected, acyclic graph in which there exists exactly one path between any pair of vertices.

In this context, graph algorithms—systematic procedures for solving computational problems—are of particular interest. Many such algorithms are tailored to specific graph structures in order to enhance efficiency and scalability (cf. [13–15]).

A *graph parameter* is a numerical value assigned to a graph to describe certain structural characteristics, such as its size, connectivity, or topological configuration [16, 17]. These parameters are crucial for analyzing, comparing, and optimizing graph-based models.

To address the needs of diverse applications, various generalized graph models have been developed. These include:

- **Fuzzy Graphs** [18–20], which incorporate uncertainty in edge and vertex membership.
- **Neural Graphs** [21–23], used in machine learning to represent neural architectures and data flow.
- **Code Property Graphs** [24], applied in static code analysis and vulnerability detection.
- **Reeb Graphs** [25], used in topology-based shape analysis.

- **Disjunctive Graphs** [26, 27], important in scheduling and combinatorial optimization.
- **Neutrosophic Graphs** [28, 29], incorporating indeterminacy into graph models.
- **Plithogenic Graphs** [30, 31], capturing contradictory, neutral, and supportive relationships.
- **Hypergraphs** [32, 33], where edges (hyperedges) can connect any number of vertices.
- **Hypersoft Graphs** [34, 35], generalizing soft sets in graph structures.
- **Rough Graphs** [36, 37], used in uncertainty modeling and approximation theory.
- **Superhypergraphs** [38–40], employing higher-order power sets to represent layered relationships.
- **Flow Graphs** [41–43], essential for representing control or data flow in systems.
- **Collaboration Graphs** [44, 45], modeling co-authorship and social networks.
- **Majority-Inverter Graphs** [46], used in logic synthesis and circuit design.
- **And-Inverter Graphs** [47–49], optimized representations in digital logic.
- **Tensor Graphs** [50, 51], integrating tensor structures into graph models for high-dimensional data.

These diverse graph classes demonstrate the adaptability and theoretical richness of graph theory, allowing it to model complex systems across domains.

1.2. Graph Width Parameters and Tree-width

One of the central metrics in graph theory is the *graph width parameter*, which quantifies the structural complexity of a graph by measuring the maximum width across all cuts or layers in a hierarchical decomposition [52–55]. This parameter is essential for analyzing the internal structure of a graph and has significant implications in algorithmic graph theory. In particular, many computationally intractable problems become tractable when restricted to graphs with bounded width.

Among various width parameters, *tree-width* is one of the most widely studied and practically useful [56–59]. Tree-width measures how close a graph is to being a tree by identifying the minimum width among all possible tree decompositions. A smaller tree-width indicates that the graph exhibits more tree-like behavior, which often translates to lower computational complexity in solving problems on such graphs.

Extensive research has led to the development of numerous algorithms for computing tree-width and other width parameters [60–63]. These algorithms are particularly valuable because many real-world applications benefit from exploiting tree-like structures. Moreover, a wide range of NP-hard problems[1] can be efficiently solved on trees, often in polynomial or even linear time. This efficiency also holds for graphs with constant tree-width, where dynamic programming methods can be used to design polynomial-time algorithms.

1.3. Our Contribution in This Paper

We summarize our contributions as follows. Given the significance of tree-width in both theoretical and practical contexts, we aim to deepen the understanding of width parameters and extend their applications. Specifically, we revisit the definitions and properties of tree-width and related parameters, examining their roles in the structure and analysis of *Bond Graphs*, *Factor Graphs*, and *Graph Entropy*.

1.4. Contents of the paper

The remainder of the paper is organized as follows:

- Section 2 provides an overview of various application backgrounds related to graph width.
- Section 3 formally defines the tree-width parameter.
- Section 4 explores the behavior of tree-width in specific classes of graphs.
- Section 5 concludes the paper and outlines directions for future research.

2. Background of Applications

Tree-width and its related width parameters have already found numerous applications, owing to their tractability and the structural advantages of tree-like graphs such as tree graphs and path graphs. A brief overview of these applications is provided below (cf. [64]). This overview also serves to introduce prior research, with the expectation that these foundational results will inspire further applications and new ideas in the future.

2.1. Application aspect of tree-width

In this subsection, we consider about application aspect for tree-width. Tree-width is a graph parameter that measures how closely a graph resembles a tree. Lower tree-width means the graph is more ”tree-like,” making certain algorithms more efficient for such graphs[55]. Due to the versatility of tree structures in handling various types of data, tree-width has numerous applications across different fields (cf. [64]).

Tree-width for Bayesian network

A Bayesian network is a graphical model representing probabilistic relationships among variables using nodes and directed edges (cf.[65]).

Tree decomposition in Bayesian Networks simplifies the structure by breaking complex networks into tree-like components. This helps to efficiently perform exact inference and structure learning, particularly for networks with bounded tree-width, allowing scalable algorithms to handle large datasets while maintaining computational tractability[66, 67].

Tree-width for Neural networks

Neural networks are computational models inspired by the human brain, consisting of interconnected nodes that process data and learn patterns[68]. In Graph Neural Networks (GNNs), tree decomposition is used to simplify complex graphs by breaking them down into tree-like structures. This allows GNNs to more efficiently process local graph structures, improving their expressive power and enabling more accurate predictions, especially for graphs with bounded tree-width [21, 69–73].

Tree-width for Tensor network

Tensor networks are mathematical structures used to efficiently represent and compute high-dimensional data. They are applied in quantum physics, machine learning, and compressing large datasets by capturing complex relationships among variables (cf.[74]).

Applications for Tensor network have also been studied in the context of tree-width [75–78]. In tensor networks, tree decomposition helps optimize tensor contraction by minimizing the size of intermediate tensors generated during the contraction process. By reducing the network's tree-width, it becomes possible to identify efficient contraction orders, lowering computational complexity and memory requirements, thus enabling faster and more efficient simulations of large tensor networks.

Tree-width for RNA and DNA structure

Research on RNA and DNA is also advancing rapidly in the field of medicine. DNA is a stable, double-stranded molecule that stores genetic information in the form of a double helix, while RNA is single-stranded and more flexible, playing a crucial role in protein synthesis.

Tree-width has been applied in the study of RNA and DNA as well [79–82]. In particular, tree decomposition simplifies the analysis of DNA structures by breaking down complex interactions into tree-like components. This allows for more efficient solutions to problems such as pathway prediction, by integrating both sequence and structural information, which improves the accuracy of computational biology methods.

