Prime Blocks and Additive Divisor Obstructions in Cyclotomic Littlewood Polynomials

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Abstract

The sign changes of a cyclotomic Littlewood polynomial depend only on the gcd of the index with its length. Divisor inversion gives canonical binary coordinates, in which standard prime-block products are exactly the divisibility chains. A local rigidity identity forces an additional correction in an explicit divisor window whenever a first block fails. Together with the exponential-equation theorem of Scott and Styer, this proves factorization at every length $2^t3^a7^b$, with all exponents unrestricted. Further algebraic arguments cover every length 2^tpq, at most three active coordinates other than 1, and a chain below two sufficiently separated highest coordinates. Reciprocal interpolation inserts a prime block whenever the prime exceeds its cofactor, with a sharper test based on signed period and rational degree. For at most four active coordinates, the arguments classify every comparable initial pair, every three-member highest layer, and every length not divisible by eight. At arbitrary dyadic height, they also classify a smallest highest coordinate with two odd lower coordinates. Midpoint representation parity supplies coprimality; runs of additive divisors force the separation needed for period extraction. Fourth-order traces settle every remaining ordering at length $4M$, with M odd. An internal block change shortens the necessary correction window. The polynomial proofs use no coefficient enumeration. The unrestricted Borwein–Choi conjecture remains unresolved.

1 The factorization problem

A *Littlewood polynomial of length N* is a polynomial

$$P(x) = \sum_{k=0}^{N-1} a_k x^k, \qquad a_k \in \{-1, 1\}.$$

Here *cyclotomic* means that every root is a root of unity; an overall sign and repeated cyclotomic factors are allowed. The irreducible mth cyclotomic polynomial is denoted by Φ_m. Write

$$G_m(x) = 1 + x + \cdots + x^{m-1}.$$

The elementary identity $G_{ab}(x) = G_a(x)G_b(x^a)$ illustrates the problem: a rigid alphabet is preserved when short coefficient blocks are placed consecutively. The question is whether every cyclotomic word is built in this way. A factorization would explain the entire coefficient sequence through shorter sign patterns, rather than only describe its roots.

Conjecture 1.1 (Borwein–Choi [1])**.** Every cyclotomic Littlewood polynomial of length N has a representation

$$P(x) = \sigma \prod_{j=1}^{s} \Phi_{p_j}(\varepsilon_j x^{p_1 \cdots p_{j-1}}), \qquad N = p_1 \cdots p_s, \tag{1.1}$$

where the p_j are primes, repetitions are permitted, and $\sigma, \varepsilon_j \in \{-1, 1\}$.

Call a polynomial admitting (1.1) *standard*. Such a product is cyclotomic and Littlewood: successive factors occupy distinct digits in a mixed-radix expansion, so each coefficient is a product of signs with exactly one contributing term. The difficult direction is to force a block from the root condition and then justify the same extraction for its quotient.

Borwein and Choi proved the conjecture for odd N and for powers of two [1]. Akhtari and Choi proved it for all $N = 2^t p^a$, with p odd prime, and for polynomials without repeated roots [2]. Hong and Cao credit Hong and Li with the case $N = 2pq$ for distinct odd primes [3, p. 66]. Sun also announced a large-prime subfamily of lengths $2^t pq$ in a 2008 abstract for the 2009 Joint Mathematics Meetings [7]. The case $2pq$ therefore has an established classification; the argument below gives an alternative proof. The divisor obstruction is [2, Lemma 3.1].

The main sufficient condition is especially simple.

Theorem 1.2 (Arithmetic criterion)**.** *Suppose that N has no coprime integers $1 < u < v$ satisfying*

$$uv(u + v) \mid N. \tag{1.2}$$

Then every cyclotomic Littlewood polynomial of length N is standard.

The three factors in (1.2) are pairwise coprime. Thus a failure requires three disjoint sets of prime divisors to support an additive relation. The criterion is sufficient, not necessary: for example, 30 contains $2 \cdot 3 \cdot 5$, but the conjecture holds at length 30.

Corollary 1.3. *Let $N = 2^t p^\alpha q^\beta$, where p, q are distinct odd primes. If none of*

$$2^u + p^v = q^w, \qquad 2^u + q^w = p^v, \qquad p^v + q^w = 2^u \tag{1.3}$$

holds with $1 \le u \le t$, $1 \le v \le \alpha$, and $1 \le w \le \beta$, then every cyclotomic Littlewood polynomial of length N is standard. In particular, if neither $p + q$ nor $|p - q|$ is a power of two, this conclusion holds for $N = 2^t pq$ for every $t \ge 0$.

For example, every length $2^t \cdot 7 \cdot 13$ is covered. The exceptional equations in (1.3) identify possible obstructions; their existence does not produce a counterexample.

To state the other results precisely, we introduce the coordinates that will organize the coefficient word. Normalize $a_0 = 1$ and put

$$\delta_k = \frac{1 - a_{k-1}a_k}{2}, \qquad \alpha_d \equiv \sum_{e \mid d} \mu(d/e)\delta_e \pmod 2, \quad \alpha_d \in \{0, 1\}, \tag{1.4}$$

where $1 \le k < N$ and d is a proper positive divisor of N. Here μ is the Möbius function. A divisor d is *active* when $\alpha_d = 1$; their set is denoted by $\mathcal{A}(P)$. Section 3 proves that these data recover every coefficient: an active d contributes a sign reversal at every multiple of d. The coordinate 1 simply records substitution $x \mapsto -x$.

Theorem 1.4 (Principal factorization results)**.** *Every cyclotomic Littlewood polynomial is standard in each of the following cases:*

(i) *its length is $2^t pq$, with p, q distinct odd primes;*
(ii) *its length is $2^t 3^a 7^b$, with $t, a, b \geq 0$;*
(iii) *it has at most three active divisor coordinates other than 1;*
(iv) *its length is $2M$, with M odd, and at most one even divisor coordinate is active;*
(v) *it has at most four active coordinates other than 1, and the first normalized active coordinate divides the second;*
(vi) *it has at most four active coordinates other than 1, its length has dyadic valuation at most two, its highest layer has two members, and the remaining coordinates form a divisor chain;*
(vii) *it has at most four active coordinates other than 1, and its highest layer has three members;*
(viii) *its length is $2M$, with M odd, and it has at most four active coordinates other than 1;*
(ix) *its length is $4M$, with M odd, and it has at most four active coordinates other than 1;*
(x) *it has at most four active coordinates other than 1, its highest layer has two members including its smallest normalized active coordinate, and its remaining coordinates are odd.*

For a length Aq^s, where $A, s \geq 1$ and $q > A$ is an odd prime, there is a factorization

$$P(x) = R(x)\Phi_q(\delta x^h)W(x^{qh}), \qquad h \mid A,$$

after normalization at zero, with R, W cyclotomic Littlewood polynomials of lengths h and $(A/h)q^{s-1}$, respectively, and $\delta \in \{-1, 1\}$.

These assertions are proved in Theorems 11.1, 12.1, 15.3, 13.5, 17.10, 18.4, 20.3, 21.3, 24.18, 22.5, and 26.1, respectively. Theorem 15.2 gives a further criterion with arbitrarily many coordinates: a chain below two separated highest coordinates yields two explicit outer blocks and a standard core.

The four-coordinate frontier admits further analytical classifications. Theorem 17.10 covers every initial pair a, b with $a \mid b$, extending the doubled-scale and even-ratio arguments of Corollary 16.6 and Theorem 17.5. Theorem 18.4 covers lengths $2M$ and $4M$, with M odd, when the two remaining coordinates below the highest dyadic layer form a divisor chain. No separation assumption is needed in this class. Theorem 20.3 classifies the entire three-member highest layer, and Theorem 21.3 classifies all supports of at most four coordinates at twice odd length. Theorem 24.18 classifies every support of at most four coordinates at length $4M$. Theorem 22.5 treats a smallest highest coordinate with two odd lower coordinates at every dyadic height. With one odd lower coordinate, Theorem 22.6 gives a corresponding coprimality criterion, while Theorem 23.2 covers a comparable highest pair both divisible by that odd coordinate. The proofs compare short coefficient prefixes with forced traces, additive divisors, and periods of residue subsequences. The adaptive window in Theorem 10.1 instead uses the first internal change of the initial blocks to force an earlier correction.

There is one common foundation and three factorization arguments. The multiplicity budgets give trace identities, and a logarithm modulo four transfers those identities to the sign changes. Divisor inversion then gives the coordinates above. The canonical criterion is that P is standard exactly when $\mathcal{A}(P)$ is a divisibility chain (Theorem 5.1). This replaces a choice among many prime factorizations by a property of a unique set of divisors.

The first route is local: the earliest failure of a constant block forces an additive triple of divisors. Two interacting corrections violate the sign alphabet, so a further correction must occur in an explicit divisor window. For support $\{2, 3, 7\}$, a complete list of primitive additive triples reduces the problem to eleven initial pairs; a common logarithmic identity excludes them. The arithmetic input is Scott–Styer [8, Theorem 8], whose proof is external to this article. The second route uses root squaring: the highest dyadic layer determines a cyclotomic divisor, and a period argument converts that root information into consecutive coefficient blocks. The third route isolates a large odd prime: reciprocity upgrades a congruence modulo four to exact periodicity, after which a prime

block can be inserted between two shorter factors. The local argument is in Sections 6–12, the norm argument in Sections 13–15, and prime insertion in Section 26.

The accompanying *Analytical Extensions and Exact Certificates* proves standardness at lengths $2^t p^a q$, with p, q distinct odd primes and $a \geq 1$, in the range $3 \cdot 2^t < 2p(p-1)$ or $t \leq 2$, the full family $2^t 3^a 5^b$, and the bound of 47 changes or nonchanges. Its references carry the prefix S. The three-prime arithmetic classification there uses Scott's algebraic theorem [4]. It also contains independent proofs, finite certificates, and examples delimiting stronger assertions. None of the proofs in this article depends on a finite search. Repeated roots are allowed throughout.

2 From signs to traces

Throughout the article $v_2(k)$ denotes the exponent of 2 in a positive integer k, and φ is Euler's totient function. Multiplication by -1 and substitution $x \mapsto -x$ preserve both cyclotomicity and standardness. For $N > 1$ we may therefore normalize

$$a_0 = a_1 = 1. \tag{2.1}$$

If $P \neq G_N$, let $i \geq 2$ be its first negative coefficient. The boundary polynomial

$$D(x) = (1-x)P(x) = \sum_{k=0}^{N} b_k x^k \tag{2.2}$$

records sign changes, rather than signs:

$$b_0 = 1, \quad b_N = -a_{N-1}, \quad b_k = a_k - a_{k-1} \in \{0, \pm 2\} \quad (0 < k < N).$$

In particular $b_1 = \cdots = b_{i-1} = 0$ and $b_i = -2$.

The following factor budgets are due to Borwein and Choi. We include their short reduction modulo two because it is the arithmetic input to every subsequent trace assertion.

Lemma 2.1 (Conductor and multiplicity budgets)**.** *Let P be cyclotomic Littlewood of length $N = 2^t M$, with M odd. Every cyclotomic factor of P has index dividing $2N$. If e_m is the multiplicity of Φ_m in P, then for every $d \mid M$,*

$$e_d + e_{2d} + 2e_{4d} + \cdots + 2^t e_{2^{t+1}d} = \begin{cases} 2^t - 1, & d = 1, \\ 2^t, & d > 1. \end{cases} \tag{2.3}$$

For $D = (1-x)P$, the corresponding multiplicities E_m satisfy

$$\sum_{r=0}^{t+1} \varphi(2^r) E_{2^r d} = 2^t \qquad (d \mid M). \tag{2.4}$$

Proof. Over $\mathbb{F}_2$, $P = G_N = (x^M - 1)^{2^t}/(x-1)$. For odd d, the reduction of Φ_d has as roots precisely the elements of order d in an algebraic closure of $\mathbb{F}_2$. These roots are simple, and different odd indices have disjoint root sets. Moreover,

$$\overline{\Phi_{2^r d}} = \overline{\Phi_d}^{\,\varphi(2^r)} \qquad (r \geq 0),$$

with the weights $1, 1, 2, 4, \ldots$. Thus an occurring odd part must divide M. Its multiplicity cannot exceed the multiplicity in G_N, so an occurring dyadic level has $r \leq t+1$. Comparison of the multiplicities of these disjoint root sets proves (2.3). Multiplication by $1 - x$ adds one Φ_1 factor and gives (2.4). □

Let S_k be the sum of the kth powers of all N roots of D, with multiplicity. A cyclotomic root multiset is invariant under inversion, so the constant-term normalization $D(0) = 1$ gives the formal identity

$$-\frac{xD'(x)}{D(x)} = \sum_{k\geq 1} S_k x^k. \tag{2.5}$$

Equating coefficients yields Newton's identities in the useful form

$$S_k + b_1 S_{k-1} + \cdots + b_{k-1} S_1 + k b_k = 0 \qquad (1 \leq k \leq N). \tag{2.6}$$

Hence $S_1 = \cdots = S_{i-1} = 0$ and $S_i = 2i$.

For a positive integer m, put

$$c_m(k) = \sum_{\substack{1\leq a\leq m \\ (a,m)=1}} e^{2\pi i a k/m}.$$

The Ramanujan sum $c_m(k)$ depends only on $\gcd(m,k)$, is multiplicative in m, and satisfies $\sum_{d|M} c_d(k) = M$ if $M \mid k$, and zero otherwise. These facts follow either by grouping all Mth roots according to their exact orders, or from $c_m(k) = \sum_{d|(m,k)} d\mu(m/d)$.

Lemma 2.2 (Trace reduction, Akhtari–Choi)**.** *For $1 \leq k < N$,*

$$2^{t+1} \nmid k \quad \Longrightarrow \quad S_k = S_{\gcd(N,k)}, \tag{2.7}$$

$$2^{t+1} \mid k \quad \Longrightarrow \quad S_k = 0. \tag{2.8}$$

In particular, the first negative coefficient index i divides N.

Proof. The factorization of D gives

$$S_k = \sum_{d|M} \sum_{r=0}^{t+1} E_{2^r d} c_{2^r d}(k).$$

If $v_2(k) \leq t$, replacing k by $\gcd(N,k)$ preserves its gcd with every $2^r d \mid 2N$, proving (2.7). If $2^{t+1} \mid k$, multiplicativity and (2.4) instead give

$$S_k = \sum_{d|M} c_d(k) \sum_{r=0}^{t+1} \varphi(2^r) E_{2^r d} = 2^t \sum_{d|M} c_d(k) = 0.$$

Indeed $M \nmid k$, since otherwise $N \mid k$, contrary to $0 < k < N$. Finally $S_i = 2i \neq 0$. The second alternative is impossible, and the first would give a smaller nonzero trace unless $\gcd(N,i) = i$. □

3 Sign changes on the divisor lattice

The trace reduction has a global consequence for the coefficient word. In this section only $a_0 = 1$ is assumed; the linear coefficient may have either sign. Recall from (1.4) that $\delta_k = 1$ precisely when the word changes sign at k. Trace reduction by itself concerns weighted sums of roots. The useful observation is that modulo four a boundary coefficient remembers only whether a sign change occurs: both 2 and -2 give the same residue. The logarithm transfers the trace identities to this binary information.

Theorem 3.1 (Gcd invariance of sign changes)**.** *For every cyclotomic Littlewood polynomial of length N,*

$$\delta_k = \delta_{\gcd(N,k)} \qquad (1 \le k < N). \tag{3.1}$$

Equivalently, $b_k \equiv b_{\gcd(N,k)} \pmod 4$.

Proof. Work modulo x^N, writing $D = 1 + 2F$. In the ring $\mathbb{Z}_{(2)}$ of rational numbers with odd denominator,

$$\frac{1}{2}\log(1+2F) \equiv F - F^2 \pmod 2.$$

Indeed $2^{r-1}/r \in 2\mathbb{Z}_{(2)}$ for $r \ge 3$. Since $F(x)^2 \equiv F(x^2) \pmod 2$ and $[x^k]F = b_k/2 \equiv \delta_k \pmod 2$, the numbers $\ell_k = -S_k/(2k)$ lie in $\mathbb{Z}_{(2)}$ and satisfy

$$\ell_k \equiv \delta_k + \mathbf{1}_{2|k}\delta_{k/2} \pmod 2. \tag{3.2}$$

Write $N = 2^t M$ with M odd and induct on k. If $v_2(k) \le t$, put $d = \gcd(N,k)$. Trace reduction gives $S_k = S_d$, while k/d is odd; hence $\ell_k \equiv \ell_d \pmod 2$. For odd k, equation (3.2) immediately gives $\delta_k = \delta_d$. For even k, the induction hypothesis and $\gcd(N, k/2) = d/2$ cancel the halved terms in that equation, with the same conclusion.

If $v_2(k) > t$, then $S_k = 0$, so $\delta_k = \delta_{k/2}$ by (3.2). Now $\gcd(N, k/2) = \gcd(N,k)$, and induction again gives (3.1). In both cases equality modulo two is equality because the quantities are zero or one. □

Corollary 3.2 (Counting changes and extracting their common block)**.** *Let $r(P)$ be the number of sign changes. Then*

$$r(P) = \sum_{\substack{d|N \\ d<N}} \delta_d \varphi(N/d). \tag{3.3}$$

If the change set is nonempty, put $h = \gcd\{d : d \mid N,\ d < N,\ \delta_d = 1\}$; otherwise put $h = N$. Then

$$P(x) = G_h(x) R(x^h) \tag{3.4}$$

for a cyclotomic Littlewood polynomial R. Moreover h is the largest positive block length in such a factorization, and $r(R) = r(P)$.

Proof. There are $\varphi(N/d)$ indices k with $1 \le k < N$ and $\gcd(N,k) = d$. This proves the count. Equation (3.1) also shows that h divides every change index. The coefficients are therefore constant on consecutive blocks of length h, giving (3.4). The quotient is cyclotomic by root lifting, and its changes are in bijection with those of P. Any positive block length divides N and every change index, so it divides h. □

The whole word can consequently be recovered from the binary coordinates α_d in (1.4). Divisor Möbius inversion first separates the contributions of nested divisor classes; summation over the change indices then gives

$$a_n = (-1)^{\sum_{\substack{d|N \\ d<N}} \alpha_d \lfloor n/d \rfloor} \qquad (0 \le n < N). \tag{3.5}$$

Indeed $\delta_k \equiv \sum_{d|\gcd(N,k)} \alpha_d \pmod 2$, and each divisor d occurs $\lfloor n/d \rfloor$ times among $1, \dots, n$. With $a_1 = 1$ one has $\alpha_1 = 0$, so the number of possible normalized words is at most $2^{\tau(N)-2}$, where $\tau(N)$ counts the positive divisors of $N > 1$.

Example 3.3 (Coordinates and visible changes)**.** At length 18, take $\mathcal{A} = \{2, 6\}$. Formula (3.5) repeats the block $(1, 1, -1, -1, 1, 1)$ three times. At the proper divisors of 18 the two kinds of binary data are

d	1	2	3	6	9
δ_d	0	1	0	0	0
α_d	0	1	0	1	0.

There is no visible change at 6: both active coordinates contribute a reversal there, and the reversals cancel. The polynomial is

$$P(x) = G_2(x)G_3(-x^2)G_3(x^6).$$

The nested scales $2 \mid 6 \mid 18$ produce successive blocks without overlap. The next structural theorem shows that this relation between divisibility chains and standard products holds at every length.

Gcd invariance alone does not imply cyclotomicity. For example,

$$Q(x) = 1 + x - x^2 + x^3 - x^4 - x^5$$

has changes at $2, 3, 4$, exactly the gcd classes 2 and 3 at length six. Nevertheless $Q(0) = 1$ and $Q(-1) = -2$, so it has a real zero in $(-1, 0)$. Formula (3.5) is a necessary parametrization, not a sufficient factorization criterion.

