Semi-FTvN systems

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National Taiwan Normal University - Taipei
June 20, 2025

The material presented here is partially based on work with Dr. David Sossa, Universidad de O'Higgins, Rancagua, Chile.

The main objective of this talk is to introduce semi-FTvN systems and describe some examples and results.

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A **semi-FTvN system** is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces, and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the following *'sharpened Cauchy-Schwarz inequality'*:

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq ||x||\,||y|| \quad (\forall\ x, y \in \mathcal{V}).$$

The above condition is equivalent to: For all $x, y \in \mathcal{V}$,

- $||\lambda(x)|| = ||x||$ (norm preserving property) and
- $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$ (inner product expanding property).

It is known that above λ is Lipschitz-continuous.

FTvN $\equiv$ Fan-Theobald-von Neumann

Two simple examples

Example 1 $(\mathcal{V}, \mathcal{R}, \lambda)$, where $\mathcal{V}$ is a real inner product space and $\lambda(x) := ||x||$.

Example 2 $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where for $x \in \mathcal{R}^n$, $\lambda(x) := x^{\downarrow}$ is the decreasing rearrangement of entries of x.

Hardy-Littlewood-Polya: For $x, y \in \mathcal{R}^n$,

(i) $||x|| = ||x^{\downarrow}||$,

(ii) $\langle x, y \rangle \leq \langle x^{\downarrow}, y^{\downarrow} \rangle$, and

(iii) Equality holds in (ii) if and only if $x = P(x^{\downarrow})$ and $y = P(y^{\downarrow})$ for some permutation matrix P.

FTvN systems

A **FTvN system** is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces, and $\lambda : \mathcal{V} \to \mathcal{W}$ is a map satisfying the conditions

(1) $||\lambda(x)|| = ||x||$ for all $x \in \mathcal{V}$,

(2) $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$ for all $x, y \in \mathcal{V}$, and

(3) $\max\{\langle c, x\rangle : x \in [a]\} = \langle \lambda(c), \lambda(a)\rangle$ for all $c, a \in \mathcal{V}$.

Here, $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$ is called the λ-orbit of a.

- It is known that $(3) \Rightarrow (1) + (2)$.
- *Every FTvN system is a semi-FTvN system.*

FTvN $\equiv$ Fan-Theobald-von Neumann

Why these names?

Let $\mathcal{S}^n$ = Set of all $n \times n$ real symmetric matrices with inner product $\langle X, Y \rangle = tr(XY)$.

- For $X \in \mathcal{S}^n$, $\lambda(X)$ = (decreasing) vector of eigenvalues of X.

  Note: $X = U\, Diag(\lambda(X))\, U^T$ for some orthogonal matrix U.

- For all $X, Y \in \mathcal{S}^n$,
  - (i) $||X|| = ||\lambda(X)||$,
  - (ii) (**Fan**, 1949) $\langle X, Y \rangle \leq \langle \lambda(X), \lambda(Y) \rangle$,
  - (iii) (**Theobald**, 1975) Equality holds in (ii) if and only if for some orthogonal U,

$$X = U\, Diag(\lambda(X))\, U^T \quad \text{and} \quad Y = U\, Diag(\lambda(Y))\, U^T.$$

Considering the space of all $n \times n$ complex matrices with $\lambda(X)$ denoting the (decreasing) vector of singular values, **von Neumann** proved analogous results.

Examples of FTvN systems include

- $(\mathcal{V}, \mathcal{R}, \lambda)$, where $\mathcal{V}$ is a real IPS and $\lambda(x) = ||x||$,
- $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, $\lambda(x) = x^{\downarrow}$,
- **Euclidean Jordan algebras**,
- Systems induced by certain hyperbolic polynomials,
- Normal decomposition systems (Eaton triples).

FTvN systems were originally introduced to transform certain optimization problems over $\mathcal{V}$ to similar problems over $\mathcal{W}$. Semi-FTvN systems turn out to be more flexible than FTvN systems, especially when dealing with subspaces and compositions.