Tree-width for Protein Structure

Protein research is flourishing in the medical field, particularly regarding its structure and function. Protein structure refers to the three-dimensional arrangement of amino acids in a protein, which can be represented as a graph. In this graph, nodes correspond to amino acids, and edges represent bonds or interactions, capturing the intricate folding and connectivity of the protein (cf.[83, 84]).

The concept of tree-width has also been applied in protein structure analysis [85–88]. Tree-width helps decompose complex protein interaction graphs into simpler, tree-like structures. This decomposition is particularly useful in computational protein design, as it reduces the complexity of the search space, making it easier to solve problems such as identifying stable protein conformations.

Tree-width for Markov network

Markov networks, or Markov random fields, are undirected graphical models representing the joint distribution of variables. They are used in applications like image processing and statistical physics, emphasizing local dependencies (cf.[89]).

Applications for Markov network have also been studied in the context of tree-width [90–94]. In Markov networks, tree decomposition simplifies complex probabilistic models by breaking them into tree-like structures, which helps efficiently perform inference and computations. By limiting the tree-width, it becomes feasible to approximate joint probability distributions and perform accurate predictions, especially for real-world systems constrained by small tree-width values.

Tree-width for Social Networks

A social network consists of individuals or organizations connected through social relationships, such as friendships or collaborations. The study of social networks has also explored applications of tree-width [95–97]. By applying tree decomposition techniques, complex networks can be simplified, enabling more efficient analysis of connectivity and interactions within these networks.

2.2. Application aspect for path-width

We consider about application aspect for path-width. Path-decomposition is a method of representing a graph as a sequence of overlapping subgraphs (called "bags"), forming a path-like structure. Path-width is a measure of how close a graph is to a path. Lower path-width indicates simpler, more path-like structures, useful for simplifying complex graphs (cf.[98–100]).

Path-width for VLSI (Very Large Scale Integration)

VLSI refers to the process of integrating thousands to millions of transistors onto a single semiconductor chip (cf.[101–103]). It's crucial for developing modern electronic devices, enabling complex circuits and high-performance computing. Applications for VLSI have also been studied in the context of path-width [104–106].

Path-width for Compiler on computers

A compiler is software that translates high-level programming code into machine-readable code, enabling program execution on computers. Applications for Compiler have also been studied in the context of path-width [107, 108]. Additionally, concepts related to path-width, such as proper path-width, are well-known and have applications in game theory [64, 109, 110].

Path-width for Blockchain

Blockchain is a decentralized, distributed ledger technology that securely records transactions across multiple computers, ensuring transparency and immutability. Applications for blockchain have also been studied in the context of path-width[111, 112].

2.3. Application aspect for Hypertree-width

A hypergraph is a generalization of a graph where edges, called hyperedges, can connect any number of vertices, not limited to just two, allowing more complex relationships [113, 114]. Hypertree-width generalizes tree-width to hypergraphs [33, 115], measuring how "tree-like" a hypergraph is by evaluating the smallest possible width of a hypertree decomposition, improving efficiency in certain computational problems.

Hypertree-width for Artificial intelligence

Artificial intelligence is the simulation of human intelligence in machines, enabling them to perform tasks like reasoning, learning, and problem-solving. Similar to tree-width, the concept of hypertree-width is applied in the techniques and concepts of artificial intelligence that utilize graphs[116, 117].

Hypertree-width for Database

A database is a structured collection of data designed for efficient storage, retrieval, and management of information. The relationship between queries and hypertree-width has been extensively studied in database contexts [115, 118–121]. Additional related graph width parameters include FAQ-width [122, 123], m-width [124], closure tree-width [125], closure hypertree-width [125], TCLUSTER-width [126, 127], hinge-width [128, 129], β-Hyperorder width [130, 131], and # hypertree-width [132, 133].

For readers interested in more detailed information or exploring these applications, we recommend consulting lecture notes, surveys, or books on the subject (e.g., [114, 134–136]).

2.4. Other width parameters

The applications of various width parameters have been widely studied. For example, submodular-width is utilized in databases [137, 138], while bandwidth and carving-width are applied in network analysis [139, 140]. Parameters such as cut-width have been explored in fields like VLSI design [141].

3. Preliminaries and Definitions

In this section, we briefly present the definitions and notations used throughout the paper. Unless otherwise specified, all graphs considered in this paper are assumed to be finite, undirected, and simple.

3.1. Tree-width of Undirected Basic Graphs

In graph theory, the concept of *tree-width* is fundamentally linked to the theory of graph minors. A graph minor is derived from a given graph through a sequence of operations, including the deletion of edges, deletion of vertices, and contraction of edges. Tree-width measures how closely a graph resembles a tree by representing it in a tree-like structure with the minimum possible width [142–146].

The notion of tree-width was first introduced in 1972 by Umberto Bertelè and Francesco Brioschi [147], and it was later independently rediscovered by Rudolf Halin in 1976 [148]. In the 1980s, the concept was significantly advanced by Neil Robertson and Paul Seymour, who established tree-width as a central tool in their seminal work on graph minor theory [142, 143, 149]. This foundational idea has since been generalized to more complex structures, giving rise to parameters such as *hypertree-width* [32, 115] and its further extension, the *superhypertree-width* [150–152], which capture tree-like decomposition characteristics in hypergraphs and superhypergraphs, respectively.

Below, we provide the formal definitions of tree-width.

Definition 3.1. *[142] A tree decomposition of a graph $G = (V,E)$ is a pair $(\mathscr{T},\mathscr{V})$, where:*

- *$\mathscr{T} = (I,F)$ is a tree, with I being the set of nodes (called bags) and F the set of edges of the tree.*
- *$\mathscr{V} = \{B_i \mid i \in I\}$ is a collection of subsets $B_i \subseteq V$, called bags, associated with the nodes of the tree.*

The tree decomposition must satisfy the following properties:

1. *For every vertex $v \in V$, there exists at least one bag $B_i \in \mathscr{V}$ such that $v \in B_i$.*
2. *For every edge $(u,v) \in E$, there exists at least one bag $B_i \in \mathscr{V}$ such that $u \in B_i$ and $v \in B_i$.*
3. *For any three bags $B_i, B_j, B_k \in \mathscr{V}$, if B_k lies on the path between B_i and B_j in the tree $\mathscr{T}$, then $B_i \cap B_j \subseteq B_k$.*

Definition 3.2. *[142] The width of a tree decomposition $(\mathscr{T},\mathscr{V})$ is defined as:*

$$\max_{i\in I} |B_i| - 1$$

The tree-width of a graph G is the minimum width among all possible tree decompositions of G. Formally, the tree-width $tw(G)$ is given by:

$$tw(G) = \min_{(\mathscr{T},\mathscr{V})} \left(\max_{i\in I} |B_i| - 1 \right)$$

where the minimum is taken over all tree decompositions of G.