4 The first failure of a block

If every sign change occurs at a multiple of i, the coefficient word is constant on blocks of length i. Newton's identities express the same condition in root traces. The point of passing between the two descriptions is that a first nonzero trace outside this support must occur at a divisor of the length. We first make the equivalence precise; the distinction between one block and a complete recursive factorization remains essential.

Lemma 4.1 (Exact first-block criterion)**.** *With the preceding normalization and first negative index i, the following are equivalent:*

(i) $S_k = 0$ *whenever* $1 \le k < N$ *and* $i \nmid k$*;*
(ii) $b_k = 0$ *whenever* $1 \le k < N$ *and* $i \nmid k$*;*
(iii) $D(x) \in \mathbb{Z}[x^i]$*;*
(iv) $P(x) = G_i(x)R(x^i)$ *for a cyclotomic Littlewood polynomial R of length N/i.*

Proof. Use (2.6) inductively at an index not divisible by i. In each convolution term, at least one index is not divisible by i; its factor is zero by the inductive hypothesis. This proves the equivalence of (i) and (ii). Since $i \mid N$, the endpoint is compatible, giving (iii). Condition (ii) says exactly that the coefficients of P are constant in consecutive blocks of length i. Reading one coefficient from each block gives R, with coefficients ± 1, and proves (iv). Every root of R is the ith power of a root of $R(x^i)$, and thus is a root of unity. The reverse implications follow from $D = (1 - x^i)R(x^i)$ and (2.5). □

The next lemma contains the common descent mechanism. At the first trace outside a chosen support $h\mathbb{Z}$, all earlier off-support terms vanish. At the sum with the first negative index, only two convolution terms remain. Their magnitude prevents cancellation by a single boundary coefficient.

Lemma 4.2 (Least defect outside a divisor of the first index)**.** *Let P be a normalized cyclotomic Littlewood polynomial of length N, and let i be its first negative coefficient index. Let $h \mid i$. If the set*

$$\{k : 1 \le k < N,\ h \nmid k,\ S_k \neq 0\}$$

is nonempty, let k be its least member. Then

$$i < k, \qquad k \mid N, \qquad i + k \mid N, \tag{4.1}$$

and

$$b_k \in \{-2, 2\}, \qquad S_k = -kb_k, \qquad S_{i+k} = -(i + k)(2b_k + b_{i+k}) \neq 0. \tag{4.2}$$

Proof. Since $S_1 = \cdots = S_{i-1} = 0$, $S_i = 2i$, and $h \mid i$, we have $k > i$. Also $h \mid N$ by Lemma 2.2. That lemma gives $k \mid N$: otherwise a smaller index $d = \gcd(N, k)$ would have $S_d = S_k \neq 0$, and $h \nmid d$ because $h \nmid k$. The alternative in (2.8) cannot occur at a nonzero trace.

Newton induction shows that $b_\ell = 0$ whenever $\ell < k$ and $h \nmid \ell$. At index k, each convolution term in (2.6) has either a coefficient index or a trace index not divisible by h, so it vanishes. Hence $S_k = -kb_k$, and $b_k \in \{-2, 2\}$.

The distinct proper divisors i, k of N satisfy $i + k < N$. At index $i + k$, the only possibly nonzero convolution terms are $b_i S_k$ and $b_k S_i$. Indeed, coefficient indices below i give zero. If $i < \ell < k$, then either $h \nmid \ell$, giving $b_\ell = 0$, or $h \mid \ell$, in which case $i + k - \ell$ is not divisible by h and is smaller than k, giving a zero trace. Indices above k are paired with traces below i. Using $b_i = -2$ and $S_i = 2i$ proves (4.2). Its last expression is nonzero because $2b_k = \pm 4$ whereas $b_{i+k} \in \{0, \pm 2\}$.

Put $d = \gcd(N, i + k)$. Lemma 2.2 gives $S_d = S_{i+k} \neq 0$. If $d < i + k$, then $d \le (i + k)/2 < k$, so minimality of k forces $h \mid d$. Consequently $h \mid i + k$; together with $h \mid i$, this gives $h \mid k$, a contradiction. Thus $i + k \mid N$. □

Theorem 4.3 (Additive divisor obstruction, Akhtari–Choi [2])**.** *If the equivalent conditions in Lemma 4.1 fail, there is an index j such that*

$$2 \le i < j, \qquad i \nmid j, \qquad i \mid N, \quad j \mid N, \quad i + j \mid N. \tag{4.3}$$

More precisely, take j least with $i \nmid j$ and $S_j \neq 0$. Then

$$b_j \in \{-2, 2\}, \qquad S_j = -jb_j, \qquad S_{i+j} = -(i + j)(2b_j + b_{i+j}) \neq 0. \tag{4.4}$$

Proof. Apply Lemma 4.2 with $h = i$ and $k = j$. □

Proof of Theorem 1.2. If (4.3) holds, put $g = \gcd(i, j)$, $u = i/g$, and $v = j/g$. Then $1 < u < v$, the integers $u, v, u + v$ are pairwise coprime, and

$$g\,uv(u + v) = \operatorname{lcm}(i, j, i + j) \mid N.$$

Thus the hypothesis excludes every failure of Lemma 4.1. We obtain $P = G_i R(x^i)$. The same arithmetic hypothesis holds for every divisor of N, in particular N/i. Induction on the length makes R standard. Factoring G_i by $G_{ab} = G_a G_b(x^a)$ gives its prime blocks and completes the induction. The all-positive polynomial and length one are immediate. □

Proof of Corollary 1.3. If (1.2) holds, the three pairwise coprime integers $u, v, u + v$, each greater than one, must allocate the three available primes $2, p, q$ to three separate factors. Each factor is therefore a power of one of those primes. Their additive relation is one of (1.3), with exponents bounded by those in N. The last assertion follows by taking $\alpha = \beta = 1$. □

Remark 4.4 (Recursive scope). A first-block criterion for one polynomial is not a proof of the full conjecture. Its quotient must also be covered. For instance, if $R(0) = 1$ and $[x]R = -1$, then $(1 + x)R(x^2)$ has its first negative coefficient at 2 and all boundary traces at odd indices vanish, irrespective of the internal structure of R. The hereditary hypothesis in Theorem 1.2 supplies exactly the missing recursive justification.

5 A canonical divisor-chain criterion

The binary coordinates in (3.5) distinguish a standard word without choosing a prime factorization of its length. Recall that $\mathcal{A}(P)$ is the set of active proper divisors. A *divisor chain* is a set whose members are pairwise comparable under divisibility; the empty set is included.

Theorem 5.1 (Canonical chain criterion)**.** *A Littlewood word with $a_0 = 1$ satisfying the gcd support law is standard if and only if $\mathcal{A}(P)$ is a divisor chain.*

Proof. Length one is immediate, so assume $N > 1$. Suppose first that P is a standard product. Write its successive prefix lengths as $D_0 = 1$ and $D_j = p_1 \cdots p_j$, and write $\varepsilon_j = (-1)^{e_j}$. The jth mixed-radix digit of an index n is

$$t_j(n) = \lfloor n/D_{j-1} \rfloor - p_j \lfloor n/D_j \rfloor.$$

Its coefficient therefore has sign $(-1)^{\sum_j e_j t_j(n)}$. Collecting the floor functions shows that all its nonzero divisor coordinates occur on the chain $D_0 \mid D_1 \mid \cdots \mid D_{s-1}$. Uniqueness of the coordinates follows from the divisor Möbius inversion defining α_d.

Conversely, adjoin 1 and N to the proposed chain and list the result as $1 = D_0 < D_1 < \cdots < D_s = N$. Put $m_j = D_j/D_{j-1}$ and regard unlisted coordinates as zero. Choose binary signs recursively by

$$e_1 = \alpha_{D_0}, \qquad e_{j+1} \equiv \alpha_{D_j} + m_j e_j \pmod 2 \quad (1 \le j < s).$$

The same digit identity shows that

$$P(x) = \prod_{j=1}^{s} G_{m_j}((-1)^{e_j} x^{D_{j-1}}).$$

Each composite geometric block splits into prime blocks by $G_{uv}(z) = G_u(z)G_v(z^u)$. This is a standard factorization. □

The chain criterion describes a finite word by its successive blocks. For trace calculations it is useful to describe the same construction by rational functions. Each added coordinate reverses alternate blocks; the next identity records exactly that operation.

Lemma 5.2 (The boundary of a chain)**.** *Let $d_1 \mid d_2 \mid \cdots \mid d_r$ be distinct positive integers, and extend their sign word to all $n \ge 0$ by*

$$\beta_n = (-1)^{\sum_{j=1}^{r} \lfloor n/d_j \rfloor}, \qquad \varepsilon_j = (-1)^{\sum_{k<j} d_j/d_k}.$$

Then, as a formal rational identity,

$$(1 - x) \sum_{n \ge 0} \beta_n x^n = \prod_{j=1}^{r} \left(\frac{1 - x^{d_j}}{1 + x^{d_j}} \right)^{\varepsilon_j}. \tag{5.1}$$

If σ_k denotes its logarithmic trace, then

$$\sigma_k = 2 \sum_{\substack{1\le j\le r \\ d_j\mid k,\ k/d_j\ \text{odd}}} \varepsilon_j d_j. \tag{5.2}$$

In particular, this trace is zero whenever $v_2(k) > \max_j v_2(d_j)$. The empty chain gives the constant boundary series 1*, whose positive traces all vanish.*

Proof. Before the coordinate d_j is introduced, the sequence has signed period d_j with sign ε_j. If B_j is its first block of length d_j, its generating series is $B_j(x)/(1-\varepsilon_j x^{d_j})$. Introducing d_j reverses the signs of alternate blocks, replacing the denominator by $1+\varepsilon_j x^{d_j}$. The quotient of the two series is $(1-x^{d_j})^{\varepsilon_j}/(1+x^{d_j})^{\varepsilon_j}$. Start with $1/(1-x)$ and multiply these quotients. For $R_d(x) = (1-x^d)/(1+x^d)$ one has

$$-\frac{xR_d'(x)}{R_d(x)} = 2d\sum_{s\ge 0} x^{(2s+1)d}.$$

Logarithmic differentiation proves (5.2) and its support assertion. □

Corollary 5.3 (Counting standard words)**.** *Let $F(N)$ be the number of ordered factorizations of N into integers greater than one, with $F(1) = 1$ for the empty factorization. Thus*

$$F(N) = \sum_{\substack{d\mid N \\ d<N}} F(d) \qquad (N>1).$$

For $N > 1$, there are exactly $2F(N)$ standard Littlewood polynomials of length N with constant coefficient one, and exactly $F(N)$ with $a_0 = a_1 = 1$.

Proof. The proper cumulative products of an ordered factorization form a chain of divisors strictly between 1 and N. Successive ratios give the inverse correspondence. By Theorem 5.1, these chains are precisely the possible active coordinates other than 1. The coordinate α_1 may independently be zero or one; normalization $a_1 = 1$ selects zero. Uniqueness of the divisor coordinates prevents double counting. □

Two reductions are visible directly in these coordinates. The following observation will also justify coordinate inheritance whenever a later argument extracts a shorter initial block.

Lemma 5.4 (Coordinates of an initial block)**.** *Let Q and P be sign words with constant coefficient one, of lengths $L \mid N$, respectively, both satisfying gcd invariance. Suppose the coefficients of Q are the first L coefficients of P. Then*

$$\alpha_d(Q) = \alpha_d(P) \quad (d \mid L,\ d < L), \qquad \mathcal{A}(Q) = \{d \in \mathcal{A}(P) : d \mid L,\ d < L\}.$$

Proof. For a proper divisor d of L, every $e \mid d$ lies below L. The two words have the same changes at these indices, so their Möbius sums in (1.4) agree. Both coordinate sets are supported on the proper divisors of their lengths, which gives the second assertion. □

Proposition 5.5 (Common divisor and common multiple)**.** *If $\mathcal{A}(P) \neq \varnothing$, put*

$$h = \gcd \mathcal{A}(P), \qquad H = \operatorname{lcm} \mathcal{A}(P), \qquad \eta = (-1)^{\sum_{d\in\mathcal{A}(P)} H/d}.$$

Then

$$P(x) = G_h(x)Q(x^h) = R(x)G_{N/H}(\eta x^H), \tag{5.3}$$

where Q and R are Littlewood polynomials of lengths N/h and H. If P is cyclotomic, so are both quotients.

Proof. All changes occur at multiples of h, giving the first factorization. For the second, the floor-function formula gives $a_{n+H} = \eta a_n$ whenever both indices lie in the word. Thus the successive blocks of length H are signed copies of the first block. The root argument of Lemma 4.1 applies to Q; the roots of R are roots of P. □

The first factorization replaces each active divisor d by d/h. The second retains the active divisors smaller than H and discards H itself if it is active. Substitution $x \mapsto -x$ toggles only α_1. Consequently these reductions preserve any upper bound on the number of active divisors other than 1.

Theorem 5.6 (Two active divisor coordinates)**.** *A cyclotomic Littlewood polynomial with $|\mathcal{A}(P) \setminus \{1\}| \le 2$ is standard.*

Proof. Constants are immediate. Normalize $a_0 = a_1 = 1$, so that $\alpha_1 = 0$. A chain is already covered by Theorem 5.1. Otherwise the two active divisors a, b are incomparable. Remove their common divisor and then their common multiple by Proposition 5.5. This produces a cyclotomic Littlewood polynomial of length uv whose only active divisors are coprime integers $1 < u < v$. Its first negative coefficient occurs at u. Below v the word changes only at multiples of u, whereas it changes at v. Newton's identities therefore make v its first off-block trace defect. The additive obstruction gives $u + v \mid uv$, impossible because $u + v > 1$ is coprime to uv. □

The unrestricted conjecture therefore has a canonical equivalent form: the active divisor coordinates of every cyclotomic Littlewood polynomial must form a chain. A least-length counterexample, normalized by $a_1 = 1$, would have gcd one and lcm equal to its length. The three-coordinate theorem in Section 15 further requires at least four active coordinates. Excluding all remaining non-chain words is necessary for unrestricted factorization.

Real-axis positivity does not supply this exclusion. For example,

$$Q(x) = 1 + x + x^2 - x^3 + x^4 + x^5 + x^6 + x^7 - x^8 + x^9 + x^{10} + x^{11}$$

has gcd-invariant changes, with active coordinates $\{3, 4, 6\}$. For $0 \le x < 1$, pairing $1 - x^3$ and $x - x^8$ shows $Q(x) > 0$. For $0 \le y < 1$,

$$Q(-y) = (1 - y)(1 + y^2 + 2y^3 + 3y^4 + 2y^5 + 3y^6 + 2y^7 + y^8 + y^{10}) > 0.$$

Nevertheless its first negative index is 3 and its first off-block defect is 4, so cyclotomicity would force $7 \mid 12$ by Theorem 4.3. Thus gcd invariance and positivity on $(-1, 1)$ together are still insufficient.

6 Removing a common dyadic scale from an obstruction

The least-defect lemma applies simultaneously to every divisor of the first negative index. At a length with two odd primes, this forces a common scale for the first additive obstruction.

Theorem 6.1 (Common-scale descent at length $2^t pq$)**.** *Let $p < q$ be distinct odd primes and let P be a normalized cyclotomic Littlewood polynomial of length $N = 2^t pq$. Let i be its first negative coefficient index. Suppose a first block at i fails, and let j be the least index with $i \nmid j$ and $S_j \neq 0$. Put $g = \gcd(i, j)$. Then*

$$D(x) \in \mathbb{Z}[x^g], \qquad P(x) = G_g(x)P_0(x^g), \tag{6.1}$$

where P_0 is cyclotomic Littlewood of length N/g. The first negative index and first off-block defect of P_0 are respectively i/g and j/g.

Proof. By Theorem 4.3, the indices $i, j, i + j$ divide N. The integers

$$i/g, \quad j/g, \quad (i + j)/g$$

are pairwise coprime, exceed one, and satisfy

$$g\,(i/g)(j/g)((i + j)/g) = \operatorname{lcm}(i, j, i + j) \mid N.$$

There are exactly three available prime supports. Thus the primitive triple consists of H, p, q, where $H = 2^u$ for some $u \geq 1$. Both odd primes are used to their first powers, so $g = 2^v$ is dyadic. The additive relation is either

$$q = p + H \qquad \text{or} \qquad H = p + q.$$

In the first relation the smaller original index is $i = gH$ when $H < p$, and $i = gp$ when $p < H$. In the second relation it is $i = gp$. These exhaust the possibilities, including the sum case.

For $g = 1$ the conclusion about D is immediate. Suppose $g > 1$ and, seeking a contradiction, let $k < N$ be the least index with $g \nmid k$ and $S_k \neq 0$. Lemma 4.2, with $h = g$, gives $i < k$ and $i, k, i + k \mid N$. Write $a = v_2(k) < v$ and $d = \gcd(i, k)$.

First suppose $i = gH = 2^{v+u}$. Here $d = 2^a$. The primitive triple

$$i/d, \quad k/d, \quad (i + k)/d$$

again consists of three pairwise coprime integers greater than one. Its first member is 2^{v+u-a}. The other two are odd and in increasing order, so they must be p and q, respectively. Hence

$$q - p = 2^{v+u-a}.$$

But the original obstruction has $q - p = H = 2^u$. This would force $a = v$, contrary to the choice of k.

Now suppose $i = gp$. If $p \mid k$, then $p \mid d$, and N/d has at most the two prime supports $2, q$. Nevertheless,

$$(i/d)(k/d)((i + k)/d) \mid N/d$$

is a product of three pairwise coprime integers greater than one, which requires three distinct prime supports. This is impossible. If $p \nmid k$, then $d = 2^a$ and

$$i/d = 2^{v-a}p$$

already uses both the 2 and p supports. The two remaining pairwise coprime integers k/d and $(i + k)/d$, both greater than one, would each have to use the sole remaining prime q. This is again impossible.

Thus every trace below N at an index not divisible by g vanishes. Newton induction gives the same support statement for the coefficients of D; its endpoint is compatible because $g \mid N$.

Therefore $D \in \mathbb{Z}[x^g]$. Equivalently, the coefficients of P are constant on consecutive blocks of length g. Reading those blocks gives (6.1); the quotient has coefficients ± 1 and is cyclotomic by the root-lifting argument in Lemma 4.1.

Put $D_0 = (1-x)P_0$. Then $D(x) = D_0(x^g)$, so

$$S_{g\ell}(D) = gS_\ell(D_0), \qquad S_k(D) = 0 \quad (g \nmid k).$$

The asserted first indices therefore scale by g. □

Corollary 6.2 (Primitive obstruction reduction)**.** *Any counterexample to the Borwein–Choi conjecture among lengths $2^t pq$ can be reduced, by removing initial positive dyadic blocks, to one whose first negative index i and first off-block defect j are coprime. The only remaining types are*

$$\begin{aligned} &(i,j) = (H,p), && q = p+H, && H < p, \\ &(i,j) = (p,H), && q = p+H, && p < H, \\ &(i,j) = (p,q), && H = p+q, \end{aligned}$$

where H is a power of two.

Proof. First remove any full first block whose quotient remains a counterexample; this strictly reduces the length and must stop at a first-block failure. A quotient losing either odd prime has at most two prime supports and is standard by Theorem 1.2; hence a surviving counterexample retains both odd primes, and each removed block has dyadic length. If the resulting obstruction has $g > 1$, Theorem 6.1 gives a shorter quotient with coprime obstruction indices. A standard quotient would make the original polynomial standard, since G_g is a positive prime chain. Thus a counterexample quotient persists. The list is the primitive-triple classification in the theorem. □

Remark 6.3. This descent does not exclude the three primitive types. It justifies restricting further proofs or exact searches to coprime obstruction indices, without assuming that a finite initial trace interval remains supported at all later indices.