A large class of semi-FTvN systems comes from hyperbolic polynomials.

Hyperbolic polynomials

On a finite dimensional real vector space $\mathcal{V}$, let p be a homogeneous polynomial of degree n (a natural number) and $0 \neq e \in \mathcal{V}$. We say that p is **hyperbolic in the direction of** e if, for for each fixed $x \in \mathcal{V}$, the univariate polynomial $t \mapsto p(x - te)$ has only real roots. Given such a polynomial p and $x \in \mathcal{V}$, we look at the roots of $p(x - te) = 0$ and rearrange them in the decreasing order to get the vector $\lambda(x)$ in $\mathcal{R}^n$. We say that p is **complete** if $\lambda(x) = 0 \Rightarrow x = 0$. Assuming this, Bauschke et al., (2001) have shown that

$$\langle x, y \rangle := \frac{1}{4}\Big[||\lambda(x+y)||^2 - ||\lambda(x-y)||^2\Big]$$

defines an inner product on $\mathcal{V}$ with

$$\langle x, y \rangle \leq \langle \lambda(x), \lambda(y) \rangle \leq ||x||\,||y|| \quad \forall\, x, y \in \mathcal{V}.$$

Thus,

Every complete hyperbolic polynomial induces a semi-FTvN system.

Examples:

(1) On $\mathcal{R}^n$, let $p(x) = x_1 x_2 \cdots x_n$. Then, $\lambda(x) = x^{\downarrow}$.

(2) On $\mathcal{S}^n$, let $p(X) := \det(X)$. Then, $\lambda(X)$ is the eigenvector of X with $\langle X, Y \rangle = trace(XY)$.

(3) On a Euclidean Jordan algebra $\mathcal{V}$ with Jordan product $x \circ y$, let $p(x) = \det(x)$. Then, $\lambda(x)$ is the vector of eigenvalues of x with $\langle x, y \rangle = trace(x \circ y)$.

Some new concepts

From now on, we fix a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$. Recall that $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces with $\lambda : \mathcal{V} \to \mathcal{W}$ satisfying the condition

$$\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle \leq ||x||\,||y|| \quad \forall\, x, y \in \mathcal{V}.$$

For simplicity, we assume that $\mathcal{V}$ is **finite-dimensional**. Let

$\mathrm{L}(\mathcal{V})$ denote the set of all linear transformations on $\mathcal{V}$,

$\mathrm{GL}(\mathcal{V})$ denote the set of all invertible linear transformations in $\mathcal{V}$.

Majorization in a semi-FTvN system

Recall: In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, for any $a \in \mathcal{V}$, the λ-orbit of a is $[a] := \{x \in \mathcal{V} : \lambda(x) = \lambda(a)\}$.
As λ is Lipschitz-continuous, $[a]$ is closed and bounded (recall $\mathcal{V}$ is finite-dimensional). By Minkowski's Theorem, its convex hull $\operatorname{conv}[a]$ is also compact.

Definition

Given $x, y \in \mathcal{V}$, we say that x is **majorized** by y if $x \in \operatorname{conv}[y]$.
Notation: $x \prec y$.

Example: In the setting of $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$, this coincides with the classical definition of majorization (thanks to Hardy-Littlewood-Polya).

Example: In the setting of a Euclidean Jordan algebra, $x \prec y$ if and only if $\lambda(x) \prec \lambda(y)$ in $\mathcal{R}^n$. In particular, this yields a result of T. Ando: In the setting of $\mathcal{H}^n$, $\lambda(A) \prec \lambda(B)$ if and only if A is in the convex hull of matrices of the form UBU^*, where U is a unitary matrix.

Example: In the setting of a semi-FTvN system coming from a complete hyperbolic polynomial, it is known that $x \prec y \Rightarrow \lambda(x) \prec \lambda(y)$. The reverse implication holds when the polynomial is also 'isometric'. The general situation is not known.