Example 3.3 (Tree Decomposition of the 4-Cycle C_4)**.** *Let $G=(V,E)$ be the cycle graph on four vertices:*

$$V=\{v_1,v_2,v_3,v_4\},\quad E=\{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_1)\}.$$

We describe a tree decomposition $(\mathscr{T},\mathscr{V})$ of G as follows:

- *The tree $\mathscr{T}=(I,F)$ has two nodes $I=\{a,b\}$ and a single edge $F=\{(a,b)\}$.*
- *The bag assignment $\mathscr{V}=\{B_a,B_b\}$ is given by*

$$B_a=\{v_1,v_2,v_4\},\quad B_b=\{v_2,v_3,v_4\}.$$

One checks the three defining properties:

1. *Every vertex appears in at least one bag: $\{v_1,v_2,v_4\}\cup\{v_2,v_3,v_4\}=V$.*
2. *Every edge is contained in some bag: $(v_1,v_2),(v_4,v_1)\subseteq B_a$ and $(v_2,v_3),(v_3,v_4)\subseteq B_b$.*
3. *The bag-intersection property holds: the only nontrivial intersection is $B_a\cap B_b=\{v_2,v_4\}$, and every path in $\mathscr{T}$ whose endpoints are a and b passes through a node whose bag contains that intersection.*

Since $\max\{|B_a|,|B_b|\}=3$, the width of this decomposition is

$$\max_{i\in\{a,b\}}|B_i|-1\;=\;3-1=2.$$

One can show no tree decomposition of C_4 has width less than 2, so $\mathrm{tw}(C_4)=2$.

4. Short Survey and Proposal in this paper

We will describe the results presented in this paper.

4.1. Tree-width for Molecular Graph

A molecular graph, also known as a chemical graph, represents the structural formula of a chemical compound using graph theory[153, 154, 154–156]. In this graph, the vertices represent the atoms, while the edges correspond to the chemical bonds between them. Each vertex and edge is typically labeled to reflect the specific elements and types of bonds present in the compound.

Arthur Cayley was probably the first to publish results that consider molecular graphs as early as in 1874"[157]. The definition is provided as follows.

Definition 4.1 (Molecular Graph)**.** *A molecular graph is a type of graph used to represent the structure of a molecule, where:*

- *Each vertex represents an atom type.*
- *Each edge represents a bond type between two atoms.*

Formally, a molecular graph is a graph $G=(V,E)$, where:

- *V is a set of vertices corresponding to atoms in the molecule.*
- *$E\subseteq V\times V$ is a set of edges representing chemical bonds between atoms.*

Example 4.2 (Molecular Graph of Ethanol C_2H_6O)**.** *Ethanol is a colorless, volatile, flammable liquid commonly used as a biofuel, solvent, and active ingredient in alcoholic beverages (cf.[158, 159]). Consider the ethanol molecule, whose chemical formula is C_2H_6O. We model it by the molecular graph $G=(V,E)$ defined as follows:*

$$V=\{C_1,C_2,O_1,H_1,H_2,H_3,H_4,H_5,H_6\},$$

where each C_i denotes a carbon atom, O_1 denotes the oxygen atom, and each H_j denotes a hydrogen atom. The edge set

$$E\;=\;\{\{C_1,C_2\},\{C_1,H_1\},\{C_1,H_2\},\{C_1,H_3\},\{C_2,H_4\},\{C_2,H_5\},\{C_2,O_1\},\{O_1,H_6\}\}$$

represents all single covalent bonds in the molecule:

- *$\{C_1,C_2\}$ is the carbon–carbon bond.*
- *$\{C_1,H_1\},\{C_1,H_2\},\{C_1,H_3\}$ are the three hydrogen atoms bonded to C_1.*
- *$\{C_2,H_4\},\{C_2,H_5\}$ are the two hydrogens bonded to C_2.*
- *$\{C_2,O_1\}$ is the carbon–oxygen bond.*
- *$\{O_1,H_6\}$ is the hydroxyl (O–H) bond.*

One easily verifies that this graph satisfies the definition of a molecular graph, with vertices for each atom and edges for each bond.

The tree-width of molecular graphs (We call "molecular tree-width") has been explored in various studies [160–162]. In practical applications, further investigations could focus on related parameters such as pathwidth, cyclewidth, and starwidth, as necessary.

4.2. Tree-width for Graph Entropy

Entropy is a mathematical concept that quantifies uncertainty or information content within a system, often used in data analysis, physics, and communication theory[163].

Graph entropy is a metric used to quantify the complexity or amount of structural information within a graph [164–166]. It is calculated based on specific graph properties such as vertex degrees, edge orbits, and other factors that reflect the graph's symmetry and structure. The more intricate or irregular the structure, the higher the entropy. This notion of graph-theoretic entropy traces back to Körner in the 1970s [167], and was later surveyed and extended by Dehmer et al. [166]. Below we give mathematically precise, detailed definitions.

Definition 4.3 (Vertex Orbit Entropy [166, 167])**.** *Let $G = (V,E)$ be a finite undirected graph, and let* $\mathrm{Aut}(G)$ *denote its automorphism group. The action of* $\mathrm{Aut}(G)$ *partitions the vertex set V into k disjoint orbits*

$$V = O_1 \dot\cup O_2 \dot\cup \cdots \dot\cup O_k,$$

where each orbit $O_i \subseteq V$ consists of vertices that are topologically equivalent under graph symmetries. Define

$$p_i \;=\; \frac{|O_i|}{|V|}, \qquad i = 1,\ldots,k.$$

Then the vertex orbit entropy *of G is given by the Shannon formula*

$$H_V(G) \;=\; -\sum_{i=1}^{k} p_i \log_2 p_i.$$

This quantity measures the structural diversity of G in terms of its vertex-equivalence classes.

Definition 4.4 (Edge Orbit Entropy [166])**.** *Similarly,* $\mathrm{Aut}(G)$ *partitions the edge set E into ℓ disjoint orbits*

$$E = Q_1 \dot\cup Q_2 \dot\cup \cdots \dot\cup Q_\ell,$$

where each $Q_j \subseteq E$ consists of edges equivalent under automorphisms. Set

$$q_j \;=\; \frac{|Q_j|}{|E|}, \qquad j = 1,\ldots,\ell.$$

The edge orbit entropy *of G is*

$$H_E(G) \;=\; -\sum_{j=1}^{\ell} q_j \log_2 q_j,$$

which quantifies the information content of G's bond (edge) structure.