7 Dyadic factors and the two-prime decomposition

A *dyadic rational function* is a product of integer powers of $1-x$ and the polynomials $1+x^{2^k}$, with negative powers permitted. It has an integral formal expansion and constant term one. Its special feature is an exact binary recursion: a linear coefficient equal to a sign can be extracted as a factor $1 \pm x$.

We first record a rigidity fact for ratios of dyadic cyclotomic factors. The degree of a nonzero rational function means its numerator degree minus its denominator degree, which is unchanged by cancellation.

Lemma 7.1 (Binary rigidity of a dyadic rational function)**.** *Let $T = 2^t$ and let*

$$R(x) = (1-x)^{e_0} \prod_{k=0}^{t} (1+x^{2^k})^{e_{k+1}}, \qquad e_0, \dots, e_{t+1} \in \mathbb{Z},$$

have degree -1. If $\varepsilon_k = [x^{2^k}]R \in \{-1, 1\}$ for $0 \le k \le t$, then

$$R(x) = \frac{\prod_{k=0}^{t}(1+\varepsilon_k x^{2^k})}{1-x^{2T}}. \tag{7.1}$$

In particular every coefficient of R is a sign, and its coefficient sequence is periodic with period $2T$.

Proof. The coefficient of x is $e_1 - e_0$. If this coefficient is 1, then

$$(1-x)^{e_0}(1+x)^{e_1} = (1+x)(1-x^2)^{e_0};$$

if it is -1, the same expression is $(1-x)(1-x^2)^{e_1}$. Thus in either case $R(x) = (1+\varepsilon_0 x)R_1(x^2)$, where R_1 has a dyadic factor expression with one fewer level and integer exponents.

After k steps, write

$$R(x) = \prod_{h=0}^{k-1}(1+\varepsilon_h x^{2^h})R_k(x^{2^k}).$$

The displayed polynomial prefix has degree $2^k - 1$ and constant term one. Hence $[x^{2^k}]R = [y]R_k(y)$. The same two-factor identity therefore applies at the next step. After $t+1$ steps this gives

$$R(x) = \prod_{k=0}^{t}(1+\varepsilon_k x^{2^k})(1-x^{2T})^m$$

for an integer m. Taking rational degrees gives $-1 = (2T-1) + 2Tm$, so $m = -1$. The numerator in (7.1) has exactly one signed monomial at every degree from zero to $2T-1$, by uniqueness of binary expansion. Its repetition by the denominator proves the final assertion. □

For the remainder of this section let $N = Tpq$, where $T = 2^t$ and $p < q$ are distinct odd primes. Normalize $P(0) = 1$, put $D = (1-x)P$, and let E_m be the cyclotomic multiplicities of D.

For $0 \le a \le t+1$ define constant-term-one dyadic factors

$$C_0(y) = 1 - y, \qquad C_1(y) = 1 + y, \qquad C_a(y) = 1 + y^{2^{a-1}} \quad (a \ge 2).$$

Using the multiplicities E_m of D, set

$$U_d(y) = \prod_{a=0}^{t+1} C_a(y)^{E_{2^a d}} \qquad (d \in \{1, p, q, pq\}).$$

Each U_d has degree T by (2.4) and constant term one. Also $E_1 \ge 1$, since $1-x$ divides D. For an odd prime r, the normalized factor of conductor $2^a r$ is $C_a(x^r)/C_a(x)$; at odd part pq it is

$$\frac{C_a(x^{pq})C_a(x)}{C_a(x^p)C_a(x^q)}.$$

These identities give the exact rational factorization

$$P(x) = B(x)F_p(x^p)F_q(x^q)U_{pq}(x^{pq}), \tag{7.2}$$

where

$$B(x) = \frac{U_1(x)U_{pq}(x)}{(1-x)U_p(x)U_q(x)}, \qquad F_p(y) = \frac{U_p(y)}{U_{pq}(y)}, \qquad F_q(y) = \frac{U_q(y)}{U_{pq}(y)}. \tag{7.3}$$

Every series here has integer coefficients and constant term one, since its denominator does.

In particular B has rational degree -1, while F_p, F_q have rational degree zero. The decomposition isolates the dyadic baseline from corrections beginning at p, q, and pq. It is valid for every $t \ge 0$, without a relative-size condition on the primes.

For a rational function R with $R(0) = 1$, define its formal traces by

$$-\frac{yR'(y)}{R(y)} = \sum_{k\ge 1}\sigma_k(R)y^k. \tag{7.4}$$

For $R = D$ these are S_k. For a ratio of dyadic cyclotomic factors they are integer linear combinations of the Ramanujan sums $c_{2^a}(k)$. In particular they depend only on $v_2(k)$ at every index.

8 A local rigidity lemma for two corrections

The coefficient obstruction below applies to an arbitrary formal block series. In particular, factors whose arguments are multiples of the block length may be absorbed into that series without changing the argument. No cyclotomic or dyadic hypothesis is required in this lemma. Its mechanism is a forced alternation. The first correction makes a second correction necessary at the sum of the two initial indices. Requiring their subsequent boundary contributions to remain in $\{0, \pm 2\}$ forces the baseline blocks to alternate. At the last tested block the coefficient condition forces a repetition instead, and the two requirements give a boundary coefficient of -4.

Lemma 8.1 (Two-correction rigidity)**.** *Let $2 \le i < j$ with $i \nmid j$, and write $j = mi + r$, where $m \ge 1$ and $0 < r < i$. Put*

$$n_* = j + (m+1)i = 2j + i - r < 3j.$$

Let C, J, K, L be formal power series over $\mathbb{R}$ with

$$C(y) = 1 - y + O(y^2), \qquad J(y) = 1 + \lambda y + O(y^2), \qquad K(0) = 1, \qquad L(x) = 1 + O(x^{n_*+1}),$$

where $\lambda \in \{-2, 2\}$. Then in

$$P(x) = G_i(x) C(x^i) J(x^j) K(x^{j+i}) L(x) \tag{8.1}$$

some coefficient $[x^n]P$, with $0 \le n \le n_*$, does not belong to $\{-1, 1\}$.

Proof. Suppose all coefficients through n_* are signs. We first derive the boundary recurrence and then use the sign condition to determine its terms. Write $C(y) = \sum_{s \ge 0} \gamma_s y^s$, where $\gamma_0 = 1$ and $\gamma_1 = -1$, and put $\gamma_{-1} = 0$. Before degree j, the coefficients are constant on blocks of length i, so $\gamma_0, \dots, \gamma_m$ are signs. At degree j,

$$[x^j]P = \gamma_m + \lambda \in \{-1, 1\}, \qquad \lambda = -2\gamma_m.$$

Let $D = (1 - x)P = \sum b_n x^n$, and write $\mu = [y]K$. Every b_n with $1 \le n \le n_*$ belongs to $\{0, \pm 2\}$. The baseline boundary series is

$$E(x) = (1 - x^i) C(x^i) = \sum_{s \ge 0} (\gamma_s - \gamma_{s-1}) x^{si}.$$

Since $j + i < 2j$, comparison at $j + i$ gives

$$b_{j+i} = \mu - 2\lambda \in \{0, \pm 2\}, \qquad \mu \in \{\lambda, 2\lambda, 3\lambda\}. \tag{8.2}$$

Thus μ is even, has the sign of λ, and has absolute value at least two; these are consequences of the sign hypothesis.

For $2 \le s \le m$, the index $j + si$ is less than $2j$ and is not divisible by i. Only the first terms of J and K can contribute to its boundary coefficient. Hence

$$b_{j+si} = \lambda(\gamma_s - \gamma_{s-1}) + \mu(\gamma_{s-1} - \gamma_{s-2}). \tag{8.3}$$

At the final index $n_* = j + (m+1)i$ the same identity remains true:

$$b_{n_*} = \lambda(\gamma_{m+1} - \gamma_m) + \mu(\gamma_m - \gamma_{m-1}). \tag{8.4}$$

Indeed, $n_* < 2j + i$ excludes the mixed term from J and K, and also excludes the second term of K. The inequality $n_* < 3j$ excludes the third term of J. Its second term, at degree $2j$, is multiplied by $[x^{n_*-2j}]E = [x^{i-r}]E = 0$. The baseline itself has zero coefficient at n_*, since $i \nmid j$. The stated order of $L - 1$ excludes every tail contribution.

Suppose first that $m \geq 2$. At $s = 2$, equation (8.3) gives

$$b_{j+2i} = \lambda(\gamma_2 + 1) - 2\mu.$$

The value $\gamma_2 = -1$ would give absolute value at least four. Thus $\gamma_2 = 1$, and the bound $|2(\lambda - \mu)| \leq 2$, together with the evenness of λ, μ, gives $\mu = \lambda$. For $3 \leq s \leq m$, equation (8.3) becomes $b_{j+si} = \lambda(\gamma_s - \gamma_{s-2})$. The two terms in parentheses are signs, so their difference must vanish. Consequently

$$\gamma_s = (-1)^s \ (0 \leq s \leq m), \qquad A := \gamma_m = (-1)^m, \qquad \lambda = \mu = -2A.$$

At degree $(m + 1)i$, which is below $j + i$ and $2j$, the coefficient of P is $\gamma_{m+1} + \lambda$. Therefore $\gamma_{m+1} \in \{A, 3A\}$. At $j + i$, the two corrections cancel, since $\lambda\gamma_1 + \mu\gamma_0 = 0$; its coefficient is γ_{m+1}. The sign condition forces $\gamma_{m+1} = A$. Equation (8.4) now yields

$$b_{n_*} = 0 + (-2A)(2A) = -4,$$

a contradiction.

Finally let $m = 1$. Then $\lambda = 2$ and $\mu \geq 2$. At degree $2i < j + i < 2j$, the sign condition gives $\gamma_2 + 2 \in \{-1, 1\}$, hence $\gamma_2 \in \{-1, -3\}$. Equation (8.4) gives

$$b_{n_*} = 2(\gamma_2 + 1) - 2\mu \leq -4,$$

again impossible. □

In particular the lemma applies whenever $L = 1 + O(x^{3j})$. Appendix SA gives a separate binary-carry proof under the additional assumption that C is dyadic rational.

Lemma 8.2 (Residue protection)**.** *Lemma 8.1 remains valid if its tail is*

$$L(x) = L_0(x) \prod_{\delta \in \Delta} R_\delta(x^\delta), \qquad L_0 = 1 + O(x^{n_*+1}), \quad R_\delta(0) = 1,$$

where every $\delta \in \Delta$ satisfies

$$\delta > j + i, \qquad \delta + j > n_*, \qquad \delta \not\equiv j \pmod{i}.$$

Proof. The proof of Lemma 8.1 uses coefficients of P before j, at $(m + 1)i$, and at $j + i$, together with boundary coefficients at $j + si$ for $1 \leq s \leq m + 1$. The first set precedes every δ. Up to n_*, an additional factor can contribute only its linear term times the baseline: its products with either correction or another additional term start later. The baseline boundary is supported on $i\mathbb{Z}$, so the residue condition makes every such contribution vanish at the specified boundary indices. All equations in the original proof therefore remain unchanged. □

9 A necessary third correction

The local rigidity lemma can be applied without isolating prime factors. Cyclotomicity confines every binomial factor to a divisor of twice the length. A failed first block therefore requires a third correction in a short, explicitly prescribed part of this divisor set.

Lemma 9.1 (Integral binomial coordinates)**.** *Let P be cyclotomic Littlewood of length N, with $P(0) = 1$, and put $D = (1-x)P$. There are integers c_d, indexed by $d \mid 2N$, such that*

$$D(x) = \prod_{d \mid 2N} (1 - x^d)^{c_d}, \qquad S_k = \sum_{d \mid \gcd(2N,k)} d c_d.$$

If i, j are its first negative and first off-block trace indices, then $c_d = 0$ whenever $d < j$ and $i \nmid d$.

Proof. The conductor bound puts every cyclotomic index among the divisors of $2N$. Writing $\widetilde{\Phi}_1 = 1 - x$ and $\widetilde{\Phi}_m = \Phi_m$ for $m > 1$, the identity $\widetilde{\Phi}_m(x) = \prod_{d \mid m} (1 - x^d)^{\mu(m/d)}$ therefore gives the first formula, with integer exponents; constant terms fix the normalization. Logarithmic differentiation gives the trace formula. Set $c_d = 0$ also for indices not dividing $2N$. For $d < j$ with $i \nmid d$, the trace S_d is zero. Induction on d in the trace formula shows $c_d = 0$: every divisor of an index outside $i\mathbb{Z}$ is itself outside $i\mathbb{Z}$, so all the proper-divisor exponents have already vanished. □

For $2 \le i < j$, $i \nmid j$, write $j = mi + r$, with $0 < r < i$, and set

$$n_* = j + (m+1)i.$$

Define the divisor window

$$\mathcal{W}_N(i,j) = \left\{ d \mid 2N : \begin{array}{ll} j < d \le n_*, & i \nmid d, \quad d \ne i + j, \\ d < i + j \text{ or } d \equiv j \pmod{i} & \end{array} \right\}. \tag{9.1}$$

Thus the window consists of off-block divisors strictly between j and $i + j$, together with divisors at $j + 2i, \dots, j + (m+1)i$.

Theorem 9.2 (The third-correction obstruction)**.** *If a normalized cyclotomic Littlewood polynomial has first indices i, j, then $c_d \ne 0$ for some $d \in \mathcal{W}_N(i,j)$. In particular $\mathcal{W}_N(i,j)$ is nonempty.*

Proof. Suppose every exponent in the window is zero. Collect all binomial factors whose arguments are multiples of i into $E(x^i)$, and put $C(y) = E(y)/(1-y)$. Since $b_i = -2$, one has $C(y) = 1 - y + O(y^2)$. Keep the factors at j and $i + j$ separately:

$$P(x) = G_i(x) C(x^i) J(x^j) K(x^{i+j}) L(x), \quad J(y) = (1-y)^{c_j}, \quad K(y) = (1-y)^{c_{i+j}}.$$

The preceding lemma removes every remaining off-block argument below j, and $[y]J = b_j \in \{-2, 2\}$. Every remaining nonzero binomial argument $d \le n_*$ satisfies

$$d > i + j, \qquad d \not\equiv j \pmod{i}.$$

Moreover $d + j > 2j + i > n_*$. The factors at these arguments satisfy Lemma 8.2; all factors with arguments above n_* form its ordinary tail. Lemma 8.1 gives a coefficient outside $\{-1, 1\}$ by degree n_*.

All tested coefficients are interior. Indeed, if $g = \gcd(i,j)$, $u = i/g$, and $v = j/g$, then $2 \le u < v$ and the additive obstruction implies

$$N \ge \operatorname{lcm}(i, j, i+j) = guv(u+v) > 3gv = 3j > n_*.$$

This contradicts the Littlewood hypothesis. □

Corollary 9.3 (A divisor-window factorization criterion)**.** *Suppose that for every pair $2 \le i < j$ with $i \nmid j$ and $i, j, i+j \mid N$, the window $\mathcal{W}_N(i,j)$ is empty. Then every cyclotomic Littlewood polynomial of length N is standard.*

Proof. The condition is hereditary: if $n \mid N$, every additive pair for n is one for N, and $\mathcal{W}_n(i,j) \subseteq \mathcal{W}_N(i,j)$. Theorem 9.2 therefore excludes every first-block failure at every divisor length. Extract a positive first block and induct on the length, as in the arithmetic criterion. □

10 An earlier correction from an internal block change

The third-correction window can be shortened when the coefficient word changes its alternating block pattern before the first off-block index. The relevant scale is the first such internal change, rather than the size of the entire first off-block index.

Theorem 10.1 (An adaptive third-correction window)**.** *Let P be a cyclotomic Littlewood polynomial of length N, normalized by $a_0 = a_1 = 1$. Let i be its first active divisor coordinate and j the first active coordinate not divisible by i. Suppose that there is an active coordinate strictly between i and j, and let b be the least such coordinate. Thus $i \mid b$, $i < b < j$, and $i \nmid j$. Write $q = b/i$ and define*

$$\mathcal{W}_N(i,j;b) = \big\{ d \mid 2N : \quad j < d \le j+b, \quad i \nmid d, \quad d \ne i+j, \quad (d < i+j \text{ or } d \equiv j \pmod{i}) \big\}.$$

In the integral binomial factorization $(1-x)P = \prod_{d \mid 2N} (1-x^d)^{c_d}$, at least one exponent c_d with $d \in \mathcal{W}_N(i,j;b)$ is nonzero. In particular, this window is nonempty. Its upper endpoint satisfies $j + b < 2j$, and is strictly smaller than the endpoint $j + (\lfloor j/i \rfloor + 1)i$ of the general third-correction window.

Proof. Assume all the displayed exponents vanish. Put $K_* = j + b$. As in Lemma 9.1, every off-i binomial exponent below j is zero. Absorb all arguments divisible by i into a formal series, and separate the factors at j and $j+i$:

$$\begin{aligned} P(x) &= G_i(x) C(x^i) J(x^j) K(x^{j+i}) L(x), \\ C(y) &= \sum_{s \ge 0} \gamma_s y^s, \\ J(y) &= 1 + \lambda y + O(y^2), \qquad K(y) = 1 + \mu y + O(y^2). \end{aligned}$$

Here $\gamma_0 = 1$, $\gamma_1 = -1$, and $\lambda = b_j \in \{-2, 2\}$. Before j, the only active coordinates are multiples of i. Before b, the sole active coordinate is i, whereas the coordinate b reverses the sign at b. Consequently

$$\gamma_s = (-1)^s \quad (0 \le s < q), \qquad \gamma_q = -(-1)^q, \qquad 2 \le q \le \lfloor j/i \rfloor.$$

These are coefficients before j, so none of the other factors affects their values.

Every remaining binomial argument $d \le K_*$ is greater than $j+i$ and is not congruent to j modulo i. Such a factor cannot affect any coefficient used below: its products with a correction or another remaining factor begin above $K_* < 2j$; its linear contribution to the boundary is supported on $d + i\mathbb{Z}$, disjoint from $j + i\mathbb{Z}$. All larger arguments are ordinary tail factors. Thus the following boundary comparisons are unchanged by L.

Write $D = (1-x)P = \sum b_n x^n$. Its baseline is

$$E(x) = (1-x^i) C(x^i) = \sum_{s \ge 0} (\gamma_s - \gamma_{s-1}) x^{si}, \qquad \gamma_{-1} = 0.$$

At $j+i$, only the two linear corrections contribute. Hence

$$b_{j+i} = \mu - 2\lambda \in \{0, \pm 2\}, \qquad \mu \in \{\lambda, 2\lambda, 3\lambda\}.$$

For $2 \le s \le q$, one has $j + si \le K_* < 2j$, so

$$b_{j+si} = \lambda(\gamma_s - \gamma_{s-1}) + \mu(\gamma_{s-1} - \gamma_{s-2}). \tag{10.1}$$

If $q = 2$, then $\gamma_2 = -1$ and this boundary coefficient is -2μ, of absolute value at least four. If $q \ge 3$, then $\gamma_2 = 1$, so the same comparison at $s = 2$ gives $b_{j+2i} = 2(\lambda - \mu)$. Since both λ and μ are even and this coefficient has absolute value at most two, one has $\mu = \lambda$. Equation (10.1) then becomes

$$b_{j+si} = \lambda(\gamma_s - \gamma_{s-2}).$$

Both terms in parentheses are signs. The boundary bound therefore forces $\gamma_s = \gamma_{s-2}$ for $3 \le s \le q$, contradicting $\gamma_q = -(-1)^q$.