The automorphism group of a semi-FTvN system

Definition

$$\mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda) := \{A \in \mathrm{GL}(\mathcal{V}) : \lambda(Ax) = \lambda(x),\ \forall\ x \in \mathcal{V}\}.$$

As $||\lambda(x)|| = ||x||$, every A in this set is norm preserving (i.e., orthogonal). In particular, this set is closed in $\mathrm{GL}(\mathcal{V})$. Thus,

$$\mathcal{G} := \mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$$

is a **matrix Lie group** with the corresponding **Lie algebra**

$$\mathrm{Lie}(\mathcal{G}) := \{D \in \mathrm{L}(\mathcal{V}) : exp(tD) \in \mathcal{G} \text{ for all } t \in \mathcal{R}\}.$$

Example: In the setting of a Euclidean Jordan algebra, $\mathcal{G}$ is the group of all Jordan algebra automorphisms and $\mathrm{Lie}(\mathcal{G})$ is the set of all derivations.

Commutativity in a semi-FTvN system

Definition

Let $x, y \in \mathcal{V}$. We say that x and y **commute** if $\langle Dx, y\rangle = 0$ for all $D \in \mathrm{Lie}(\mathcal{G})$.

Example: In the setting of the semi-FTvN system $(\mathcal{R}^n, \mathcal{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$, $\mathcal{G}$ is the permutation group and its Lie algebra is $\{0\}$. Thus, in this system, any two elements commute.

Example: In the setting of $(\mathcal{S}^n, \mathcal{R}^n, \lambda)$, above commutativity is the same as the usual commutativity of two matrices.

Example: In the setting of a Euclidean Jordan algebra, above commutativity is the same as operator commutativity.

Strong commutativity in a semi-FTvN system

Definition

Let $x, y \in \mathcal{V}$. We say that x and y **strongly commute** if

$$\langle x, y \rangle = \langle \lambda(x), \lambda(y) \rangle.$$

Theorem

Strong commutativity $\Rightarrow$ commutativity.

Example: In $(\mathcal{R}^2, \mathcal{R}^2, \lambda)$ with $\lambda(x) = x^{\downarrow}$, vectors $x = (-1, 2)$ and $y = (-2, 3)$ strongly commute. However, $u = (1, 0)$ and $v = (0, 1)$ commute, but not strongly.

Example: In a Euclidean Jordan algebra, strong commutativity is the same as strong operator commutativity.

A commutation principle for $\mathcal{S}^n$

- $\mathcal{S}^n$ = All $n \times n$ real symmetric matrices, $\langle X, Y \rangle = tr(XY)$.
- For $X \in \mathcal{S}^n$, $\lambda(X)$ = (decreasing) vector of eigenvalues of X.
- $E \subseteq \mathcal{S}^n$ is a **spectral set** if $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$.
- $F : \mathcal{S}^n \to \mathcal{R}$ is a **spectral function** if $F = f \circ \lambda$ for some $f : \mathcal{R}^n \to \mathcal{R}$.

Theorem

(Iusem and Seeger - 2007)
In $\mathcal{S}^n$, let E be a spectral set, F be a spectral function, and $C \in \mathcal{S}^n$. If A is a local optimizer of

$$\underset{X \in E}{max/min} \quad \langle C, X \rangle + F(X),$$

then A and C commute, that is, $AC = CA$.

Theorem of Ramírez, Seeger, and Sossa (2013)

- $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ = Euclidean Jordan algebra of rank n.
- For $x \in \mathcal{V}$, $\lambda(x)$ = (decreasing) vector of eigenvalues of x.
- $E \subseteq \mathcal{V}$ is a **spectral set** if $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{R}^n$.
- $F : \mathcal{V} \to \mathcal{R}$ is a **spectral function** if $F = f \circ \lambda$ for some $f : \mathcal{R}^n \to \mathcal{R}$.

Theorem

In $\mathcal{V}$, let E be a spectral set, F be a spectral function, and $\Phi : \mathcal{V} \to \mathcal{R}$ be Fréchet differentiable. If a is a local optimizer of

$$\underset{x \in E}{max/min} \quad \Phi(x) + F(x),$$

then a and $b := \Phi'(a)$ operator commute, that is, $L_a L_b = L_b L_a$.