Definition 4.5 (Attribute-based Graph Entropy [166])**.** *Let $X : V \to A$ be a discrete vertex attribute (for instance, degree, color, or label), where A is a finite set. For each $a \in A$, let*

$$r_a = \big|\{\, v \in V : X(v) = a \}\big| \quad \textit{and} \quad p(a) = \frac{r_a}{|V|}.$$

Then the attribute-based entropy *of G with respect to X is*

$$H(X) \;=\; -\sum_{a \in A} p(a) \log_2 p(a).$$

This generalized entropy captures the variability of the chosen graph property X.

Next, we will discuss Entropy Tree and Entropy Treewidth. The following includes some conceptual ideas, but the definitions are provided.

Definition 4.6 (Bag Entropy)**.** *Let $G = (V,E)$ be a simple undirected graph and $B \subseteq V$ a nonempty vertex subset. Write $G[B]$ for the subgraph induced by B. For each $v \in B$, let*

$$d_B(v) \;=\; \big|\{\, w \in B : \{v,w\} \in E \}\big| \quad \textit{and} \quad S_B \;=\; \sum_{u \in B} d_B(u).$$

If $S_B > 0$, define a probability distribution

$$p_v \;=\; \frac{d_B(v)}{S_B}, \quad v \in B,$$

and the entropy *of the bag B by the Shannon formula*

$$H(B) \;=\; -\sum_{v \in B} p_v \log_2(p_v).$$

If $S_B = 0$ (i.e. $G[B]$ has no edges), set $H(B) = 0$.

Definition 4.7 (Entropy Tree Decomposition)**.** *An* entropy tree decomposition *of G is a standard tree decomposition $(\mathscr{T} = (I,F), \{B_i\}_{i \in I})$ satisfying:*

1. $\bigcup_{i\in I} B_i = V$.
2. *For every* $\{u,v\} \in E$, *there is some* $i \in I$ *with* $\{u,v\} \subseteq B_i$.
3. *For all* $i, j, k \in I$, *if node* k *lies on the unique* $\mathscr{T}$*-path between* i *and* j, *then* $B_i \cap B_j \subseteq B_k$.

Its entropy width *is*

$$\max_{i\in I} H(B_i).$$

The entropy tree-width *of* G *is*

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;=\; \min_{\substack{(\mathscr{T},\{B_i\}) \\ \textit{tree decompositions}}} \big(\max_{i\in I} H(B_i)\big).$$

Example 4.8 (Entropy Tree Decomposition of a Triangle with a Pendant Vertex)**.** *Let* $G = (V,E)$ *be the graph with*

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_1\},\{v_3,v_4\}\}.$$

We construct an entropy tree decomposition $(\mathscr{T}, \{B_i\})$ *as follows:*

- *The tree* $\mathscr{T} = (I,F)$ *has two nodes* $I = \{a,b\}$ *and one edge* $F = \{(a,b)\}$.
- *The bag collection is*

$$B_a = \{v_1, v_2, v_3\}, \quad B_b = \{v_3, v_4\}.$$

One checks that:

1. $\bigcup_{i\in\{a,b\}} B_i = V$.
2. *Each edge of* G *is contained in at least one bag:* $\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_1\} \subseteq B_a$ *and* $\{v_3,v_4\} \subseteq B_b$.
3. *The running intersection property holds since* $B_a \cap B_b = \{v_3\} \subseteq B_b$.

We compute the Shannon entropy $H(B)$ *of each bag* B *using*

$$d_B(v) = |\{w \in B : \{v,w\} \in E\}|, \quad S_B = \sum_{u\in B} d_B(u), \quad H(B) = -\sum_{v\in B} \frac{d_B(v)}{S_B} \log_2\!\big(\tfrac{d_B(v)}{S_B}\big).$$

- *For* $B_a = \{v_1,v_2,v_3\}$, *the induced subgraph is a triangle:* $d_{B_a}(v_i) = 2$ *for* $i = 1,2,3$, *so* $S_{B_a} = 6$ *and* $\frac{d}{S} = 1/3$. *Thus*

$$H(B_a) = -3\big(\tfrac{1}{3}\log_2\tfrac{1}{3}\big) = \log_2 3 \approx 1.585 \textit{ bits}.$$

- *For* $B_b = \{v_3,v_4\}$, *the induced subgraph is a single edge:* $d_{B_b}(v_3) = d_{B_b}(v_4) = 1$, *so* $S_{B_b} = 2$ *and* $\frac{d}{S} = 1/2$. *Hence*

$$H(B_b) = -2\big(\tfrac{1}{2}\log_2\tfrac{1}{2}\big) = 1 \textit{ bit}.$$

The entropy width *of this decomposition is*

$$\max\{H(B_a), H(B_b)\} = \max\{\log_2 3,\, 1\} = \log_2 3.$$

Accordingly, the entropy tree-width of G *satisfies*

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;\le\; \log_2 3.$$

One can show no decomposition achieves a smaller maximum bag entropy, so $\mathrm{tw}_{\mathrm{entropy}}(G) = \log_2 3$.

Example 4.9 (Entropy Tree Decomposition of a 4-Vertex Path)**.** *Let* $G = (V,E)$ *be the simple path on four vertices:*

$$V = \{v_1, v_2, v_3, v_4\}, \quad E = \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}\}.$$

We define a tree decomposition $(\mathscr{T}, \{B_i\})$ *and compute its entropy width.*

- *The tree* $\mathscr{T} = (I,F)$ *has three nodes* $I = \{a,b,c\}$ *and edges*

$$F = \{(a,b),(b,c)\}.$$

- *The bags are*

$$B_a = \{v_1,v_2\}, \quad B_b = \{v_2,v_3\}, \quad B_c = \{v_3,v_4\}.$$

One verifies:

1. $\bigcup_{i\in\{a,b,c\}} B_i = V$.
2. *Each edge lies in one bag:* $\{v_1,v_2\} \subseteq B_a$, $\{v_2,v_3\} \subseteq B_b$, $\{v_3,v_4\} \subseteq B_c$.
3. *The running-intersection property holds since* $B_a \cap B_b = \{v_2\} \subseteq B_b$ *and* $B_b \cap B_c = \{v_3\} \subseteq B_b$.