All the compared boundary coefficients are interior. Indeed the additive obstruction gives $i, j, i + j \mid N$, and its usual coprime reduction yields $N > 3j > K_*$. This proves the theorem. □

Example 10.2 (Strict refinement of the divisor window)**.** At length 630, the primitive coordinate word with active set $\{2, 6, 7, 45\}$ has first negative index $i = 2$, first internal change $b = 6$, and first off-block index $j = 7$. The general window contains the divisor 15 of 1260, whereas the adaptive window is empty: its only possible integers are 11 and 13. Thus the adaptive criterion excludes this word even though its general correction window is nonempty.

11 The two-prime factorization theorem

The common-scale descent and the local rigidity lemma together remove the restriction on the dyadic exponent.

Theorem 11.1. *Let $p < q$ be distinct odd primes and let $t \ge 0$. Every cyclotomic Littlewood polynomial of length $2^t pq$ is standard.*

Proof. We first exclude every primitive first-block obstruction. Set $T = 2^t$ and use the exact factorization (7.2)–(7.3):

$$P(x) = B(x)F_p(x^p)F_q(x^q)U_{pq}(x^{pq}), \qquad A(x) = (1 - x)B(x).$$

Here B, A, F_p, F_q are ratios of dyadic cyclotomic factors, and every factor is regular at zero with constant term one. The primitive possibilities in Corollary 6.2 are

$$(i, j) = (H, p), \quad q = p + H, \quad H < p;$$

$$(i, j) = (p, H), \quad q = p + H, \quad p < H; \qquad \text{or} \qquad (i, j) = (p, q), \quad H = p + q,$$

where H is a power of two. We will put each case in the exact form of Lemma 8.1.

First suppose $i = H < j = p$. Before degree p, the coefficients of B equal those of P. Thus they are 1 below H and -1 at H. The first $\log_2 H$ steps of the binary recursion in Lemma 7.1 consequently give

$$B(x) = G_H(x)C(x^H),$$

where C is a dyadic rational function and $[y]C = -1$. It also has rational degree -1: the degree of B is -1, so $-1 = (H-1) + H \deg C$. Take

$$J = F_p, \qquad K = F_q, \qquad L(x) = U_{pq}(x^{pq}).$$

Since $q = j + i$, this is exactly (8.1).

Now suppose $i = p$, encompassing both remaining primitive types. We claim that

$$A(x) = A_0(x^H) \tag{11.1}$$

for a formal series A_0 with integer coefficients and constant term one. Every power of two strictly below H is smaller than j: in the difference case $j = H$, while in the sum case $j = q > H/2$. Such an index is not divisible by p, so its trace for D vanishes by the definition of the first off-block index. At this index the logarithmic derivatives of the three factors with odd arguments x^p, x^q, x^{pq} contribute zero. Thus

$$\sigma_{2^a}(A) = 0 \qquad (2^a < H).$$

For a dyadic cyclotomic ratio, the trace at any index with $v_2(k) = a$ equals its trace at 2^a. Hence all formal traces of A at indices not divisible by H vanish. Formal integration of the logarithmic derivative proves (11.1). The coefficients of A_0 are integral because those of A are.

Because $p < H, q$, the coefficient of x^p in D comes only from $F_p(x^p)$. Since $b_p = -2$, we have $[y]F_p = -2$. Consequently

$$C(y) = \frac{F_p(y)}{1-y}$$

is a dyadic rational function with $[y]C = -1$. Its rational degree is -1, since $F_p = U_p/U_{pq}$ is a ratio of polynomials of equal degree T. Moreover,

$$\frac{F_p(x^p)}{1-x} = G_p(x)C(x^p).$$

In the difference case take

$$j = H, \qquad J = A_0, \qquad K = F_q,$$

since $q = H + p = j + i$. In the sum case take

$$j = q, \qquad J = F_q, \qquad K = A_0,$$

since $H = q + p = j + i$. In both cases set $L(x) = U_{pq}(x^{pq})$. These are again exact factorizations of the form (8.1).

It remains to verify the coefficient and cutoff hypotheses of that lemma uniformly. Write $J(y) = 1 + \lambda y + O(y^2)$. The baseline boundary series is

$$(1-x)G_i(x)C(x^i) = (1-x^i)C(x^i).$$

It is supported on multiples of i, has constant term one, and has coefficient -2 at i. Since $i \nmid j$, comparison at the first off-block index gives

$$\lambda = b_j \in \{-2, 2\}.$$

No separate hypothesis on the linear coefficient of K is needed: the boundary comparison at $j + i$ in Lemma 8.1 forces its required sign and size.

In the first primitive type $j = p$ and $pq \geq 3j$. In the second $j = H < q$, so $pq \geq 3q > 3j$. In the sum type $j = q$ and again $pq \geq 3j$. It follows in every case that $L(x) = 1 + O(x^{3j})$. Finally the primitive integers $i, j, i + j$ are pairwise coprime and their product divides $N = Tpq$. Therefore

$$N \geq ij(i + j) > 3j.$$

All coefficients at indices below $3j$ are consequently interior coefficients of P and are signs. Lemma 8.1 gives a contradiction. No primitive first-block obstruction exists.

For completeness, pass from this assertion to the full standard factorization. The odd-length case is covered by Theorem 1.2. For an even length, a first-block failure would, by Theorem 6.1, descend to one of the primitive obstructions just excluded. Thus every normalized polynomial has a first block, unless it is already all positive. Induct on the length after extracting that block. A quotient retaining both odd primes has length $2^{t'}pq$ with $t' < t$; a quotient losing either odd prime is covered by Theorem 1.2. The quotient is standard in either case, and a positive prime-chain factorization of the extracted G_i completes the factorization of P. The usual overall sign and substitution $x \mapsto -x$ restore the initial normalization. □

This theorem allows arbitrary powers of two and the first powers of two distinct odd primes. Section S4 extends the method to unbounded odd exponents in specified ranges. General lengths $2^t p^a q^b$ and unrestricted additional prime supports are not settled by these results. Section S7 separately treats all exponents for $\{p, q\} = \{3, 5\}$.

12 The prime support $\{2,3,7\}$

A complete arithmetic classification of signed prime-power equations makes it possible to settle a second three-prime support with all exponents unrestricted. The coefficient argument has one mechanism: a zero logarithmic coefficient removes an intervening boundary, after which the first-return identity gives a contradiction.

Theorem 12.1. *For all integers $t, a, b \geq 0$, every cyclotomic Littlewood polynomial of length $2^t 3^a 7^b$ is standard.*

Lemma 12.2 (The four primitive relations)**.** *For lengths supported on $\{2,3,7\}$, the complete set of primitive additive pairs is*

$$\mathcal{B} = \{(2,7), (3,4), (7,9), (32,49)\}.$$

Proof. The three pairwise coprime members of a primitive triple exceed one, so they are powers of $2, 3, 7$, one prime for each member. Scott and Styer [8, Theorem 8] classify the signed positive-exponent solutions for the prime triple $(7, 3, 2)$ and give exactly four. The four identities

$$2 + 7 = 9, \qquad 3 + 4 = 7, \qquad 7 + 9 = 16, \qquad 32 + 49 = 81$$

therefore exhaust the possibilities. Sorting the two summands gives the asserted list. □

The cited equation classification is an external input. The finite arithmetic reduction and the coefficient exclusions below are proved here; they do not require a search over Littlewood coefficients.

Lemma 12.3 (The eleven possible first pairs)**.** *A least-length nonstandard cyclotomic Littlewood polynomial supported on* $\{2,3,7\}$ *would have first indices in the following list:*

$$(2,7),\ (3,4),\ (6,8),\ (7,9),\ (14,18),\ (21,27),\ (32,49),$$
$$(42,54),\ (96,128),\ (224,288),\ (672,864).$$

Proof. For the first pair i,j, let U consist of the first entries u of primitive pairs $(u,v)\in\mathcal{B}$ such that $u \mid i$ and $v/u \geq j/i$. This set is nonempty. Put $h = i/\operatorname{lcm} U$. Every possible first off-h trace index k would be at least j, since $h \mid i$, and Lemma 4.2 would give $k \mid N$ and $i+k \mid N$. Its primitive pair belongs to $\mathcal{B}$, so $k = iv/u$ for some $u\in U$, and hence $h \mid k$, a contradiction. Thus every off-h trace below N vanishes. Newton induction gives $D\in\mathbb{Z}[x^h]$, and the block argument yields $P = G_h Q(x^h)$ with Q cyclotomic Littlewood. Minimality forces $h=1$, since otherwise Q is a shorter standard polynomial. Consequently $i = \operatorname{lcm} U$, in particular $i \mid 672$.

For any subset of available primitive pairs, equivalently,

$$\gcd(i,\{iv/u\}) = i/\operatorname{lcm}\{u\},$$

by prime valuations and $\gcd(u,v)=1$. For a fixed slope j/i, retain only primitive slopes at least that large. The joint gcd condition says exactly that i is the lcm of their available first entries. In decreasing slope order this gives

j/i	available first entries	possible i
$7/2$	2	2
$49/32$	$32, 2$	32
$4/3$	$3,\ 2$ if $2 \mid i,\ 32$ if $32 \mid i$	$3, 6, 96$
$9/7$	$7,\ 2$ if $2 \mid i,\ 3$ if $3 \mid i,\ 32$ if $32 \mid i$	$7, 14, 21, 42, 224, 672.$

In the second row $32 \mid i$ automatically makes 2 available. Multiplying each first entry by its slope gives the eleven pairs. □

Write $\rho(k) = 2^{v_2(k)}3^{v_3(k)}7^{v_7(k)}$, and call k *smooth* when $\rho(k)=k$. Trace reduction and gcd invariance of sign changes give

$$\rho(k) < i \quad\Longrightarrow\quad S_k = b_k = 0 \qquad (k<N). \tag{12.1}$$

Indeed $\gcd(N,k) \leq \rho(k) < i$ gives a zero initial boundary; Theorem 3.1 and Lemma 2.2 give the boundary and trace conclusions, respectively. In particular every nonsmooth boundary below $5i$ vanishes, because five is the least prime outside $\{2,3,7\}$.

Lemma 12.4 (Excluding the ten short slopes)**.** *None of the ten pairs in Lemma 12.3 with* $j < 2i$ *can occur.*

Proof. Set $n = 2i+j$. Before $2i$, the only possibly nonzero boundaries are $b_i = -2$, $b_j = 2$, and those at smooth indices in $(j,2i)$. Indeed there is no multiple of i strictly between i and j, and every nonsmooth boundary in this range vanishes by (12.1).

For an intervening smooth index d, choose $k = i+d$ or $k = j+d$ with $k<3i$ and $\rho(k)<i$. If the chosen pair is the unique unordered representation of k by possibly nonzero boundary indices, then $S_k = b_k = 0$, cubic logarithmic terms start at $3i$, and

$$0 = [x^k]\log D = -b_i b_d \quad\text{or}\quad -b_j b_d.$$

Thus $b_d = 0$.

The following table lists every intervening smooth index. An entry in A uses $k = i + d$, and an entry in B uses $k = j + d$. Every indicated pair is already unique in the original set of possible boundaries; no previous elimination is needed. The last column records the smooth part of the first-return index.

i	j	A	B	$\rho(2i + j)$
3	4	$\varnothing$	$\varnothing$	2
6	8	9	$\varnothing$	4
7	9	12	$\varnothing$	1
14	18	$21, 24, 27$	$\varnothing$	2
21	27	$32, 36$	28	3
32	49	$54, 56, 63$	$\varnothing$	1
42	54	$63, 64, 72, 81$	56	6
96	128	$144, 162, 168, 189$	147	64
224	288	$294, 324, 336, 378, 384, 392, 432, 441$	343	32
672	864	$882, 972, 1008, 1024, 1134, 1152, 1176, 1296, 1323$	$896, 1029$	96

The assertions in the table are finite integer identities: list the numbers $2^u 3^v 7^w$ between j and $2i$, then test the sums in $\{i, j\} \cup A \cup B$. For example, in the last row the boundary at 896 is isolated by $1760 = 864 + 896$, while the boundary at 1029 is isolated by $1893 = 864 + 1029$. Both witnesses lie below $3i = 2016$ and have smooth part smaller than 672.

All boundaries strictly between j and $2i$ have now vanished. The last column and (12.1) give $S_n = b_n = 0$. The only surviving quadratic products in $[x^n] \log D$ are $b_i b_{i+j}$ and $b_j b_{2i}$; the sole cubic product is $b_i^2 b_j$, and higher products start at $4i > n$. Thus

$$0 = [x^n] \log D = 2b_{i+j} - 2b_{2i} + 8, \qquad b_{2i} - b_{i+j} = 4.$$

But $a_{2i-1} = 1$, so $b_{2i} \le 0$, while $b_{i+j} \ge -2$. Their difference is at most two, a contradiction. This is the first-return identity in its present form. Every tested index is interior: if $g = \gcd(i, j)$ and $(u, v) = (i/g, j/g)$, then $N \ge guv(u + v) > 3j > 2i + j$. □

Lemma 12.5 (Excluding the remaining slope)**.** *The pair $(i, j) = (2, 7)$ cannot occur.*

Proof. For this pair the divisor-window cutoff is $n_* = 15$. The only odd positive integers at most 15 supported on $\{2, 3, 7\}$ are $1, 3, 7, 9$. The window requires $d > 7$ and excludes $d = i + j = 9$; even divisors are multiples of i. Thus $\mathcal{W}_N(2, 7) = \varnothing$, contradicting Theorem 9.2. □

Proof of Theorem 12.1. A least counterexample would have one of the eleven first pairs. Lemmas 12.4 and 12.5 exclude all of them. □

13 Dyadic norms and the highest divisor layer

Root squaring separates the dyadic levels of the factor budgets. The construction is classical [1]; its relation to the binary divisor coordinates gives an exact way to read the last level from the coefficient word. The intervening polynomials have odd coefficients, which need not be signs. There are three steps. First construct the norm without assuming that its coefficients remain Littlewood. Next identify what happens to the sign coordinates modulo four. Finally recover the terminal cyclotomic factors from those coordinates. This order keeps the coefficient information separate from any unproved flatness assertion.

Proposition 13.1 (The normalized norm sequence)**.** *Let P be cyclotomic with odd coefficients, $P(0) = 1$, and length $N = 2^t M$, where M is odd and $t \geq 1$. There are unique constant-term-one polynomials $P_0 = P, P_1, \ldots, P_t$ such that*

$$P_r(x)P_r(-x) = (1 - x^2)P_{r+1}(x^2)^2 \qquad (0 \leq r < t). \tag{13.1}$$

Each P_r is cyclotomic, has odd coefficients, and has length $N/2^r$. If e_m is the multiplicity of Φ_m in P, then

$$P_t(y) = \prod_{\substack{d \mid M \\ d>1}} \Phi_d(y)^{1-s_d}\Phi_{2d}(y)^{s_d}, \qquad s_d = e_{2^{t+1}d} \in \{0, 1\}. \tag{13.2}$$

Proof. It suffices to describe one step. For an odd $d > 1$, the new multiplicities at d and $2^r d$ are respectively

$$\frac{e_d + e_{2d}}{2}, \qquad e_{2^{r+1}d} \quad (r \geq 1).$$

At $d = 1$, the first expression is replaced by $(e_1 + e_2 - 1)/2$. The budgets make these nonnegative integers: $e_d + e_{2d}$ is even for $d > 1$ and is odd for $d = 1$. The identities $\Phi_d(x)\Phi_{2d}(x) = \Phi_d(x^2)$ and $\Phi_{2^{r+1}d}(x) = \Phi_{2^r d}(x^2)$ prove (13.1), including the factor $1 - x^2$ at $d = 1$. The new multiplicities satisfy the budget at half the length, so their product is congruent to the corresponding geometric polynomial modulo two. Its degree is one less than that length. Uniqueness follows by taking the square root with constant term one.

After t steps, the length is M. The budget there chooses exactly one of Φ_d, Φ_{2d} for every $d > 1$ dividing M. Its second choice is the original highest-level multiplicity, giving (13.2). □

For any polynomial Q with odd coefficients and $Q(0) = 1$, let $\widehat{Q}$ be the sign word whose coefficients are congruent to those of Q modulo four. The proof of Theorem 3.1 applies to this word whenever Q is cyclotomic: its boundary coefficients are even, and their halves determine the changes of $\widehat{Q}$ modulo two. Thus $\widehat{Q}$ has binary divisor coordinates even when Q itself is not Littlewood.

Lemma 13.2 (Coordinates under a norm step)**.** *In (13.1), the divisor coordinates satisfy*

$$\alpha_d(\widehat{P}_{r+1}) = \alpha_{2d}(\widehat{P}_r) \qquad \left(d \mid N/2^{r+1},\ d < N/2^{r+1}\right).$$

Proof. Write $D_r = (1 - x)P_r$, and let $S_r(k)$ denote its root trace. Equation (13.1) gives $D_r(x)D_r(-x) = D_{r+1}(x^2)^2$, whence

$$S_{r+1}(k) = \tfrac{1}{2}S_r(2k).$$

If $\delta_r(k)$ records a change in $\widehat{P}_r$, the logarithmic congruence (3.2) therefore gives

$$\delta_{r+1}(k) + \mathbf{1}_{2|k}\delta_{r+1}(k/2) \equiv \delta_r(2k) + \delta_r(k) \pmod 2.$$

Insert $\delta_r(k) \equiv \sum_{d|k} \alpha_d(\widehat{P}_r)$, with coordinates supported on divisors of the length. To see the cancellation explicitly, write $k = 2^s m$ with m odd. In the sum of the expressions for $\delta_r(2k)$ and $\delta_r(k)$, each divisor of k occurs twice. Only the new dyadic level remains:

$$\delta_r(2k) + \delta_r(k) \equiv \sum_{d|m} \alpha_{2^{s+1}d}(\widehat{P}_r) \pmod 2.$$

Now set $\gamma(k) = \sum_{d|k} \alpha_{2d}(\widehat{P}_r)$ modulo two. The same cancellation shows that $\gamma(k) + \gamma(k/2)$ equals the last display when k is even; for odd k the equation is $\gamma(k) = \delta_r(2k) + \delta_r(k)$. These recurrences determine the values uniquely by successive halving. Thus $\delta_{r+1}(k) \equiv \gamma(k)$. Divisor Möbius inversion proves the claim. Every index used is strictly below the appropriate length. □

For a Littlewood polynomial of length $2^t M$, define its highest divisor layer by

$$\mathcal{B} = \{h : h \mid M,\ h < M,\ \alpha_{2^t h}(P) = 1\}.$$

After t norm steps these, and only these, are the active coordinates of $\widehat{P}_t$. Every lower dyadic coordinate has disappeared. At odd length each remaining multiplicity budget is one, so determining the factors of P_t amounts to deciding between Φ_d and Φ_{2d} for each odd $d > 1$ dividing M.

Theorem 13.3 (The highest-layer identity)**.** *For every $d > 1$ dividing M, the multiplicity in (13.2) is exactly*

$$s_d = (\#\{h \in \mathcal{B} : d \nmid h\}) \bmod 2. \tag{13.3}$$

The terminal norm P_t is Littlewood if and only if $\mathcal{B}$ is a divisor chain.

Proof. Put $H = P_t$ and $R_e(y) = (1-y^e)/(1+y^e)$. The product formula for cyclotomic polynomials rewrites (13.2) as

$$H(y) = G_M(y) \prod_{e \mid M} R_e(y)^{m_e}, \qquad m_e = -\sum_{\substack{d \mid M \\ e \mid d,\ d>1}} \mu(d/e) s_d.$$

These integers satisfy $\sum_{e|M} m_e = 0$, since $\sum_{e|d} \mu(d/e) = 0$ for each $d > 1$. For any integer exponent m_e,

$$R_e(y)^{m_e} \equiv 1 + 2m_e \sum_{j \geq 1} y^{ej} \pmod 4.$$

Cross-products of two nonconstant terms vanish modulo four. Since $(1-y)G_M = 1 - y^M$, every interior boundary coefficient therefore gives

$$\delta_k(\widehat{H}) \equiv \sum_{e \mid k} m_e \pmod 2 \qquad (1 \leq k < M),$$

where only $e \mid M$ enter the sum. Divisor inversion yields $m_e \equiv \alpha_e(\widehat{H}) \pmod 2$ for $e < M$. The preceding lemma identifies these coordinates with $\alpha_{2^t e}(P)$. Möbius inversion on multiples gives $s_d = -\sum_{d|e|M} m_e$. Eliminating m_M by the zero-sum identity proves (13.3), because s_d is either zero or one. The same calculation shows that an odd-length cyclotomic polynomial with odd coefficients is uniquely determined by the divisor coordinates of its reduction to signs modulo four.