Theorem of Gowda and Jeong (2017)

- $\mathcal{V}$ = Euclidean Jordan algebra of rank n.
- $\mathrm{Aut}(\mathcal{V})$ = **Automorphism group** of $\mathcal{V}$.
- $E \subseteq \mathcal{V}$ is a **weakly spectral set** if $A(E) \subseteq E$ for all $A \in \mathrm{Aut}(\mathcal{V})$.
- $F : \mathcal{V} \to \mathcal{R}$ is a **weakly spectral function** if $F(Ax) = F(x)$ for all $A \in \mathrm{Aut}(\mathcal{V})$.

Theorem

In $\mathcal{V}$, let E be a weakly spectral set, F be a weakly spectral function, and $\Phi : \mathcal{V} \to \mathcal{R}$ be Fréchet differentiable. If a is a local optimizer of

$$\max_{x \in E} / \min \quad \Phi(x) + F(x),$$

then a and $b := \Phi'(a)$ operator commute, that is, $L_a L_b = L_b L_a$.

A commutation principle for semi-FTvN systems

Theorem

Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system and $\mathcal{G} = \mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$. Suppose

- *$E \subseteq \mathcal{V}$ is $\mathcal{G}$-invariant, i.e., $A(E) \subseteq E$ for all $A \in \mathcal{G}$,*
- *$F : \mathcal{V} \to \mathcal{R}$ is $\mathcal{G}$-invariant, i.e., $F(Ax) = F(x)$ for all $A \in \mathcal{G}$ and $x \in \mathcal{V}$, and*
- *$\Phi : \mathcal{V} \to \mathcal{R}$ is Fréchet differentiable.*

If a is a local optimizer of

$$\max_{x \in E} / \min \quad \Phi(x) + F(x),$$

then a commutes with $b := \Phi'(a)$ in the semi-FTvN system.

Commutativity in a variational inequality problem

In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, let $E \subseteq \mathcal{V}$ and $h : \mathcal{V} \to \mathcal{R}$. Then, the variational inequality problem $VI(h, E)$ is to find $a \in E$ such that

$$\langle h(a), x - a \rangle \geq 0 \text{ for all } x \in E.$$

Corollary

Let E be $\mathcal{G}$-invariant. If a solves the variational inequality problem $\mathrm{VI}(h, E)$, then a and $h(a)$ commute.

Commutativity in a linear complementarity problem

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is a semi-FTvN system, K be a closed convex cone in $\mathcal{V}$ that is $\mathcal{G}$-invaraint. (For example, this could be the hyperbolicity cone corresponding to a complete hyperbolic polynomial or a symmetric cone in a Euclidean Jordan algebra.) Let $h : K \to \mathcal{V}$. Consider the *complementarity problem* $\mathrm{CP}(h, K)$, which is to find

$$a \in K \text{ such that } h(a) \in K^* \text{ and } \langle h(a), a \rangle = 0,$$

where $K^* := \{y \in \mathcal{V} : \langle y, x \rangle \geq 0, \, \forall \, x \in K\}$ is the dual of K. If a is such a solution, by the previous corollary specialized to $E = K$, we see that a commutes with $h(a)$. In particular, in the setting of a Euclidean Jordan algebra with its symmetric cone K, we (recover) the operator commutativity of a and $h(a)$.

Strong commutativity in FTvN systems

Suppose $(\mathcal{V}, \mathcal{W}, \lambda)$ is an FTvN system, where $\mathcal{V}$ is finite dimensional. Then, for any $c \in \mathcal{V}$,

$$\max_{[a]} \langle c, x \rangle = \langle \lambda(c), \lambda(a) \rangle.$$

Suppose the maximum above is attained at $u \in [a]$. Then, c strongly commutes with u. Also, in this setting, when E is a $\mathcal{G}$-invariant set and $h : E \to \mathcal{V}$, if a solves $\mathrm{VI}(h, E)$, then a and $-h(a)$ strongly commute.

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