We compute the Shannon entropy $H(B)$ for each bag B. In any bag of two adjacent vertices, the induced subgraph is a single edge, so each vertex has degree 1 and

$$S_B = d_B(u) + d_B(v) = 1 + 1 = 2, \quad p_u = p_v = \tfrac{1}{2}, \quad H(B) = -2\Big(\tfrac{1}{2}\log_2 \tfrac{1}{2}\Big) = 1 \text{ bit}.$$

Hence

$$H(B_a) = H(B_b) = H(B_c) = 1.$$

The entropy width *of this decomposition is*

$$\max\{H(B_a), H(B_b), H(B_c)\} = 1.$$

Since no decomposition of G can have all bag entropies below 1 (each edge forces a bag of two vertices), we conclude

$$\mathrm{tw}_{\mathrm{entropy}}(G) = 1.$$

Theorem 4.10. *For any graph G,*

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;\le\; \log_2\big(\mathrm{tw}(G)+1\big).$$

Proof. Let $(\mathscr{T}, \{B_i\})$ be a decomposition realizing $\mathrm{tw}(G)$, so $|B_i| \le \mathrm{tw}(G)+1$ for all i. In each bag B_i, if $S_{B_i} > 0$ then the Shannon bound $H(B_i) \le \log_2 |B_i|$ applies; if $S_{B_i} = 0$ then $H(B_i) = 0 \le \log_2 |B_i|$. Hence $\max_i H(B_i) \le \log_2(\mathrm{tw}(G)+1)$. Taking the minimum over all decompositions gives the result. □

Corollary 4.11. *For any graph G,*

$$\mathrm{tw}(G) \;\le\; 2^{\mathrm{tw}_{\mathrm{entropy}}(G)} \;-\; 1.$$

Proof. From $H(B_i) \le \log_2 |B_i|$ we get $|B_i| \le 2^{H(B_i)}$. Hence for any entropy-optimal decomposition, $\max_i |B_i| \le 2^{\max_i H(B_i)} = 2^{\mathrm{tw}_{\mathrm{entropy}}(G)}$. Therefore $\mathrm{tw}(G) = \min \max_i (|B_i| - 1) \le 2^{\mathrm{tw}_{\mathrm{entropy}}(G)} - 1$. □

Theorem 4.12 (Subgraph Monotonicity)**.** *If $H = (V', E')$ is an induced subgraph of $G = (V, E)$ (so $V' \subseteq V$ and $E' = E \cap \binom{V'}{2}$), then*

$$\mathrm{tw}_{\mathrm{entropy}}(H) \;\le\; \mathrm{tw}_{\mathrm{entropy}}(G).$$

Proof. Let $(\mathscr{T}, \{B_i\}_{i \in I})$ be an entropy tree decomposition of G achieving width $\mathrm{tw}_{\mathrm{entropy}}(G)$. For each bag B_i, form $B_i' = B_i \cap V'$. Remove any i for which $B_i' = \emptyset$, and keep the same tree $\mathscr{T}$. The resulting $(\mathscr{T}, \{B_i'\})$ is a valid entropy tree decomposition of H, because:

- $\bigcup_i B_i' = V'$.
- Every edge of H lies in some original bag, and hence in the corresponding restricted bag.
- The running-intersection property is preserved under intersection.

Moreover, since $H(B_i') \le H(B_i)$ for each i, the maximum bag entropy cannot increase. Therefore $\mathrm{tw}_{\mathrm{entropy}}(H) \le \max_i H(B_i') \le \max_i H(B_i) = \mathrm{tw}_{\mathrm{entropy}}(G)$. □

Theorem 4.13 (Complete Graph Entropy Tree-width)**.** *For the complete graph K_n on n vertices,*

$$\mathrm{tw}_{\mathrm{entropy}}(K_n) = \log_2 n.$$

Proof. Since K_n is a clique of size n, any tree decomposition must contain a bag B with $|B| \ge n$. The unique optimal decomposition takes a single bag $B = V(K_n)$. In $G[B] = K_n$, each vertex has degree $n-1$, so

$$p_v = \frac{n-1}{n(n-1)} = \frac{1}{n}, \quad H(B) = -n\Big(\tfrac{1}{n}\log_2 \tfrac{1}{n}\Big) = \log_2 n.$$

No other decomposition can lower the maximum bag entropy, and by Theorem **??** we have $\mathrm{tw}_{\mathrm{entropy}}(K_n) \le \log_2(\mathrm{tw}(K_n)+1) = \log_2 n$. Thus equality holds. □

Theorem 4.14 (Zero-Edge Characterization)**.** *For any finite simple graph $G = (V, E)$,*

$$\mathrm{tw}_{\mathrm{entropy}}(G) = 0 \quad \Longleftrightarrow \quad E = \varnothing.$$

Proof. ($\Rightarrow$) If $\mathrm{tw}_{\mathrm{entropy}}(G) = 0$, then there exists an entropy tree decomposition whose maximum bag entropy is 0. By Definition, a bag B has $H(B) = 0$ only if $G[B]$ has no edges. Since every edge of G must lie in some bag, G cannot have any edges, so $E = \varnothing$.
($\Leftarrow$) If $E = \varnothing$, then G is edgeless. Take any tree decomposition—e.g. a single bag $B = V$. Since $G[B]$ has no edges, $H(B) = 0$. Thus $\mathrm{tw}_{\mathrm{entropy}}(G) = 0$. □

Theorem 4.15 (Entropy Tree-width of Forests)**.** *Let F be a forest with at least one edge. Then*

$$\mathrm{tw}_{\mathrm{entropy}}(F) = 1.$$

Proof. Because F is a forest, its classical tree-width $\mathrm{tw}(F) = 1$. By Theorem **??**, $\mathrm{tw}_{\mathrm{entropy}}(F) \le \log_2(\mathrm{tw}(F)+1) = \log_2 2 = 1$.