If $\mathcal{B}$ is a chain, Theorem 5.1 makes $\widehat{H}$ standard. It is therefore an odd-length cyclotomic Littlewood polynomial. The uniqueness just proved gives $H = \widehat{H}$. Conversely, if H is Littlewood, the established odd-length theorem makes it standard; its active coordinates $\mathcal{B}$ must form a chain. The case $M = 1$ is immediate. □

The highest multiplicities exhaust a whole budget whenever they are one. This converts the parity identity into a block extraction. The degree of the proposed factor is essential: after it is removed, the quotient has exactly the length of one coefficient block. Its coefficients can then be read directly from P.

Corollary 13.4 (An outer block from the highest layer)**.** *Suppose $|\mathcal{B}|$ is odd and $h = \operatorname{lcm} \mathcal{B} < M$. Then*

$$P(x) = Q(x) G_{M/h}(-x^{2^t h}), \tag{13.4}$$

where Q is cyclotomic Littlewood of length $2^t h$. Consequently a least-length counterexample cannot have a highest layer of odd cardinality and lcm smaller than M; in particular, that layer cannot be a singleton.

Proof. If $d \mid M$ but $d \nmid h$, then d divides no member of $\mathcal{B}$. Equation (13.3) gives $s_d = 1$. Its weight 2^t exhausts the budget, so every other multiplicity with odd part d is zero. Hence P has the factor

$$\prod_{\substack{d \mid M \\ d \nmid h}} \Phi_{2^{t+1}d}(x) = \frac{G_M(-x^{2^t})}{G_h(-x^{2^t})} = G_{M/h}(-x^{2^t h}).$$

The quotient has degree $2^t h - 1$ and is cyclotomic. Its coefficients are the initial block of P, because consecutive shifts by $2^t h$ do not overlap. They are therefore signs. Minimality makes Q standard, and the geometric factor splits into prime blocks. □

Theorem 13.5 (At most one active even divisor at twice odd length)**.** *Let $N = 2M$ with M odd. If a cyclotomic Littlewood polynomial has at most one active even divisor coordinate, it is standard. There is no restriction on its active odd coordinates or on the number of prime divisors of M.*

Proof. First suppose the even layer is empty. The highest-layer identity gives $P_1 = G_M$, so $D(x)D(-x) = (1 - x^{2M})^2$. Every even interior trace of D is zero. Normalize $a_0 = a_1 = 1$. If $P \neq G_N$, its first negative index i is odd. A first off-block defect j would also be odd, whereas the additive obstruction forces the even trace S_{i+j} to be nonzero. Thus $P = G_i R(x^i)$. Here $i \mid M$, and the norm identity gives

$$R(z)R(-z) = (1 - z^2)G_{M/i}(z^2)^2.$$

Induction proves standardness. This includes the known case with no factors Φ_{4d} [3, Corollary 3.12(e)].

If the unique active even divisor is $2h$, then $h < M$. Corollary 13.4 extracts $G_{M/h}(-x^{2h})$. The quotient of length $2h$ has no active even coordinate by Lemma 5.4: its proper-divisor coordinates are inherited, and $2h$ is now the endpoint. The empty-layer case applies to it. □

Proposition 13.6 (Three-coordinate reduction)**.** *If there is a nonstandard cyclotomic Littlewood polynomial with at most three active coordinates other than 1, a least-length such polynomial, normalized by $a_0 = a_1 = 1$, has*

$$\mathcal{A}(P) = \{a, b, c\}, \quad 1 < a < b < c, \quad \gcd(a, b) = 1, \quad a + b \mid c, \quad N = \operatorname{lcm}(a, b, c).$$

Within the subfamily $v_2(N) = 1$, the same conclusion holds for a least-length counterexample, and exactly two coordinates are even. In that case a, b have opposite parity and $c = 2(a + b)w$ for an odd positive integer w.

Proof. The two-coordinate theorem gives exactly three coordinates. Coordinate block extraction and minimality give $\gcd(a, b, c) = 1$ and $\operatorname{lcm}(a, b, c) = N$. These reductions preserve the coordinate bound. In the twice-odd subfamily, any quotient of odd length is already standard, so the same minimality argument applies.

If $a \mid b$, the first off-a defect is c. Now $\gcd(a, c) = 1$, and the additive obstruction gives $a + c \mid N$. Since $a + c$ is coprime to ac, it must divide b, contrary to $b < c$. Thus the first off-block defect is b.

We next show that the first two coordinates cannot retain a common factor. The obstruction at b puts the coprime sum of their primitive parts into c. A second obstruction, now outside that common factor, would put an even larger coprime sum back into b.

Put $g = \gcd(a, b)$ and suppose $g > 1$. Write $a = gu$, $b = gv$. Then $\gcd(g, c) = 1$. The divisibility $a + b \mid N$ forces $u + v \mid c$. More precisely, for a prime $p \mid u + v$, coprimality gives $p \nmid uv$. The exponent of p in $a + b = g(u + v)$ exceeds its exponent in $\operatorname{lcm}(a, b) = guv$. Since $N = \operatorname{lcm}(a, b, c)$, the full

required exponent must come from c. The first off-g defect is c, so Lemma 4.2 gives $a+c \mid N$ as well. Let $h = \gcd(a,c) = \gcd(u,c)$. Then $\gcd(h,b) = 1$. Writing $a = hA$, $c = hC$, with $\gcd(A,C) = 1$, each prime power in $A + C$ must be supplied by b in $h(A+C) \mid \operatorname{lcm}(a,b,c)$. Thus $(a+c)/h \mid b$. This quotient is coprime to g, so it divides v. But $c = (u+v)w$ gives $h = \gcd(u,w) \le w$, whence

$$\frac{a+c}{h} > \frac{c}{h} \ge u + v > v,$$

a contradiction. Hence $g = 1$, and $a + b \mid c$ follows.

At twice odd length, Theorem 13.5 requires at least two even coordinates. All three cannot be even. Since a, b are coprime, exactly one of them is even and c is the other even coordinate. Now $a + b$ is odd and $v_2(c) = 1$, which gives the asserted form of c. □

The configuration in Proposition 13.6 is excluded in Section 15. The norm identities alone do not show that the intermediate norms P_r are Littlewood, nor that a standard terminal norm forces P to be standard. The highest layer controls only the highest cyclotomic multiplicities. The remaining layers and their orientations must still be related to one another for unrestricted factorization.

14 A geometric divisor forces a common period

The highest layer can also constrain the layers below it. The mechanism has two parts. A two-member highest layer determines a short norm identity. When those two members are comparable, their cyclotomic factors give a signed geometric divisor. Reduction modulo $1 + x^c$ then turns that divisor into a period of the remaining coefficient sequence. Together these observations exclude every non-chain support with three active coordinates.

Lemma 14.1 (A two-member highest layer)**.** *Let P be cyclotomic Littlewood of length $N = TM$, with $P(0) = 1$, $T = 2^t \ge 2$, and M odd. Suppose its highest layer is $\mathcal{B} = \{u, v\}$, with $u < v$. Put $g = \gcd(u,v)$, $H = P_t$, and $R_d(y) = (1-y^d)/(1+y^d)$. Then*

$$H(y) = G_M(y)\frac{R_g(y)^2}{R_u(y)R_v(y)}. \tag{14.1}$$

If $g < u$, the boundary trace of H at g is $4g$. If $u \mid v$, then

$$G_{v/u}(-x^{Tu}) \mid P(x). \tag{14.2}$$

Proof. The highest-layer identity says that $s_d = 1$ precisely when d divides exactly one of u, v. Thus

$$s_d = \mathbf{1}_{d\mid u} + \mathbf{1}_{d\mid v} - 2\mathbf{1}_{d\mid g}.$$

For odd h,

$$\prod_{\substack{d\mid h\\ d>1}} \frac{\Phi_{2d}(y)}{\Phi_d(y)} = \frac{G_h(-y)}{G_h(y)} = \frac{R_1(y)}{R_h(y)}.$$

Substitution into (13.2) proves (14.1). If $g < u$, then $(1-y)H(y) = 1 - 4y^g + O(y^{g+1})$, because $g < u < v < M$. Its trace at g is consequently $4g$.

When $u \mid v$, the selected odd parts are exactly $d \mid v$ with $d \nmid u$. Each corresponding highest factor occurs in P, and their product is

$$\prod_{\substack{d \mid v \\ d \nmid u}} \Phi_{2Td}(x) = \frac{G_v(-x^T)}{G_u(-x^T)} = G_{v/u}(-x^{Tu}).$$

□

Lemma 14.2 (A lower chain orders the highest pair)**.** *Suppose a cyclotomic Littlewood polynomial of length $N = TM$, $T = 2^t \geq 2$, with $P(0) = 1$, has exactly two highest active coordinates $e < c$. If all its other active coordinates form a divisor chain, then $e \mid c$.*

Proof. Put $u = e/T$, $v = c/T$, and $g = \gcd(u, v)$. If $g < u$, Lemma 14.1 and the norm trace identity give $S_{Tg}((1-x)P) = 4Tg$. Up to this index, however, neither highest coordinate has entered the word. Its coefficients agree with the infinite word of the lower chain. Every member of that chain has dyadic valuation less than t, so Lemma 5.2 gives $S_{Tg} = 0$, a contradiction. Thus $g = u$. □

The exact interval is the classical Fine–Wilf threshold [6, Theorem 1]: agreement of sequences of periods e and ℓ on $e + \ell - \gcd(e, \ell)$ consecutive terms forces agreement everywhere. Here the geometric divisor first supplies $c - e$ local equalities between entries a distance e apart. A graph on residues modulo ℓ measures when these equalities join an entire period, and also gives the dimension of the solution space below the threshold.

Theorem 14.3 (Exact rank of the period constraint)**.** *Let $e \mid c \mid N$, with c/e and N/c odd, and let $\ell \mid N$. For an ℓ-periodic real sequence β, put*

$$P_\beta(x) = \sum_{n=0}^{N-1} (-1)^{\lfloor n/e \rfloor + \lfloor n/c \rfloor} \beta_n x^n, \qquad F(x) = \frac{1 + x^c}{1 + x^e}.$$

Write $g = \gcd(e, \ell)$ and $m = c - e$. The real vector space of sequences β for which $F \mid P_\beta$ has dimension

$$\max\{g, \ell - m\}. \tag{14.3}$$

Every such sequence has period e if and only if

$$c - e \geq \ell - \gcd(e, \ell). \tag{14.4}$$

When this inequality holds, $P_\beta = Q(x) G_{N/c}(x^c)$ with $\deg Q < c$.

Proof. The proof separates the algebra of divisibility from the combinatorics of periods. We first replace the divisor condition by equalities in an invertibly transformed sequence. We then count the independent equalities.

Set $q = N/c$ and

$$A_r = \sum_{j=0}^{q-1} (-1)^j \beta_{r+jc}.$$

The sign multiplier in P_β has period c. Its remainder modulo $1 + x^c$ is therefore

$$C(x) = \sum_{r=0}^{c-1} (-1)^{\lfloor r/e \rfloor} A_r x^r. \tag{14.5}$$

Since $F \mid 1 + x^c$, divisibility of P_β is equivalent to $C = FQ_0$ with $\deg Q_0 < e$. The nonzero coefficients of F are alternating signs at multiples of e; comparison of these disjoint blocks gives

$$F \mid P_\beta \quad \Longleftrightarrow \quad A_{r+e} = A_r \quad (0 \le r < m). \tag{14.6}$$

This includes the reverse implication: under the equalities, the remainder is F times its initial block of length e. Since q is odd and β has period N, adjacent sums telescope:

$$A_r + A_{r+c} = \beta_r + \beta_{r+N} = 2\beta_r.$$

This identity proves that the linear map $\beta \mapsto A$ on the ℓ-dimensional space of periodic sequences is injective, and hence invertible. It commutes with every shift, so it carries the subspace of e-periodic sequences onto itself.

Consider residues modulo ℓ, joining r to $r + e$ for $0 \le r < m$. With all starting residues present, the graph consists of g cycles, one for each residue modulo g. If $m < \ell - g$, the starting residues are distinct and no cycle is complete: the largest representative in the class $r \pmod g$, $0 \le r < g$, is $\ell - g + r$, which is missing. The selected edges therefore form a forest of m edges and $\ell - m$ components. Their equality constraints have rank m.

At $m = \ell - g$, exactly one edge is missing from each cycle. Each class modulo g is then connected, and further edges add no equality constraint. The rank is consequently $\ell - g$ for every $m \ge \ell - g$. When $g = \ell$, the full graph consists of loops; the same conclusion has rank zero. Thus the rank in all cases is $\min\{m, \ell - g\}$, proving (14.3).

The e-periodic subspace has dimension g and is always contained in the solution space. These spaces coincide exactly under (14.4). Otherwise the solution space has larger dimension, so there are real sequences satisfying the divisibility without having period e. Finally, an e-periodic β makes the full coefficient sequence c-periodic, giving the stated factorization. □

The sharpness here concerns all real periodic sequences. It is not a claim that a cyclotomic Littlewood counterexample exists below the bound.

For example, take $e = 2$ and $\ell = 10$. With $m = c - e = 4$, the equalities join the residue sets $\{0, 2, 4\}$ and $\{1, 3, 5\}$, leaving $6, 7, 8, 9$ isolated. There are six independent component values, as predicted by $\max(2, 10 - 4) = 6$. With $m = 8$, each parity class is connected, leaving just two values. The sequence then has period two. Both cases fit the theorem with $N = 30$, using $c = 6$ and $c = 10$, respectively. The transformed sequence A and the original sequence β have the same period constraint because the transformation commutes with shifts and is invertible.

The length hypothesis cannot simply be omitted. For $e = 2$, $c = 6$, $N = 30$, let β have period ten, with $\beta_5 = -1$ and all its other entries equal to 1. Then

$$P(x) \equiv (1 - x)(1 - x^2 + x^4) \pmod{1 + x^6}.$$

Thus $G_3(-x^2)$ divides P, but β does not have period two. Here $c - e = 4 < 10$. This P is not cyclotomic: it changes sign at 6 but not at 12, contrary to gcd invariance at length 30. The example concerns the period lemma without its length condition, rather than the cyclotomic factorization conjecture.

15 Highest-layer traces and further block extractions

The period theorem has a direct consequence for any two comparable highest coordinates. When the lower coordinates form a chain, it gives a complete factorization.

Corollary 15.1 (A sharpened outer-block criterion)**.** *Let P be cyclotomic Littlewood of length $N = 2^t M$, with $P(0) = 1$ and M odd, with $t \geq 1$. Suppose its only highest coordinates are $e < c$, with $e \mid c$. Let $\mathcal{C} = \mathcal{A}(P) \setminus \{e, c\}$, and put*

$$L = \operatorname{lcm} \mathcal{C}, \qquad \eta = (-1)^{\sum_{d \in \mathcal{C}} L/d}, \qquad \ell = \begin{cases} L, & \eta = 1, \\ 2L, & \eta = -1, \end{cases}$$

where $L = 1$ for the empty set. If

$$c - e \geq \ell - \gcd(e, \ell),$$

then $P = Q(x) G_{N/c}(x^c)$ for a cyclotomic Littlewood polynomial Q of length c. A least-length counterexample with such a highest pair must satisfy the strict reverse inequality.

Proof. The sequence $\beta_n = (-1)^{\sum_{d \in \mathcal{C}} \lfloor n/d \rfloor}$ satisfies $\beta_{n+L} = \eta \beta_n$, so ℓ is a period. Every coordinate in $\mathcal{C}$ has dyadic valuation less than t; thus $2L \mid N$ when $\mathcal{C}$ is nonempty, and $\ell \mid N$ also in the empty case. Lemma 14.1 supplies $F = G_{c/e}(-x^e)$ as a divisor of P. Theorem 14.3 applies. The quotient is the first coefficient block and its roots are roots of P, proving the last assertions. □

Theorem 15.2 (A chain below two highest coordinates)**.** *Let P be cyclotomic Littlewood, with $P(0) = 1$ and length $N = 2^t M$, where M is odd and $t \geq 1$. Suppose its highest active coordinates are exactly $e < c$, and the remaining coordinates $\mathcal{C}$ form a divisor chain. Define L, η, ℓ as in Corollary 15.1. Then $e \mid c$. If*

$$c - e \geq \ell - \gcd(e, \ell), \tag{15.1}$$

there is a standard polynomial R of length e such that

$$P(x) = R(x) G_{c/e}(-x^e) G_{N/c}(x^c). \tag{15.2}$$

In particular P is standard, with no bound on $|\mathcal{C}|$.

Proof. Lemma 14.2 gives $e \mid c$. The period criterion gives $P = Q G_{N/c}(x^c)$ with Q cyclotomic Littlewood of length c. Since c/e is odd, Q has the same dyadic height, and its only highest active coordinate is e, by Lemma 5.4. Corollary 13.4 now gives $Q = R G_{c/e}(-x^e)$ with R cyclotomic Littlewood of length e. The same inheritance lemma makes the active coordinates of R a subset of $\mathcal{C}$, so Theorem 5.1 makes R standard. Both displayed geometric factors split into prime blocks. □

The factorization separates the roles of the three arguments: the chain boundary forces comparability, the geometric divisor forces a period, and the saturated highest budget removes the remaining highest coordinate.

Theorem 15.3 (Three active divisor coordinates)**.** *Every cyclotomic Littlewood polynomial with at most three active divisor coordinates other than 1 is standard.*

Proof. Suppose otherwise and choose a least-length counterexample within this class, normalized by $a_0 = a_1 = 1$. Proposition 13.6 gives

$$\mathcal{A}(P) = \{a, b, c\}, \qquad 1 < a < b < c, \qquad \gcd(a, b) = 1, \qquad a + b \mid c, \qquad N = \operatorname{lcm}(a, b, c).$$

Odd length is already classified, so write $N = TM$ with $T = 2^t \geq 2$ and M odd. The highest layer is nonempty because the active coordinates have lcm N. It cannot be a singleton, by Corollary 13.4; that extraction preserves the coordinate bound. All three coordinates cannot be even. If a, b were

both odd, the highest layer would again be a singleton. Thus exactly one of a, b is even. Denote it by e, and denote the odd member by o. The highest layer must consist of e, c, so

$$v_2(e) = v_2(c) = t, \qquad \gcd(e, o) = 1.$$

The remaining support is the singleton $\{o\}$, a chain. Lemma 14.2 therefore gives $e \mid c$.

Now both e and $e + o$ divide c, and they are coprime. Hence

$$c \geq e(e + o), \qquad c - e \geq 2o,$$

the latter inequality following from $e \geq 2$. The lower sequence has period $2o$. Theorem 15.2 therefore gives a standard factorization, contradicting the choice of P. □

Consequently a normalized counterexample must have at least four active divisor coordinates. The separation criterion also applies when there are arbitrarily many lower coordinates. It does not exclude general incomparable highest coordinates, a comparable two-member layer below the threshold $\ell - \gcd(e, \ell)$, or all layers with more members. The subset-gcd formula below supplies further trace constraints. Compatibility of all dyadic layers remains unproved.