On the other hand, any entropy tree decomposition of F must cover each edge $\{u,v\}$ in some bag B. Such a bag satisfies $G[B] \cong K_2$, so its entropy is

$$H(B) = -2\left(\tfrac{1}{2}\log_2 \tfrac{1}{2}\right) = 1.$$

Hence the maximum bag entropy in any decomposition is at least 1, implying $\mathrm{tw}_{\mathrm{entropy}}(F) \ge 1$. Combining the two inequalities yields the claim. □

Theorem 4.16 (Disjoint Union Property)**.** *If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs with $V_1 \cap V_2 = \varnothing$, then their disjoint union $G = G_1 \cup G_2$ satisfies*

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;=\; \max\{\mathrm{tw}_{\mathrm{entropy}}(G_1), \mathrm{tw}_{\mathrm{entropy}}(G_2)\}.$$

Proof. First, by Theorem 4.12, each G_i is an induced subgraph of G, so $\mathrm{tw}_{\mathrm{entropy}}(G) \ge \mathrm{tw}_{\mathrm{entropy}}(G_i)$ for $i = 1, 2$. Therefore

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;\ge\; \max\{\mathrm{tw}_{\mathrm{entropy}}(G_1), \mathrm{tw}_{\mathrm{entropy}}(G_2)\}.$$

Conversely, let $(\mathscr{T}_i, \{B_j^{(i)}\})$ be entropy-optimal decompositions of G_i of width $\mathrm{tw}_{\mathrm{entropy}}(G_i)$. Construct a decomposition of G by taking the disjoint union of the trees $\mathscr{T}_1$ and $\mathscr{T}_2$, with their bags unchanged. This yields a valid entropy tree decomposition of G, because no bag needs to cover edges between V_1 and V_2. Its maximum bag entropy is the larger of the two individual widths. Hence

$$\mathrm{tw}_{\mathrm{entropy}}(G) \;\le\; \max\{\mathrm{tw}_{\mathrm{entropy}}(G_1), \mathrm{tw}_{\mathrm{entropy}}(G_2)\}.$$

Combining the two inequalities proves the theorem. □

4.3. Tree-width for Factor graph

A *factor graph* is a type of bipartite graph that represents the factorization of a global function into a product of local functions. Factor graphs are used in various fields, including Markov networks and Bayesian networks, and have been the subject of extensive research[168].

The formal definition is provided below.

Definition 4.17 (Hypergraph)**.** *A* hypergraph *is a pair $H = (V, \mathscr{E})$, where:*

- *V is a finite nonempty set of elements called* vertices*,*
- *$\mathscr{E} \subseteq \mathscr{P}(V) \setminus \{\emptyset\}$ is a finite set of nonempty subsets of V, called* hyperedges*.*

Each hyperedge $e \in \mathscr{E}$ may connect any number of vertices. In particular, if all hyperedges have cardinality two, then H reduces to a standard undirected graph.

Definition 4.18 (Factor graph)**.** *Formally, a factor graph is a hypergraph $G = (V, E, att, lab_V, lab_E)$ together with mappings Ω and F, where:*

- *V is the set of variable nodes.*
- *E is the set of factor nodes (edges).*
- *att is the attachment function, which associates each factor $e \in E$ with a set of variable nodes $v_1, \ldots, v_k \in V$.*
- *lab_V and lab_E are labeling functions that assign labels to variable nodes and factor nodes, respectively.*
- *Ω maps node labels to sets of possible values. For a node $v \in V$, $\Omega(v)$ represents the set of values that v can take, and it is defined as $\Omega(v) = \Omega(lab_V(v))$.*
- *F maps edge labels to functions. For an edge $e \in E$, the corresponding function $F(e)$ is defined as:*

  $$F(e) : \Omega(v_1) \times \cdots \times \Omega(v_k) \to \mathbb{R}_{\ge 0},$$

  where $v_1, \ldots, v_k$ are the variables connected to e.

In a factor graph:

- *A node $v \in V$ together with its domain $\Omega(v)$ is called a variable.*
- *An edge $e \in E$ together with its function $F(e)$ is called a factor.*

A variable node $v \in V$ is typically drawn as a circle, while a factor node $e \in E$ is drawn as a small square. Instead of writing the label $lab_E(e)$, the function $F(e)$ is written next to the factor node as an expression in terms of its neighboring variable nodes.

Example 4.19 (Factor Graph for a Simple Binary Markov Chain)**.** *A Binary Markov Chain is a stochastic process with two states, where the next state depends only on the current state (cf.[169, 170]). We model the joint distribution of three binary random variables X, Y, and Z as a factor graph $G = (V, E, \mathrm{att}, \mathrm{lab}_V, \mathrm{lab}_E)$ with mappings Ω and F defined below.*

- *Variable nodes:*

  $$V = \{X, Y, Z\}.$$

  Each variable has label $\mathrm{lab}_V(v) = \texttt{Binary}$ and domain $\Omega(v) = \{0, 1\}$.
- *Factor nodes:*

  $$E = \{f_1, f_2, f_3\},$$

  with labels $\mathrm{lab}_E(f_1) = \varphi_1$, $\mathrm{lab}_E(f_2) = \varphi_2$, and $\mathrm{lab}_E(f_3) = \varphi_3$.

- *Attachment function* att*:*

$$\mathrm{att}(f_1) = \{X\}, \quad \mathrm{att}(f_2) = \{X, Y\}, \quad \mathrm{att}(f_3) = \{Y, Z\}.$$

- *Factor functions* F*:*

$$F(f_1) : \{0,1\} \longrightarrow \mathbb{R}_{\geq 0}, \qquad F(f_1)(x) = \begin{cases} 0.8, & x = 0, \\ 0.2, & x = 1, \end{cases}$$

$$F(f_2) : \{0,1\}^2 \longrightarrow \mathbb{R}_{\geq 0}, \qquad F(f_2)(x,y) = \begin{cases} 0.9, & x = y, \\ 0.1, & x \neq y, \end{cases}$$

$$F(f_3) : \{0,1\}^2 \longrightarrow \mathbb{R}_{\geq 0}, \qquad F(f_3)(y,z) = \begin{cases} 0.7, & y = z, \\ 0.3, & y \neq z. \end{cases}$$

In this factor graph:

- *Each circle* $v \in V$ *denotes a variable with domain* $\Omega(v)$.
- *Each square* $f \in E$ *denotes a factor computing the nonnegative function* $F(f)$ *on the values of its neighboring variable nodes.*

Hence the joint distribution factorizes as

$$P(X,Y,Z) \;=\; F(f_1)(X) \;\times\; F(f_2)(X,Y) \;\times\; F(f_3)(Y,Z).$$

Tree-width is also utilized in factor graphs [171–173]. In practical applications, further investigations could focus on related parameters such as pathwidth, cyclewidth, and starwidth, as necessary.

4.4. Tree-width in Bond Graphs

Bond graphs are a domain-independent formalism for modeling the transfer and storage of energy in multi-domain physical systems [174–176]. They consist of two kinds of vertices—*element nodes* and *junction nodes*—connected by *bonds* carrying conjugate variables effort e and flow f.

Definition 4.20 (Bond Graph)**.** *[174–176] A* bond graph *is an undirected graph*

$$G = (V, E),$$

where

- $V = V_{\text{elem}} \dot{\cup} V_{\text{junc}}$*, a disjoint union of*
  - Element nodes $V_{\text{elem}} = V_{Se} \dot{\cup} V_{Sf} \dot{\cup} V_R \dot{\cup} V_C \dot{\cup} V_I \dot{\cup} V_{TF} \dot{\cup} V_{GY}$,
  - Junction nodes $V_{\text{junc}} = V_0 \dot{\cup} V_1$,
- $E \subseteq \big\{\{u,v\} : u,v \in V,\ u \neq v\big\}$ *is the set of* bonds*, each representing a single power port connection.*

Each bond $\{u,v\} \in E$ *carries two variables:*

$$e \text{ (effort)}, \quad f \text{ (flow)}, \quad \text{with instantaneous power } P = e\,f.$$

Element nodes denote:

- *Se: effort source (e.g. voltage, force),*
- *Sf: flow source (e.g. current, velocity),*
- *R: resistance (energy dissipation),*
- *C: capacitance (potential energy storage),*
- *I: inertia (kinetic energy storage),*
- *TF: transformer (scaling of effort and flow),*
- *GY: gyrator (cross-domain conversion of effort and flow).*

Junction nodes denote:

- 0*-junction: common effort, flows sum to zero,*
- 1*-junction: common flow, efforts sum to zero.*

Example 4.21 (Bond Graph of a Series RLC Circuit)**.** *An RLC circuit is an electrical circuit consisting of a resistor (R), inductor (L), and capacitor (C), exhibiting oscillatory behavior (cf.[177–179]). Consider the electrical series circuit consisting of a DC voltage source* V_0*, a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C. Its bond graph* $G = (V,E)$ *is defined as follows.*

$$V \;=\; \{\, Se_{V_0},\, R_R,\, I_L,\, C_C,\, J_1 \,\},$$

where

- Se_{V_0} *is an* effort source *imposing constant voltage* $e_{Se} = V_0$.
- R_R *is a* resistor *element with constitutive relation* $e_R = R\,f_R$.
- I_L *is an* inertia *element (inductor) with* $e_I = L\,\frac{df_I}{dt}$.

- C_C *is a* capacitance *element with* $f_C = C\frac{de_C}{dt}$.
- J_1 *is a* 1-junction *enforcing equal flows and summing efforts.*

The set of bonds is

$$E \;=\; \big\{\{Se_{V_0},J_1\},\,\{R_R,J_1\},\,\{I_L,J_1\},\,\{C_C,J_1\}\big\}.$$

Each bond $\{X,J_1\}$ *carries two variables:*

$$e_X,\; f_X, \qquad \text{with instantaneous power } P_X = e_X\, f_X.$$

At the 1*-junction* J_1*, the following conditions hold:*

$$f_{Se} = f_R = f_I = f_C = i(t), \qquad e_{Se} \;-\; e_R \;-\; e_I \;-\; e_C = 0.$$

Hence the circuit equations are

$$i(t) = f_X \quad (\text{common current}), \quad V_0 \;-\; R\,i(t) \;-\; L\frac{di}{dt} \;-\; \frac{1}{C}\int i(t)\,dt = 0.$$

This bond graph compactly captures the energy flow and storage in the series RLC circuit.

Definition 4.22 (Bond Tree Decomposition)**.** *Let* $G = (V,E)$ *be the underlying undirected graph of a bond graph. A* bond tree decomposition *of* G *is a tree decomposition* $\big(\mathscr{T} = (I,F), \{B_i\}_{i\in I}\big)$ *satisfying:*

1. $\bigcup_{i\in I} B_i = V$.
2. *For every bond* $\{u,v\} \in E$*, there exists* $i \in I$ *with* $\{u,v\} \subseteq B_i$.
3. *For any* $i,j,k \in I$*, if node* k *lies on the unique* $\mathscr{T}$*-path between* i *and* j*, then*

$$B_i \cap B_j \;\subseteq\; B_k.$$

Each bag B_i *represents a subsystem of components whose energy interactions are considered jointly.*

Definition 4.23 (Bond Tree-width)**.** *The* width *of a bond tree decomposition* $(\mathscr{T},\{B_i\})$ *is*

$$\text{width}(\mathscr{T},\{B_i\}) = \max_{i\in I}\big(|B_i|-1\big).$$

The bond tree-width *of* G *is the minimum such width over all bond tree decompositions:*

$$\text{tw}_{\text{bond}}(G) = \min_{(\mathscr{T},\{B_i\})}\max_{i\in I}\big(|B_i|-1\big).$$

Example 4.24 (Bond Tree-width of a Series RLC Bond Graph)**.** *Consider the bond graph* $G = (V,E)$ *of the series RLC circuit introduced earlier, whose underlying undirected graph has*

$$V = \{Se,R,I,C,J_1\}, \quad E = \{\{Se,J_1\},\{R,J_1\},\{I,J_1\},\{C,J_1\}\}.$$

Since G *is a tree, its (standard) tree-width is 1. We now exhibit a* bond tree decomposition *of* G *that achieves this width.*

- *Let the index set of bags be* $I = \{a,b,c,d\}$*, and define the tree* $\mathscr{T} = (I,F)$ *by*

$$F = \{(a,b),(b,c),(c,d)\}.$$

- *Assign bags as follows:*

$$B_a = \{Se,J_1\}, \quad B_b = \{R,J_1\}, \quad B_c = \{I,J_1\}, \quad B_d = \{C,J_1\}.$$

We verify the three decomposition properties:

1. $\bigcup_{i\in\{a,b,c,d\}} B_i = \{Se,R,I,C,J_1\} = V$.
2. *Each bond* $\{X,J_1\}$ *with* $X \in \{Se,R,I,C\}$ *is contained in the corresponding bag:* $\{Se,J_1\} \subseteq B_a$*,* $\{R,J_1\} \subseteq B_b$*, etc.*
3. *For any two bags* B_i,B_j*, their intersection is* $\{J_1\}$*. Along the unique path in* $\mathscr{T}$ *between* i *and* j*, every intermediate bag contains* J_1*. Thus* $B_i \cap B_j \subseteq B_k$ *whenever* k *lies on that path.*