Even gcd invariance does not remove the short-interval difficulty. Take $N = 18$ and active coordinates $\{2, 3, 6, 9\}$. The corresponding word is

$$(1, 1, -1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, 1).$$

Its changes are gcd-invariant by the divisor-coordinate formula. For $e = 2$, $c = 6$, the remaining sequence has period nine, with one period $(1, 1, 1, -1, -1, -1, 1, 1, 1)$. Direct multiplication gives

$$\begin{aligned} P(x) &= (1 - x^2 + x^4)Q(x), \\ Q(x) &= 1 + x + 2x^3 - 2x^4 - x^6 - x^7 - 2x^9 + 2x^{10} + x^{12} + x^{13}. \end{aligned}$$

Nevertheless P is not 6-periodic, and the remaining sequence is not 2-periodic. Here $c - e = 4 < 9 - \gcd(2, 9) = 8$. The word is not cyclotomic: its first negative index is 2 and its first off-block defect is 3, which would require $5 \mid 18$. Thus the geometric divisor, the sign alphabet, and gcd invariance together are still insufficient for unrestricted outer extraction.

Theorem 15.4 (Subset-gcd formula for the highest traces)**.** *Let P be cyclotomic Littlewood with $P(0) = 1$ and length $N = TM$, where $T = 2^t \geq 2$ and M is odd. Write $\mathcal{B}$ for its highest layer, $b = |\mathcal{B}|$, $H = P_t$, and $R_h(y) = (1 - y^h)/(1 + y^h)$. For a nonempty subset $U \subseteq \mathcal{B}$, define*

$$g_U = \gcd U, \qquad w_U = (-1)^{b+|U|} 2^{|U|-1}.$$

Then, as a rational identity,

$$(1 - y)H(y) = (1 - (-1)^b y^M) \prod_{\varnothing \neq U \subseteq \mathcal{B}} R_{g_U}(y)^{w_U}. \tag{15.3}$$

In particular, for $1 \leq k < M$,

$$S_{Tk}((1 - x)P(x)) = 2T \sum_{\substack{\varnothing \neq U \subseteq \mathcal{B} \\ g_U \mid k,\ k/g_U \text{ odd}}} w_U g_U. \tag{15.4}$$

Empty products and sums give the corresponding formulas when $\mathcal{B}$ is empty.

Proof. Put $\varepsilon = (-1)^b$, and for $d \mid M$, $d > 1$, write $z_h = \mathbf{1}_{d|h}$. The highest-layer identity is equivalently

$$s_d = \frac{1 - \varepsilon \prod_{h \in \mathcal{B}} (1 - 2z_h)}{2} = \frac{1-\varepsilon}{2} + \varepsilon \sum_{\varnothing \neq U \subseteq \mathcal{B}} (-2)^{|U|-1} \mathbf{1}_{d|g_U}.$$

Substitute this expression in $H/G_M = \prod_{d|M, d>1} (\Phi_{2d}/\Phi_d)^{s_d}$, using

$$\prod_{\substack{d|h \\ d>1}} \frac{\Phi_{2d}}{\Phi_d} = \frac{R_1}{R_h}.$$

The powers of R_1 cancel. The remaining power of R_M is $-(1-\varepsilon)/2$, and the power at g_U is $-\varepsilon(-2)^{|U|-1} = w_U$. Since

$$(1 - y^M) R_M^{-(1-\varepsilon)/2} = 1 - \varepsilon y^M,$$

this proves (15.3). The logarithmic derivative of R_h contributes $2h$ at positive odd multiples of h and zero elsewhere. The endpoint factor has no contribution below M. Finally the norm trace relation gives $S_{Tk}((1-x)P) = TS_k((1-y)H)$, proving (15.4). □

Corollary 15.5 (Three highest coordinates and one lower coordinate)**.** *Under the hypotheses of Theorem 15.4, suppose P has exactly four active coordinates, of which exactly three are highest. Write their odd parts as $u < v < w$, and put $g = \gcd(u, v, w)$. If $g < u$, exactly two of*

$$\gcd(u, v), \quad \gcd(u, w), \quad \gcd(v, w)$$

equal g, and the remaining one is at least u.

Proof. Let o be the remaining coordinate. Before index Tu, the coefficient word is $(-1)^{\lfloor n/o \rfloor}$, whose boundary series is $R_o(x)$. Since $v_2(o) < t$, its trace at every multiple of T is zero. Equation (15.4) therefore forces the total exponent of each R_h, $h < u$, to vanish: induct on h, observing that every proper divisor contributing to the trace at h is smaller.

For a three-member layer the exponent below u is

$$-2\#\{\text{pairs with gcd } h\} + 4\mathbf{1}_{h=g}.$$

At $h = g < u$ this is zero only when exactly two pair gcds equal g. The third pair gcd is larger than g; if it were below u, its exponent would be -2, again impossible. □

These identities constrain all highest layers without making their compatibility with the lower layers automatic. The supplement, Section SG, records a complete finite exclusion of primitive four-coordinate words through length 3000. No universal four-coordinate classification, or unrestricted factorization theorem, follows from that finite range.

16 Four active coordinates

The three-coordinate theorem permits a precise reduction of the next case. The two exclusions below hold at arbitrary length within their stated hypotheses; the remaining configurations require further arguments.

Proposition 16.1 (The two possible highest-layer sizes)**.** *If a nonstandard cyclotomic Littlewood polynomial with at most four active coordinates other than 1 exists, a least-length such polynomial, normalized by $a_0 = a_1 = 1$, has exactly four active coordinates, gcd one, and lcm equal to its length. Its highest dyadic layer has two or three members.*

Proof. The three-coordinate theorem gives exactly four coordinates. The common-divisor and common-multiple factorizations preserve the coordinate bound, and a coordinate becoming 1 can be removed by substitution $x \mapsto -x$. Minimality therefore gives gcd one and full lcm. Odd length is already classified. At even length the highest layer is nonempty by full lcm. A singleton admits the highest-layer outer extraction, leaving at most three coordinates in the initial quotient. All four coordinates cannot be highest, since they would have a common even divisor. This leaves exactly two or three members. □

A nested initial pair forces a second additive divisor. This constraint uses only three terms of a formal logarithm, regardless of the later coordinates.

Proposition 16.2 (The prefix $a, 2a$)**.** *Let P be a normalized cyclotomic Littlewood polynomial whose first two active coordinates are $a, 2a$, where $a \geq 2$. If an active coordinate not divisible by a exists, let j be the least one. Then $j > 2a$ and*

$$j \mid N, \qquad j + a \mid N, \qquad j + 2a \mid N.$$

Proof. Below j, all boundary indices are multiples of a. The first coordinate gives $b_a = -2$, while the coordinate $2a$ cancels the reversal there, giving $b_{2a} = 0$. Every boundary coefficient and trace at an index below j not divisible by a vanishes. At j, $b_j \in \{-2, 2\}$. The first two divisibilities are the additive obstruction.

The formal logarithm at $j + 2a$ has the exact identity

$$[x^{j+2a}] \log D = b_{j+2a} + 2b_{j+a} + 4b_j.$$

Indeed every contributing monomial has an index not divisible by a, necessarily at least j; two such indices already exceed $j + 2a < 2j$. The other indices are multiples of a with total at most $2a$. The only possibilities are the linear term, the pair $(a, j + a)$, and the multiset (a, a, j); a term using $2a$ vanishes because $b_{2a} = 0$. Since $|4b_j| = 8$ while $|b_{j+2a} + 2b_{j+a}| \leq 6$, the trace S_{j+2a} is nonzero. This index is interior: the additive obstruction gives $N > 3j$. If $j + 2a \nmid N$, trace reduction either makes this trace zero, or identifies it with S_d for $d = \gcd(N, j + 2a) \leq (j + 2a)/2 < j$. Since $d \mid j + 2a$ and $a \nmid j$, one has $a \nmid d$. Thus $S_d = 0$, again a contradiction. □

Theorem 16.3 (A coprime odd third coordinate)**.** *Every cyclotomic Littlewood polynomial with at most four active divisor coordinates other than* 1 *is standard if its first three normalized active coordinates are*

$$a, \quad 2a, \quad j, \qquad j > 2a, \quad j \text{ odd}, \quad \gcd(a, j) = 1.$$

Proof. The three-coordinate theorem disposes of at most three coordinates. Otherwise write the support as $a, 2a, j, c$. The common-multiple block extraction reduces to length $H = \operatorname{lcm}(a, 2a, j, c)$. If it discards an endpoint coordinate, its quotient has at most three coordinates and is standard. Otherwise it retains the same support. Thus it suffices to work with $N = H$; the support already has gcd one. Write $T = 2^{v_2(N)}$. Since j is odd, a singleton highest layer would extract a polynomial with at most three active coordinates, and hence prove standardness. Thus $v_2(c) = v_2(2a)$, and the highest layer is exactly $\{2a, c\}$.

Put $e = 2a$. If $h = \gcd(e, c) < e$, the highest two-coordinate norm requires $S_h = 4h$. Yet $h < e < j$, so the coefficient prefix through h is that of the single active lower coordinate a. Moreover $v_2(h) = v_2(e) > v_2(a)$, so the chain-boundary formula gives $S_h = 0$, a contradiction. Hence $2a \mid c$, and $N = \operatorname{lcm}(j, c)$.

The additive obstruction gives $a + j \mid N$. Since $\gcd(a + j, j) = 1$, we obtain $a + j \mid c$, and therefore

$$c \geq \operatorname{lcm}(2a, a + j) = \frac{2a(a + j)}{\gcd(2, a + j)} > j + 2a.$$

Proposition 16.2 applies. Since $\gcd(j + 2a, j) = 1$, it gives $j + 2a \mid c$. Since $\gcd(a + j, j + 2a) = 1$, these divisibilities imply

$$c \geq \begin{cases} 2a(a + j)(j + 2a), & a \text{ even}, \\ a(a + j)(j + 2a), & a \text{ odd}. \end{cases}$$

In either case $c - 2a \geq 2aj$.

The lower-coordinate sign sequence $\beta_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/j \rfloor}$ has period $\ell = 2aj$. The comparable highest pair supplies the geometric divisor. The period criterion applies because $c - 2a \geq \ell - \gcd(2a, \ell)$, and extracts an outer block of length c. Its initial quotient has at most three active coordinates, hence is standard. The outer geometric block splits into prime blocks, a contradiction. □

Lemma 16.4 (Coprimality after the doubled first coordinate)**.** *Suppose a normalized cyclotomic Littlewood word has exactly four active coordinates $a, 2a, j, c$, where $2a < j < c$ and $a \nmid j$, and has primitive support (gcd one and lcm equal to its length N). Then* $\gcd(a, j) = 1$.

Proof. Suppose $a = gu$, $j = gv$, with $g > 1$ and $\gcd(u, v) = 1$. Primitivity gives $\gcd(g, c) = 1$. The first defect outside $a\mathbb{Z}$ occurs at j, so $a + j \mid N$. In particular $j + 2a < N$: the proper divisor $a + j$ is at most $N/2$. Proposition 16.2 therefore gives $j + 2a \mid N$. Dividing these relations by g and using

$$N/g = \operatorname{lcm}(2u, v, c)$$

shows that $(u + v)/\gcd(u + v, 2)$ and $(v + 2u)/\gcd(v + 2u, 2)$ divide c. To justify this step prime by prime, each sum is coprime to u, and its gcd with $\operatorname{lcm}(2u, v)$ is at most two. The two sums are coprime and at most one is even. Consequently

$$c \geq \frac{(u + v)(v + 2u)}{2} > 2uv.$$

On the other hand, c is the first defect outside $g\mathbb{Z}$, so the generalized additive obstruction gives $a + c \mid N$. Put $h = \gcd(a, c) = \gcd(u, c)$ and $z = (a + c)/h$. The coprime integers $a/h, c/h$ show prime by prime that

$$hz \mid \operatorname{lcm}(2a, j, c) \quad \Longrightarrow \quad z \mid 2j.$$

Indeed a prime in z cannot occur in either of the two reduced coprime summands, so any valuation beyond h must be supplied by j, except for one possible factor two from $2a$. Moreover $\gcd(z, g) = 1$, since $\gcd(c, g) = \gcd(h, g) = 1$. Thus $z \mid 2v$, giving

$$c = hz - a < 2hv \leq 2uv,$$

a contradiction. □

Theorem 16.5 (An off-block third coordinate)**.** *Every cyclotomic Littlewood polynomial with at most four active divisor coordinates other than* 1 *is standard if its first three normalized active coordinates are*

$$a, \quad 2a, \quad j, \qquad j > 2a, \quad a \nmid j.$$

Proof. First extract the common divisor of the active coordinates. If the first coordinate becomes 1, changing x to $-x$ removes it and leaves at most three coordinates other than 1. Otherwise the initial scales retain their doubled relation, with $a > 1$. Next use the common-multiple extraction. If a coordinate becomes the endpoint, the quotient has at most three coordinates. Otherwise retain the primitive support $\{a, 2a, j, c\}$ with $N = \operatorname{lcm}(2a, j, c)$. Lemma 16.4 gives $\gcd(a, j) = 1$. Odd j is covered by the preceding theorem, so assume j even; then a is odd.

If $v_2(j) > 1$, a singleton highest layer would reduce to at most three coordinates, so $v_2(c) = v_2(j)$ and the highest pair is j, c. Its lower coordinates $a, 2a$ form a chain, and hence $j \mid c$. The two additive divisors $a + j, j + 2a$ divide $N = \operatorname{lcm}(2a, c)$. The first is odd and coprime to $2a$, and the second has dyadic valuation one and gcd two with $2a$. Since $4 \mid c$, both divide c. Their pairwise gcds with j imply

$$c \geq \operatorname{lcm}(j, a + j, j + 2a) = \frac{j(a + j)(j + 2a)}{2}.$$

Thus $c - j \geq 4a$. The lower-chain word has period $4a$, so the period criterion extracts an outer block of length c; its quotient has at most three active coordinates and is standard.

It remains that $v_2(j) = 1$. Since a is odd, $j + 2a$ is then divisible by four. The second additive divisor forces $4 \mid N$. Thus c has strictly larger dyadic valuation than the other three coordinates and is the unique highest coordinate. Its outer extraction gives a quotient with at most three active coordinates, which is standard. This concludes every parity case. □

Corollary 16.6 (The doubled-initial-scale classification)**.** *Every cyclotomic Littlewood polynomial with at most four active divisor coordinates other than* 1 *is standard if its first two normalized active coordinates are* $a, 2a$*.*

Proof. Extract the common coordinate divisor and the common multiple. If the first coordinate becomes 1, toggle it by $x \mapsto -x$; if a coordinate becomes the length, discard that endpoint. In either case at most three coordinates remain, and the known theorem applies. Otherwise the support is primitive and its first two coordinates still have the form $a, 2a$, with $a > 1$. The cases of at most three coordinates are already known, so write its support as $a, 2a, b, c$.

If $a \nmid b$, the preceding theorem applies. If $a \mid b$, write $b = ka$, where $k \geq 3$. Primitivity forces $a \nmid c$ and $\gcd(a, c) = 1$. Thus c is the first defect outside $a\mathbb{Z}$, and the additive obstruction gives $a + c \mid N$. But

$$N = \operatorname{lcm}(2a, ka, c) \mid a \operatorname{lcm}(2, k)c, \qquad \gcd(a + c, ac) = 1.$$

Hence $a + c \mid \operatorname{lcm}(2, k) \leq 2k$, whereas $a + c > c > b = ka \geq 2k$, a contradiction. □

17 Even ratios between the first two coordinates

A comparable initial pair supplies more than its first additive obstruction. When its ratio is even, the reciprocal boundary has a strictly dominant coefficient at the second scale. This forces a run of additive divisors; the least common multiple of that run makes the highest coordinates far enough apart for period extraction.

Lemma 17.1 (Least common multiple of an arithmetic progression)**.** *Let* u, v *be coprime positive integers and let* $r \geq 1$*. Then*

$$\operatorname{lcm}_{1 \leq s \leq r}(v + su) \geq \frac{\prod_{s=1}^{r}(v + su)}{(r - 1)!}.$$

If $q \geq 4$ *and* $v > qu$*, then*

$$\operatorname{lcm}_{1 \leq s \leq q}(v + su) > q^2 u v^2.$$

Proof. A prime dividing u divides none of the progression. For every other prime p, remove a progression member of largest p-adic valuation. For each k, among the remaining $r-1$ members the number divisible by p^k is at most $\lfloor (r-1)/p^k \rfloor$. Thus the product of all members divided by their lcm divides $(r-1)!$.

For the second assertion put $F(v) = v^{-2} \prod_{s=1}^{q} (v+su)$, viewing v as a real variable. For $v \geq qu$,

$$\frac{F'(v)}{F(v)} = \sum_{s=1}^{q} \frac{1}{v+su} - \frac{2}{v} > 0,$$

since every denominator is at most $2v$, and $q \geq 4$. Since $v > qu$, comparison with $F(qu)$ and the first bound gives

$$\frac{1}{v^2} \frac{\prod_{s=1}^{q}(v+su)}{(q-1)!} > \frac{u^{q-2}}{q} \binom{2q}{q} > q^2 u.$$

The last inequality follows from $\binom{2q}{q} > q^3$ for $q \geq 4$: it holds at $q = 4$, and the ratio of successive central binomial coefficients is $2(2q+1)/(q+1) > 3$, exceeding the successive cubic ratio. Also $u^{q-3} \geq 1$. □

Lemma 17.2 (A forced additive run)**.** *Suppose a normalized cyclotomic Littlewood polynomial has at most four active coordinates, with first two a, qa, where $a > 1$ and $q \geq 4$ is even. Let j be its first coordinate outside $a\mathbb{Z}$. Then $j + qa \mid N$. If the first off-a coordinate is the fourth, all $j + sa$ $(1 \leq s \leq q)$ divide N. If it is the third, write its fourth coordinate as c. Unless*

$$c = j + ra \quad \textit{for some } 1 \leq r \leq q,$$

all those q divisibilities hold. In the exceptional case they hold at least for $1 \leq s \leq r$.

Proof. The usual additive obstruction gives $a + j \mid N$ and $a + j < N$; hence $j + qa < 2j < N$. Split $D = (1-x)P = U + V$ into its terms on and off $a\mathbb{Z}$. Before $2j$, the off-support part has only the term $b_j x^j$ and, if it exists and is off-support, $b_c x^c$. All b's are in $\{0, \pm 2\}$ and $b_j = \pm 2$. Up to degree qa, U is the boundary of the two-coordinate chain, so

$$U^{-1} \equiv \frac{1+x^a}{1-x^a} \frac{1+x^{qa}}{1-x^{qa}} \pmod{x^{qa+1}}.$$

Its coefficients at sa are 2 for $1 \leq s < q$, and 4 at qa; at degree zero it is one, and all other degrees vanish. Since V^2 starts at degree $2j$, the off-support logarithmic coefficient at $j + sa$ is the coefficient of V/U. Without the fourth term this is $2b_j$ $(s < q)$ or $4b_j$ $(s = q)$. A fourth term can contribute only if $c = j + ra$ with $r \leq s$. At $s = r < q$ its contribution is b_c, which cannot cancel $2b_j$; at $s = q$ it has absolute value at most four, which cannot cancel $4b_j$. All asserted traces are therefore nonzero. For any such index $n < 2j$, a proper gcd $d = \gcd(n, N)$ lies below j and outside $a\mathbb{Z}$. Its trace is zero, contradicting trace reduction. Thus $n \mid N$. □

Lemma 17.3 (Coprimality of the first defect)**.** *Suppose the primitive four-coordinate support is $\{a, qa, j, c\}$ with $q \geq 4$ even, $qa < j < c$, and $a \nmid j$. Then $\gcd(a, j) = 1$.*

Proof. Suppose $a = gu, j = gv$, where $g > 1$ and $\gcd(u, v) = 1$. Primitivity gives $\gcd(g, c) = 1$, so c is not congruent to j modulo a. The forced run gives $g(v + su) \mid N$ for every $1 \leq s \leq q$. Since

$$N/g = \operatorname{lcm}(qu, v, c), \qquad \gcd\left(u, \operatorname{lcm}_{1 \leq s \leq q}(v+su)\right) = 1,$$

the progression lcm divides $\operatorname{lcm}(q, v, c) \leq qvc$. Its strict lower bound yields $c > quv$.