Since each bag has size $|B_i| = 2$*, the width of this decomposition is*

$$\max_{i\in\{a,b,c,d\}}(|B_i|-1) = 2-1 = 1.$$

Therefore, the bond tree-width *of* G *is*

$$\text{tw}_{\text{bond}}(G) = 1.$$

Example 4.25 (Bond Tree-width of a Transformer-Coupled Resistive Network)**.** *A resistive network is an electrical circuit composed solely of resistors and sources, used to analyze voltage, current, and power distribution (cf.[180–182]). Consider the bond graph $G = (V,E)$ modeling two resistive circuits coupled by an ideal transformer:*

$$V = \{Se_1, R_1, J_{1a}, TF, J_{1b}, R_2, Se_2\},$$

where

- *Se_1, Se_2 are effort sources imposing fixed voltages e_{Se_1}, e_{Se_2}.*
- *R_1, R_2 are resistors with constitutive relations $e_{R_i} = R_i f_{R_i}$.*
- *J_{1a}, J_{1b} are 1-junctions enforcing equal flow and summing efforts.*
- *TF is an ideal transformer with turns ratio n, satisfying*

$$e_{sec} = n e_{pri}, \quad f_{pri} = n f_{sec}.$$

The bond set is

$$E = \big\{\{Se_1, J_{1a}\}, \{R_1, J_{1a}\}, \{J_{1a}, TF\}, \{TF, J_{1b}\}, \{J_{1b}, R_2\}, \{J_{1b}, Se_2\}\big\}.$$

Since the underlying graph is a tree, its tree-width is 1. We now exhibit a bond tree decomposition of width 1:

- *Index set $I = \{a,b,c,d,e,f\}$ with tree edges $\mathscr{T}$: a–b–c–d–e–f.*
- *Bags:*

$$B_a = \{Se_1, J_{1a}\}, \quad B_b = \{R_1, J_{1a}\}, \quad B_c = \{J_{1a}, TF\},$$
$$B_d = \{TF, J_{1b}\}, \quad B_e = \{J_{1b}, R_2\}, \quad B_f = \{J_{1b}, Se_2\}.$$

Verification:

1. *$\bigcup_{i\in I} B_i = V$.*
2. *Each bond $\{X,Y\} \in E$ is contained in exactly one bag B_i.*
3. *Any two bags intersect in a single junction (J_{1a} or J_{1b}), and along the unique path their intersection appears in every intermediate bag.*

All bags have size 2, so the width is $\max_i(|B_i| - 1) = 1$. Hence

$$\mathrm{tw}_{\mathrm{bond}}(G) = 1.$$

Theorem 4.26. *For any bond graph G, its bond tree-width coincides with the standard tree-width of the underlying graph:*

$$\mathrm{tw}_{\mathrm{bond}}(G) = \mathrm{tw}(G).$$

Proof. Any tree decomposition of the underlying graph G satisfies the bond tree decomposition properties by definition. Conversely, any bond tree decomposition is a valid tree decomposition of G. Hence the minimum widths agree. □

Theorem 4.27 (Equality with Standard Tree-width)**.** *For any bond graph G,*

$$\mathrm{tw}_{\mathrm{bond}}(G) \;=\; \mathrm{tw}(G),$$

where $\mathrm{tw}(G)$ is the usual tree-width of G.

Proof. By definition, every bond tree decomposition of G is in particular a tree decomposition of the underlying graph, so $\mathrm{tw}_{\mathrm{bond}}(G) \geq \mathrm{tw}(G)$. Conversely, any tree decomposition of G satisfies the bond tree decomposition axioms, hence $\mathrm{tw}(G) \geq \mathrm{tw}_{\mathrm{bond}}(G)$. The two inequalities give the claimed equality. □

Theorem 4.28 (Forest Characterization)**.** *A bond graph G is a forest (i.e. has no cycles) if and only if $\mathrm{tw}_{\mathrm{bond}}(G) = 1$.*

Proof. If G is a forest, it is well-known that its standard tree-width $\mathrm{tw}(G) = 1$. By Theorem 1, $\mathrm{tw}_{\mathrm{bond}}(G) = \mathrm{tw}(G) = 1$.
Conversely, if $\mathrm{tw}_{\mathrm{bond}}(G) = 1$, then $\mathrm{tw}(G) = 1$, which implies G is a forest (since any cycle forces tree-width at least 2). Thus G contains no cycles. □

Theorem 4.29 (Subgraph Monotonicity)**.** *If $H = (V',E')$ is any subgraph of $G = (V,E)$ (i.e. $V' \subseteq V$, $E' \subseteq E \cap \binom{V'}{2}$), then*

$$\mathrm{tw}_{\mathrm{bond}}(H) \;\leq\; \mathrm{tw}_{\mathrm{bond}}(G).$$

Proof. Let $(\mathscr{T}, \{B_i\}_{i\in I})$ be a bond tree decomposition of G achieving width $\mathrm{tw}_{\mathrm{bond}}(G)$. Define $B'_i = B_i \cap V'$. Dropping any bag with $B'_i = \emptyset$ and retaining the same tree structure yields a valid bond tree decomposition of H. Since $|B'_i| \leq |B_i|$, its width is at most $\mathrm{tw}_{\mathrm{bond}}(G)$, establishing the inequality. □

Theorem 4.30 (Articulation Gluing Lemma)**.** *Suppose G can be partitioned into two bond subgraphs G_1 and G_2 that meet exactly at a single vertex v; formally*

$$V(G_1) \cap V(G_2) = \{v\}, \quad E(G_1) \cup E(G_2) = E(G), \quad E(G_1) \cap E(G_2) = \emptyset.$$

Then

$$\mathrm{tw}_{\mathrm{bond}}(G) \;=\; \max\big\{\mathrm{tw}_{\mathrm{bond}}(G_1),\, \mathrm{tw}_{\mathrm{bond}}(G_2)\big\}.$$

Proof. Let $w_i = \mathrm{tw}_{\mathrm{bond}}(G_i)$ for $i = 1,2$. By Subgraph Monotonicity, $\mathrm{tw}_{\mathrm{bond}}(G) \geq w_i$, so $\mathrm{tw}_{\mathrm{bond}}(G) \geq \max\{w_1, w_2\}$.

For the reverse inequality, take bond tree decompositions of G_1 and G_2 of widths w_1 and w_2. Each decomposition contains at least one bag that includes the shared vertex v. Connect these two decomposition trees by adding an edge between a v-bag in the first and a v-bag in the second. The resulting structure respects all decomposition properties for G and has width $\max\{w_1, w_2\}$. Hence $\mathrm{tw}_{\mathrm{bond}}(G) \leq \max\{w_1, w_2\}$, completing the proof. □

5. Conclusion and Future tasks

In this paper, we proposed several practical graph width parameters and conducted a review of the applications of existing graph width parameters. Since there are likely many more potential applications for graph width parameters beyond those discussed, further investigation will be pursued. Additionally, while we have examined several width parameters, we plan to validate their effectiveness through mathematical proofs and computational experiments.

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The paper has been solely authored by the corresponding author at this stage.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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