The first defect outside $g\mathbb{Z}$ is c, so $a + c \mid N$. For $h = \gcd(a, c) = \gcd(u, c)$, put $z = (a + c)/h$. Prime-by-prime comparison in $hz \mid \operatorname{lcm}(qa, j, c)$ gives $z \mid qj$. As $\gcd(z, g) = 1$, it follows that $z \mid qv$. Consequently $c = hz - a < qhv \le quv$, a contradiction. □

Lemma 17.4 (The short resonance is impossible)**.** *Under the preceding primitive hypotheses, the fourth coordinate cannot satisfy $c = j + ra$ with $1 \le r \le q$.*

Proof. Coprimality gives $\gcd(a, j) = 1$. The support is not a divisor chain, since a and j are incomparable. Thus any reduction establishing standardness contradicts Theorem 5.1. Suppose first $r = 1$. If a is even, both j and c are odd, so qa is the unique highest coordinate. Its outer extraction leaves at most three active coordinates and proves standardness. If a is odd, exactly one of j, c is even. A singleton highest layer has the same consequence. Otherwise the highest layer is the pair qa, d, where d is that even coordinate, and both have the same dyadic valuation. If $h = \gcd(qa, d) < qa$, its highest-layer trace is $4h$, whereas before qa the only active coordinate is a and its trace at h vanishes. Thus $qa \mid d$, contradicting $\gcd(a, d) = 1$. Hence this case also supplies no nonstandard word.

For $r = 2$, the additive divisor $m = j + a$ is coprime to aj. Since $m \mid N \mid qajc$ and $c = m + a$, it follows that $m \mid q$, impossible because $m > j > qa$.

For $3 \le r \le q$, the forced run gives

$$L_r = \operatorname{lcm}_{1 \le s \le r}(j + sa) \mid \operatorname{lcm}(q, j, c) \le qjc.$$

The progression estimate and $c = j + ra$ therefore imply

$$\frac{\prod_{s=1}^{r-1}(j + sa)}{(r - 1)!} \le qj.$$

But $j > qa \ge 2q$, and $(r - 1)! \le 2^{r-2} q^{r-3}$ for $q \ge r \ge 3$. Thus $j^{r-2} > q(r - 1)!$, contradicting the displayed inequality. □

Theorem 17.5 (An even ratio between the first two coordinates)**.** *Every cyclotomic Littlewood polynomial with at most four active divisor coordinates other than 1 is standard if its first two normalized active coordinates are a, qa, with q even.*

Proof. The case $q = 2$ is Corollary 16.6. Assume $q \ge 4$. Remove the common divisor and common multiple of the support. If the first coordinate becomes 1, toggle it by $x \mapsto -x$; if an active coordinate becomes the length, discard that endpoint. Either event leaves at most three active coordinates and is settled. Otherwise the same first-pair ratio remains, with primitive support and $a > 1$. A primitive non-chain support must have a coordinate outside $a\mathbb{Z}$.

Suppose its first such coordinate j is fourth, so its support is $\{a, qa, ra, j\}$, where $r > q$ and $j > ra$. Primitivity gives $\gcd(a, j) = 1$. The full forced run gives

$$L_q = \operatorname{lcm}_{1 \le s \le q}(j + sa) \mid \operatorname{lcm}(q, r, j) \le qrj.$$

Its strict lower bound gives $r > qaj$, contradicting $r < j/a$.

It remains that the support is $\{a, qa, j, c\}$, with $qa < j < c$ and $a \nmid j$. The coprimality and resonance lemmas reduce a nonstandard word to the full forced run, and hence

$$L_q \mid \operatorname{lcm}(q, j, c) \le qjc, \qquad c > qaj.$$

Write $t = v_2(qa)$. If $v_2(j) = t$, the forced divisor $j + qa$ has dyadic valuation greater than t; thus c is the unique highest coordinate, whose extraction leaves at most three. If $v_2(j) < t$, absence of a

singleton highest layer forces $v_2(c) = t$. The highest pair is qa, c. Its gcd cannot be smaller than qa: the ghost trace would occur before qa, where the prefix contains only a and has zero trace at that dyadic level. Thus $qa \mid c$. The lower word on a, j has period $2aj$, and $c - qa > qaj - qa \geq 2aj$. The period criterion extracts an outer block of length c. If $v_2(j) > t$, absence of a singleton similarly gives highest pair j, c, whose lower coordinates a, qa form a chain. Hence $j \mid c$. That lower word has period $2qa$, and $c - j > qaj - j \geq 2qa$, so the same period criterion applies. In both cases the initial quotient has at most three active coordinates and is standard. The geometric outer block splits into prime blocks. □

17.1 The remaining odd-ratio configuration

An odd ratio leaves a different boundary coefficient. Its possible highest layers can nevertheless be reduced to one precise arrangement.

Proposition 17.6 (Highest-layer reduction for an odd initial ratio)**.** *Let P be normalized cyclotomic Littlewood with at most four active coordinates other than 1, and suppose its first two are a, qa, with $q > 1$ odd. If a is odd, then P is standard. More generally, after the common-divisor and common-multiple reductions, a nonstandard word in this class must have even a, exactly four active coordinates, and highest layer exactly $\{a, qa\}$. Its other two coordinates exceed qa and have strictly smaller dyadic valuation than a.*

Proof. The three-coordinate theorem and the common block extractions reduce to a primitive four-coordinate support $\{a, qa, j, c\}$ with $qa < j < c$. An extraction producing coordinate 1 is followed by $x \mapsto -x$; this leaves at most three coordinates. Otherwise the odd initial ratio is preserved. A singleton highest layer likewise extracts a quotient with at most three coordinates.

Suppose the highest dyadic valuation exceeds $v_2(a)$. Its only possible non-singleton layer is $\{j, c\}$. The lower coordinates a, qa form a chain, so the lower-chain ordering lemma gives $j \mid c$. The common dyadic valuation makes c/j odd, and therefore at least three. The lower coefficient word has period qa, because q is odd. Consequently $c - j \geq 2j > qa$, and the period criterion gives an outer block with a quotient having at most three coordinates. Thus P is standard. If a is odd, this settles every even length, while odd length is already classified.

It remains to consider a highest layer containing a, qa. Primitivity excludes a four-member highest layer, since a is now even. Suppose it has three members a, qa, d, and write $T = 2^{v_2(a)}$. Put $u = a/T$, $v = qa/T$, $w = d/T$, and

$$g = \gcd(u, w), \qquad h = \gcd(v, w).$$

Here $u \mid v$, and the triple gcd is g. If $g < u$, the subset-gcd identity at $Tg < a$ forces $h = g$: otherwise its nonzero exponent is $-2 + 4 = 2$, whereas the actual trace is zero before the first active coordinate. At a, the two gcd terms at g then cancel the triple term, leaving highest-layer exponent $1 - 2 = -1$. The required trace is $-2a$, but the actual first trace is $2a$, a contradiction. Thus $u \mid w$, or $a \mid d$.

Set $H = \gcd(qa, d)$. The highest-layer trace at qa is

$$2a + 2qa - 4H.$$

The actual prefix has only coordinates a, qa, whose chain boundary is R_a/R_{qa}. Its trace at qa is $2a - 2qa$. Equality forces $H = qa$, so $qa \mid d$. The three highest coordinates are therefore a chain. Their lcm is d, a proper divisor of the length, and their cardinality is odd. Highest-layer outer extraction again leaves at most three coordinates. Thus the only possible highest layer of a nonstandard word is $\{a, qa\}$, as asserted. □

17.2 Closing the odd initial ratio

Lemma 17.7 (The second lower reversal is nearby)**.** *Suppose a primitive cyclotomic Littlewood polynomial has active support $\{a, qa, j, c\}$, with $a > 1$, $q \geq 3$ odd, and $qa < j < c$. Suppose its highest coordinates are exactly a, qa. Then*

$$c < j + 2a.$$

Proof. Put $d = qa$ and

$$\gamma_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/d \rfloor}, \qquad G(x) = \sum_{n \geq 0} \gamma_n x^n, \qquad G^{[j]}(x) = \sum_{n \geq 0} \gamma_{j+n} x^n.$$

For a series $F(0) = 1$, put $\sigma_k(F) = -k[x^k] \log((1-x)F(x))$. The highest-layer formula requires $S_k = \sigma_k(G)$ at every positive multiple of a below N. The common required value is

$$2a\mathbf{1}_{a|k,\ k/a \text{ odd}} - 2d\mathbf{1}_{d|k,\ k/d \text{ odd}}.$$

Here the boundary of G is R_a/R_d, where $R_h = (1-x^h)/(1+x^h)$.

The lower coordinate j is not divisible by a. Set $k_0 = a\lceil j/a \rceil$. Choose $k = k_0$ if $d \nmid k_0$, and $k = k_0 + a$ otherwise. Then

$$j < k < j + 2a < 2j.$$

Suppose $c > k$. Below degree $2j$, the reversal at j occurs only once, and no reversal at c has occurred through degree k. Consequently, through that degree,

$$P = G - 2x^j G^{[j]}, \qquad S_k - \sigma_k(G) = 2k[x^{k-j}] \frac{G^{[j]}}{G}.$$

Since $q \geq 3$,

$$\frac{1}{G} = (1-x)(1 + 2x^a + O(x^{2a})).$$

Writing $\Delta\gamma_n = \gamma_n - \gamma_{n-1}$, the required coefficient is $\Delta\gamma_k$ in the first case, and $\Delta\gamma_k + 2\Delta\gamma_{k-a}$ in the second. In the second case the latter difference vanishes, since $k - a = k_0$ is a multiple of d and both highest reversals cancel there. In either case the remaining difference has absolute value two. Thus the trace discrepancy has absolute value $4k$, a contradiction. The tested index is interior because $k < c < N$ under the assumed inequality. Therefore $c \leq k < j + 2a$. □

Lemma 17.8 (Coprimality for an odd initial ratio)**.** *Under the hypotheses of Lemma 17.7, one has* $\gcd(a, j) = 1$.

Proof. Suppose $a = gu, j = gv$, where $g > 1$ and $\gcd(u, v) = 1$. Primitivity gives $\gcd(g, c) = 1$. Since j has smaller dyadic valuation than a, u is even and v is odd. In particular $u \geq 2$ and $v > qu$.

For $1 \leq s < q$, the off-support logarithmic coefficient at $j + sa$ is nonzero. Indeed the reciprocal boundary of the initial chain is

$$\frac{1}{(1-x)G(x)} = \frac{1+x^a}{1-x^a} \frac{1-x^{qa}}{1+x^{qa}},$$

whose coefficient at sa is two for $1 \leq s < q$. A contribution from the fourth coordinate would require $c \equiv j \pmod{a}$, impossible because $\gcd(g, c) = 1$. All tested indices lie below $2j$, so at most one off-support boundary term occurs in each logarithmic monomial. Thus these coefficients equal

$2b_j \neq 0$. Trace reduction then forces $j + sa \mid N$: a proper gcd index is below j and outside $a\mathbb{Z}$, where its trace vanishes. It follows that

$$L = \operatorname{lcm}_{1\le s<q}(v + su) \mid \operatorname{lcm}(q, v, c), \qquad \gcd(L, u) = 1.$$

Here $N/g = \operatorname{lcm}(qu, v, c)$ and the coprimality with u removes its factor from the divisor assertion.

The first coordinate outside $g\mathbb{Z}$ is c, so the additive obstruction gives $a + c \mid N$. Put $h = \gcd(a, c) = \gcd(u, c)$ and $z = (a + c)/h$. Comparison of prime valuations gives $z \mid qj$; since $\gcd(z, g) = 1$, one has $z \mid qv$. Hence

$$c/h = z - a/h < qv.$$

Since $h \mid u$ and $\gcd(L, u) = 1$, the relation $L \mid qvc$ therefore gives

$$L \mid qv(c/h), \qquad L < q^2v^2.$$

For $q \ge 5$, Lemma 17.1 with $q - 1 \ge 4$ terms gives

$$L > (q - 1)^2uv^2 \ge 2(q - 1)^2v^2 > q^2v^2,$$

a contradiction.

If $q = 3$, the two terms $v + u, v + 2u$ are coprime to one another and, because v is odd, both are coprime to v. Thus their product L divides $\operatorname{lcm}(3, c)$, and consequently $L \mid 3(c/h)$ and $L < 9v$. But

$$L = (v + u)(v + 2u) \ge (v + 2)(v + 4) > 9v,$$

since $v^2 - 3v + 8 > 0$. This is again a contradiction. □

Lemma 17.9 (Exclusion of the odd-ratio highest pair)**.** *There is no primitive cyclotomic Littlewood polynomial with active support $\{a, qa, j, c\}$, where $a > 1$, $q \ge 3$ is odd, $qa < j < c$, and the highest coordinates are exactly a, qa.*

Proof. The preceding lemmas give $\gcd(a, j) = 1$ and $c < j + 2a$. The highest coordinate a is even, so j is odd. Let $m_1 = j + a$ and $m_2 = j + 2a$. Both are coprime to a and j, and they are coprime to one another.

Unless $c = j+a$, the reciprocal-boundary argument in the preceding proof gives both $m_1, m_2 \mid N$. Indeed its first two coefficients are two because $q \ge 3$, and the only possible intervening resonance would be $c = j + a$; the alternative $c = j + 2a$ is excluded by nearness. Thus

$$m_1m_2 \mid \operatorname{lcm}(q, c), \qquad m_1m_2 \le qc < qm_2,$$

which would imply $m_1 < q$, contrary to $j > qa$.

It remains to consider $c = j + a$. If $b_c = b_j$, the same nonzero trace at m_2 supplies the same contradiction. Thus suppose $b_c = -b_j$. The high-chain signs at j and $j + a$ show that this equality holds only when an endpoint of a qa-block is crossed: writing $m = \lfloor j/a \rfloor$, one has $m \equiv q - 1 \pmod q$. Since $j > qa$, it follows that $j > (2q - 1)a$. The reciprocal boundary has coefficients zero at qa and minus two at $(q + 1)a$. Its convolution with the two off-support terms $b_jx^j - b_jx^{j+a}$ therefore has nonzero coefficients at

$$n = j + qa, \qquad p = j + (q + 1)a.$$

Both indices are below $2j$, so trace reduction gives $n, p \mid N$.

Since $\gcd(a, j) = 1$, n is coprime to a. The relation $n \mid \operatorname{lcm}(q, j, c)$ can be sharpened to $n \mid \operatorname{lcm}(q, c)$: at any prime, if the valuation in j exceeds that in q, the valuation in $n = j + qa$ equals that in q; otherwise j contributes no valuation beyond q. Moreover $\gcd(n, c) \mid q - 1$, so $n \mid q(q-1)$. Similarly $p = c + qa \mid \operatorname{lcm}(q, j)$ and $\gcd(p, j) \mid q + 1$, giving $p \mid q(q+1)$. Put

$$R = \frac{q(q-1)}{n}, \qquad S = \frac{q(q+1)}{n+a}.$$

These are positive integers, and subtraction gives

$$2q = (S - R)n + Sa.$$

But $n > 2qa \geq 4q$, while $Sa < (q+1)/2 < 2q$. If $S - R \geq 1$, the right side exceeds $2q$; if $S - R \leq -1$, it is negative. Thus $S = R$, giving $Sa = 2q$ and $n = a(q-1)/2$, again contradicting $n > 2qa$. This excludes the final case. □

Theorem 17.10 (Comparable first coordinates)**.** *Every cyclotomic Littlewood polynomial with at most four active coordinates other than* 1 *is standard if its first two normalized active coordinates are comparable by divisibility.*

Proof. Write the first two coordinates as a, qa, with $q \geq 2$. An even ratio is covered by Theorem 17.5. For odd q, Proposition 17.6 reduces any counterexample to the primitive highest-pair configuration excluded by Lemma 17.9. □

Consequently, the first two coordinates of any least-length four-coordinate counterexample must be incomparable. This conclusion does not classify all such incomparable supports.

18 Two highest coordinates at low dyadic height

A different exclusion applies to a lower divisor chain. The separation bound in Theorem 15.2 can be removed at lengths $2M$ and $4M$, with M odd, when the lower layer is a chain. The additional input is local: the first reversal across an incompatible scale creates a trace which the highest layer cannot supply.

Lemma 18.1 (Ordering the lower chain)**.** *Let P be a normalized primitive cyclotomic Littlewood polynomial of length $N = 2^t M$, with M odd and $t \geq 1$. Suppose its active coordinates are $\{a, b, e, c\}$, where $a \mid b$ are the lower coordinates and $e < c$ are the highest coordinates. Here primitive means that the active coordinates have gcd one and lcm N. Then $a < e$.*

Proof. Lemma 14.2 gives $e \mid c$. Thus c/e is odd and at least three. Moreover $\gcd(a, e) = \gcd(a, b, e, c) = 1$, so a is odd and $a \neq e$. Write $T = 2^t$ and set

$$\gamma_n = (-1)^{\lfloor n/e \rfloor + \lfloor n/c \rfloor}, \qquad G(x) = \sum_{n \geq 0} \gamma_n x^n, \qquad G^{[a]}(x) = \sum_{n \geq 0} \gamma_{a+n} x^n.$$

The chain-boundary identity gives

$$(1 - x)G(x) = \frac{R_e(x)}{R_c(x)}, \qquad R_d(x) = \frac{1 - x^d}{1 + x^d}.$$

The highest-layer trace formula therefore requires

$$S_k = \sigma_k(G) = 2e\mathbf{1}_{e \mid k,\ k/e \text{ odd}} - 2c\mathbf{1}_{c \mid k,\ k/c \text{ odd}}, \qquad T \mid k, \quad 0 < k < N. \tag{18.1}$$

where $\sigma_k(F) = -k[x^k]\log((1-x)F(x))$ for a series $F(0) = 1$.

Suppose $a > e$. Since $b \geq 2a$, only the first lower reversal has entered before degree $2a$. Consequently

$$P = G - 2x^a G^{[a]} + O(x^{2a}), \qquad S_k - \sigma_k(G) = 2k[x^{k-a}]\frac{G^{[a]}}{G} \quad (a < k < 2a).$$

Put $n_0 = e\lceil a/e\rceil$. Coprimality gives $a < n_0 < a + e$. If $c \nmid n_0$, choose $k = n_0$; otherwise choose $k = n_0 + e$. In both cases $T \mid k$ and $k < 2a$. For the second choice, n_0 is a positive multiple of c, so $a > c - e \geq 2e$, whence $k < a + 2e < 2a$.

To evaluate the ratio, note that $c \geq 3e$ gives

$$\frac{1}{G(x)} = (1-x)(1 + 2x^e + O(x^{2e})).$$

Write $\Delta\gamma_n = \gamma_n - \gamma_{n-1}$. For the first choice, $0 < k - a < e$, so the required coefficient is $\Delta\gamma_k$, of absolute value two. For the second, $e < k - a < 2e$, and it is

$$\Delta\gamma_k + 2\Delta\gamma_{k-e}.$$

The second term vanishes because $k - e = n_0$ is a multiple of c: the reversals at e and c cancel there. The first term again has absolute value two. Thus $|S_k - \sigma_k(G)| = 4k$, contradicting (18.1). All tested indices are interior because $k < 2a \leq b < N$. Hence $a < e$. □

Theorem 18.2 (A lower chain at twice odd length)**.** *Let P be cyclotomic Littlewood of length $2M$, with M odd, and with at most four active coordinates other than 1. If its highest layer has two members and its remaining coordinates form a divisibility chain, then P is standard.*

Proof. Normalize $a_0 = a_1 = 1$. The theorem for at most three coordinates leaves exactly four. Suppose a counterexample exists and choose one of least length within the stated class. The common-divisor and common-multiple factorizations preserve the two-member highest layer and the lower chain unless a coordinate disappears; in that event the quotient has at most three active coordinates and is standard. A coordinate becoming 1 can likewise be removed by $x \mapsto -x$. Thus minimality gives a primitive support

$$\mathcal{A}(P) = \{a, b, e, c\}, \qquad a \mid b, \quad e \mid c, \quad a < b, \quad e < c, \quad \gcd(a, e) = 1.$$

Here a, b are odd and e, c have dyadic valuation one. The preceding lemma gives $a < e$.

The lower-chain sequence has period b, since b/a is odd. If $c - e \geq b - \gcd(e, b)$, Theorem 15.2 already gives standardness. Otherwise

$$b > c - e + \gcd(e, b) \geq 2e.$$

Write $e = 2u$ and $e = qa + r$, with $0 < r < a$. Put

$$k = 2a\lceil u/a\rceil.$$

Coprimality and $a < 2u$ give $e < k < 2e$ and $e \nmid k$. Indeed, if $a \leq u$, then $a\lceil u/a\rceil < u + a \leq 2u$; if $u < a < 2u$, the ceiling is one. Also $\lceil u/a\rceil < u$, so $u \mid k/2$ would contradict $\gcd(a, u) = 1$.

Both b and c exceed $2e$, so below that degree only the coordinates a, e affect the word. Define

$$f(x) = \frac{G_a(x)}{1 + x^a}, \qquad f_e(x) = \sum_{n \geq 0} (-1)^{\lfloor (e+n)/a \rfloor} x^n.$$

The shifted square wave gives the exact identity

$$\frac{f_e(x)}{f(x)} = (-1)^q \frac{1 - 2x^{a-r} + x^a}{1 - x^a}.$$

The reversal at e therefore gives

$$\log((1-x)P) = \log R_a - 2x^e \frac{f_e}{f} + O(x^{2e}).$$

For $d = k - e$, one has $d > 0$, $a \nmid d$, and $d \equiv -r \pmod a$. Hence $[x^d](f_e/f) = -2(-1)^q$. Since k/a is even, the trace of R_a at k vanishes, leaving

$$S_k = -4k(-1)^q \neq 0.$$

But (18.1) gives zero: $e \nmid k$ and $k < c$. This contradiction proves the theorem. □

The same local trace argument reaches the second dyadic level. The only additional point is that the second lower coordinate may enter before the tested trace. Its contribution can be retained explicitly, and has the wrong dyadic valuation to affect that trace.

Lemma 18.3 (A lower-chain trace below the second highest reversal)**.** *Let P be a primitive cyclotomic Littlewood polynomial with active coordinates $\{a, b, e, c\}$, where $a \mid b$ are the lower coordinates and $e < c$ are highest. Write $T = 2^{v_2(N)}$ and $e = Tu$. If the separation criterion does not apply, then*

$$a < e, \qquad b > e, \qquad \left\lfloor \frac{e}{a} \right\rfloor \equiv 0 \pmod 2, \qquad a \geq \frac{2e}{T}.$$

Proof. Lemma 18.1 gives $a < e$; Lemma 14.2 gives $e \mid c$, with $c/e \geq 3$ odd. Primitivity gives $\gcd(a, e) = 1$, so a is odd. The lower-chain sequence has period at most $2b$. If $b \leq e$, the inequality $c - e \geq 2e \geq 2b$ would imply the separation criterion. Thus $b > e$.

Set $f(x) = G_a(x)/(1 + x^a)$ and $f_e(x) = \sum_{n\geq 0}(-1)^{\lfloor (e+n)/a \rfloor} x^n$. Writing $e = qa + r$, with $0 < r < a$, gives

$$\frac{f_e}{f} = (-1)^q \frac{1 - 2x^{a-r} + x^a}{1 - x^a}.$$

Since $b > e$ and $c \geq 3e$, comparison at the first highest coordinate gives $S_e = 2e(-1)^q$. The highest-layer identity requires $S_e = 2e$, so q is even.

Suppose $a < 2u$. Define $k = Ta\lceil u/a \rceil$. Coprimality gives $e < k < 2e$ and $e \nmid k$. The same ceiling inequalities used in the twice-odd proof apply, with 2 replaced by T in the definition of k. Indeed $a\lceil u/a \rceil < 2u$, and the ceiling is smaller than u, while $\gcd(a, u) = 1$. The lower coordinate b may lie below k, so it must be retained explicitly. Put $s = b/a$ and $f_b(x) = \sum_{n\geq 0}(-1)^{\lfloor (b+n)/a \rfloor} x^n$. One has $f_b = (-1)^s f$. Up to degree $2e - 1$, the reversal at e occurs once, and, if $b < 2e$, its reversal occurs once after it. Thus

$$\frac{P}{f} = 1 - 2x^e \frac{f_e}{f} + 2(-1)^s x^b + O(x^{2e}),$$

where the displayed b-term may be omitted when $b \geq 2e$. Every product of two correction terms has degree at least $2e$. Taking the logarithm therefore gives

$$\log((1-x)P) = \log R_a - 2x^e \frac{f_e}{f} + 2(-1)^s x^b + O(x^{2e}).$$

At the chosen index k, the b-term vanishes because $v_2(b) < v_2(k)$. The trace of R_a vanishes because $k/a = T\lceil u/a \rceil$ is even. As $k - e$ is positive, not divisible by a, and congruent to $-r$ modulo a,

$$S_k = -4k(-1)^q = -4k \neq 0.$$

The highest-layer trace at k is zero, since $e \nmid k$ and $k < c$. This contradiction proves $a \geq 2u = 2e/T$. All indices are interior because $k < 2e < c < N$. □

Theorem 18.4 (A lower chain through the second dyadic level)**.** *Let P be cyclotomic Littlewood with at most four active coordinates other than 1. Suppose its length has dyadic valuation at most two. If its highest layer has two members and the remaining coordinates form a divisibility chain, then P is standard.*

Proof. Odd length and at most three active coordinates are already classified. Assume otherwise and use the common-divisor and common-multiple factorizations to choose a primitive least-length counterexample. These reductions preserve the lower-chain configuration unless a coordinate disappears or becomes 1, in which case at most three coordinates remain after normalization. They do not increase the dyadic valuation of the length.

Write the primitive support as $\{a, b, e, c\}$ in the preceding lemma's notation. If the separation criterion applies, the standard factorization is already known. Otherwise that lemma gives $a < e$ and an even positive integer $\lfloor e/a \rfloor$. Hence $e/a > 2$ and $a < e/2$. But T is either two or four, so

$$a < \frac{e}{2} \leq \frac{2e}{T},$$

contradicting the same lemma. This proves the theorem. □

Proposition 18.5 (Additive divisors in the lower-chain sector)**.** *In the primitive lower-chain configuration of Lemma 18.3, suppose the separation criterion fails. Write $e = Tu$, $b = as$, and $c = ev$. For every integer j with $1 \leq j \leq \lfloor e/a \rfloor$,*

$$e + ja \mid N.$$

If $T \geq 4$, then $s > T$, and therefore $b > Ta$.

Proof. The preceding lemma gives $b > e$, $a < e/2$, and even $q = \lfloor e/a \rfloor$. Its expansion modulo x^{2e} applies even if b lies in that interval. At $n = e + ja < 2e$, the b-term vanishes since $a \nmid n$, and the shifted-ratio coefficient is $2(-1)^q = 2$. Hence $S_n = 4n \neq 0$. If $n \nmid N$, trace reduction either gives zero or identifies this trace with S_d, where $d = \gcd(N, n) \leq n/2 < e$. But $a \nmid d$, since $a \nmid n$. Before e, only the coordinate a has entered, so $S_d = 0$. This contradiction proves every displayed divisibility.

In particular $e + a$ and $e + 2a$ divide N. They are coprime, and, since $T \geq 4$, the integer

$$Q = \frac{(e + a)(e + 2a)}{2}$$

is coprime to ae. Consequently $Q \mid \operatorname{lcm}(s, v)$ and $Q \leq sv$. Failure of separation also gives $e(v - 1) < 2b = 2as$. If $s \leq T$, then

$$Q \leq sv < s\left(\frac{2as}{e} + 1\right) \leq T\left(\frac{2a}{u} + 1\right).$$

This is impossible. If $u \geq 3$, the term $3ae/2$ in Q alone exceeds $2aT/u$, while $e^2/2 > T$. If $u = 1$, use $a < T/2$ to obtain

$$Q - (2aT + T) = \frac{T^2 - aT}{2} + a^2 - T > \frac{T^2}{4} - T \geq 0.$$

Thus $s > T$. □

A primitive four-coordinate counterexample at dyadic valuation one or two must therefore have either two incomparable lower coordinates or three highest coordinates and one lower coordinate. At greater dyadic height, the lower-chain case is confined to

$$\frac{2e}{T} \le a < \frac{e}{2}, \qquad b > Ta, \qquad \left\lfloor \frac{e}{a} \right\rfloor \equiv 0 \pmod 2,$$

with the separation criterion still failing. The unrestricted four-coordinate classification remains open.

19 A three-member highest layer with one small odd coordinate

The first few highest-layer traces can exclude a whole non-chain configuration, rather than only impose a divisibility condition.

Theorem 19.1 (The odd coordinate 3 and a dyadic highest coordinate)**.** *There is no normalized cyclotomic Littlewood polynomial whose active coordinates are exactly*

$$\{3, T, Tv, Tw\}, \qquad T = 2^t \ge 2, \qquad 1 < v < w, \qquad v, w \text{ odd.}$$

The length and the odd integers v, w are unrestricted.

Proof. Proposition 5.5 reduces the common multiple to length $N = T\,\mathrm{lcm}(3, v, w)$. If this extraction removes an endpoint coordinate, at most three coordinates remain. The three-coordinate theorem and the canonical chain criterion then contradict the incomparable coordinates $3, T$ in the original support. We may therefore suppose that all four coordinates remain and $N = T\,\mathrm{lcm}(3, v, w)$. The highest dyadic layer has odd parts $1, v, w$.

We first calculate the requisite traces from the coefficient word. Put

$$B(x) = \sum_{n \ge 0} (-1)^{\lfloor n/3 \rfloor} x^n = \frac{1 + x + x^2}{1 + x^3}.$$

Let $B_r(x)$ be the generating series of the shifted sequence $(-1)^{\lfloor (n+r)/3 \rfloor}$, and write $Q_r = B_r/B$. Only r modulo six matters. In particular,

$$Q_0 = 1, \qquad Q_2 = \frac{1 - x - x^2}{1 + x + x^2} = -1 + \frac{2(1 - x)}{1 - x^3},$$

$$Q_4 = \frac{-1 - x + x^2}{1 + x + x^2} = 1 - \frac{2(1 - x^2)}{1 - x^3}.$$

Before the next highest coordinate enters, multiplying the coefficient sequence by $(-1)^{\lfloor n/T \rfloor}$ reverses it at $T, 2T, 3T$. Thus its generating series P_0 satisfies

$$\frac{P_0(x)}{B(x)} = 1 - 2x^T Q_T(x) + 2x^{2T} Q_{2T}(x) - 2x^{3T} Q_{3T}(x) + O(x^{3T+1}).$$

Here P_0 denotes the infinite word with coordinates $3, T$; the displayed coefficient calculation does not assume it cyclotomic. Since the boundary of B is $R_3(x)$, all its traces at positive even indices vanish.

If t is odd, then $T \equiv 2 \pmod 6$. At degree $2T$, the logarithm of the last display gives

$$[x^{2T}] \log(P_0/B) = -2[x^T]Q_2 + 2Q_4(0) - 2Q_2(0)^2 = -4.$$

Indeed $T \equiv 2 \pmod 3$ gives $[x^T]Q_2 = 0$. Hence $S_{2T} = 8T$. The other two highest coordinates are at least $3T$, so they do not change this trace. The cyclotomic zero-trace identity instead requires $S_{2T} = 0$, a contradiction.

Now let t be even, so $T \equiv 4 \pmod 6$ and $T \geq 4$. At degree T the same calculation gives $S_T = -2T$. Write $h = \gcd(v, w)$. Theorem 15.4 simplifies the highest-layer product identity to

$$(1-y)H(y) = (1+y^M)\frac{R_1(y)R_v(y)R_w(y)}{R_h(y)^2}, \qquad M = N/T,$$

where H is the terminal norm. Its trace at one, multiplied by T, is $-2T$ if $h = 1$ and $2T$ if $h > 1$. Therefore $h = 1$.

It remains to compare the trace at $3T$. The logarithmic expansion is

$$\begin{aligned}[x^{3T}]\log(P_0/B) = &-2[x^{2T}]Q_4 + 2[x^T]Q_2 - 2[x^T]Q_4^2 \\ &- 2Q_0(0) + 4Q_4(0)Q_2(0) - \frac{8}{3}Q_4(0)^3.\end{aligned}$$

The rational forms above give

$$[x^{2T}]Q_4 = 2, \qquad [x^T]Q_2 = -2, \qquad [x^T]Q_4^2 = \frac{4(T-1)}{3}.$$

The last identity follows by squaring $Q_4 = 1 - 2(1-x^2)/(1-x^3)$ and using $T \equiv 1 \pmod 3$. Since $Q_4(0) = -1$, $Q_2(0) = Q_0(0) = 1$, the logarithmic coefficient is $-(8T+26)/3$. Consequently

$$S_{3T}(P_0) = T(8T+26).$$

If $v > 3$, neither remaining coordinate has entered by $3T$, and this is the trace of P. The highest-layer identity with $h = 1$ instead requires $S_{3T} = -2T$. If $v = 3$, the coordinate $3T$ reverses its coefficient from -1 to 1; no earlier coefficient changes. It therefore decreases S_{3T} by $6T$, giving $S_{3T} = T(8T+20)$. The highest-layer identity instead requires $S_{3T} = 4T$. Both comparisons are impossible. All tested indices are interior, because $M \geq w \geq 5$. This completes the proof. □

Remark 19.2 (Scope). This is a classification of an infinite three-highest-coordinate sector. It does not classify a general odd coordinate, nor a smallest highest coordinate Tu with odd $u > 1$. In those cases another highest coordinate can enter before $2Tu$, and its boundary reversal can alter the trace used here.

20 Three highest coordinates

Lemma 20.1 (Rotation of an odd conductor)**.** *For odd positive integers o, d, put $D = \operatorname{lcm}(o, d)$. Every primitive $2d$th root is an oth-root rotation of a primitive $2D$th root.*

Proof. After removing a minus sign, start with a primitive dth root η and seek an oth root ξ such that $\eta\xi$ has order D. At each prime, write $b = v_p(d)$ and $c = v_p(o)$. If $b > c$, a component of order at most p^c cannot cancel the primitive p^b component of η. If $b = c$, choose the component of ξ to be one. If $b < c$, choose that component to have exact order p^c. These choices combine into an oth root by the Chinese remainder theorem. Restoring the minus sign gives the assertion, since all these orders are odd. □

Lemma 20.2 (The period of a square-wave quotient)**.** *Let q be odd, let $\gcd(a,q)=1$, and put*

$$A_\ell = (-1)^{(J+a\ell) \bmod q},$$

where residues are taken in $\{0,\dots,q-1\}$. For an integer p, the sequence $A_{\ell+p}/A_\ell$ is constant if $q \mid p$, and otherwise has least positive period q.

Proof. Put $b = ap \bmod q$. If $b=0$, the quotient is one. Otherwise, with $z=(J+a\ell) \bmod q$, it equals

$$(-1)^b(-1)^{\mathbf{1}_{z\ge q-b}}.$$

This is the sign of a nonempty proper cyclic interval, up to a constant sign and an invertible reindexing. A nonempty proper cyclic interval has exactly two transitions on the cycle. If its period were a proper divisor s of the odd integer q, these transitions would consist of $q/s \ge 3$ identical copies, which is impossible. Every period can be reduced to its gcd with q, so its least period is q. An invertible affine reindexing preserves this property. □

Theorem 20.3 (Classification of a three-member highest layer)**.** *Every cyclotomic Littlewood polynomial with at most four active divisor coordinates other than 1, of which exactly three are highest, is standard.*

Proof. Suppose otherwise and choose a least-length counterexample in this class. Theorem 15.3 and Proposition 5.5 give primitive support. Thus

$$N = TM, \qquad T = 2^t \ge 2, \qquad \mathcal{A}(P) = \{o, Tu, Tv, Tw\},$$

where $o>1$ is odd, u,v,w are distinct odd proper divisors of M, and $\gcd(o,u,v,w)=1$. Indeed an even remaining coordinate would give a common factor two; a coordinate becoming 1 may be toggled away, leaving at most three coordinates. Corollary 13.4 also gives

$$M = \operatorname{lcm}(u,v,w), \qquad o \mid M.$$

To justify this last reduction, its smaller initial quotient either loses a coordinate, and is classified, or retains the same three highest coordinates and the odd coordinate. If the odd coordinate lies inside that quotient, its change at the odd index o and gcd invariance force o to divide the quotient length. Thus the reduction preserves this class whenever four coordinates remain.

Define the odd-length sign word

$$\beta_k = (-1)^{\lfloor k/u\rfloor+\lfloor k/v\rfloor+\lfloor k/w\rfloor}, \qquad B(z) = \sum_{k=0}^{M-1} \beta_k z^k, \qquad \sigma(n) = (-1)^{\lfloor n/o\rfloor}.$$

The residue polynomials of P modulo T are

$$P_r(z) = \sum_{k=0}^{M-1} \beta_k \sigma(Tk+r) z^k, \qquad P(x) = \sum_{r=0}^{T-1} x^r P_r(x^T).$$

For the selected highest multiplicities put

$$K(z) = \prod_{\substack{d\mid M,\ d>1\\ s_d=1}} \Phi_{2d}(z), \qquad s_d = \big(\#\{h\in\{u,v,w\} : d \nmid h\}\big) \bmod 2.$$

Theorem 13.3 and $\Phi_{2d}(x^T) = \Phi_{2Td}(x)$ give $K(x^T) \mid P(x)$, hence $K \mid P_r$ for every r. In particular,

$$P_1(z) - P_0(z) = \varepsilon\, 2C_j(z), \qquad C_j(z) = \sum_{\substack{0 \le k < M \\ k \equiv j \pmod{o}}} \beta_k z^k, \quad j \equiv -T^{-1} \pmod{o}, \quad \varepsilon \in \{-1, 1\}.$$

The square wave changes exactly when $o \mid Tk+1$. Its change has constant sign on this residue class, since T is even. Therefore $K \mid C_j$. If α is a root of K and $\xi^o = 1$, then $C_j(\xi\alpha) = \xi^j C_j(\alpha) = 0$.

If $s_o = 1$, a primitive $2o$th root and an oth-root rotation would force $C_j(-1) = 0$. But this value is a sum of the odd number M/o of signs. Thus $s_o = 0$. Exactly one of u, v, w is divisible by o: an odd number must be divisible, and all three would contradict primitivity. Rename this member u.

For any $d \mid M$ with $d \nmid u$, put $D = \operatorname{lcm}(o, d)$. None of u, v, w is divisible by D, so $s_D = 1$. Lemma 20.1 and the rotation property of C_j imply $\Phi_{2d} \mid C_j$. Hence

$$F(z) := \prod_{\substack{d \mid M \\ d \nmid u}} \Phi_{2d}(z) = G_{M/u}(-z^u) \mid C_j(z). \tag{20.1}$$

Since $o \mid u$, this divisor is a polynomial in z^o. The quotient in (20.1) is supported on the residue j modulo o and has degree $u - o + j < u$. The nonoverlapping shifts by u therefore force the sampled sequence $\beta_{j+\ell o}$ to reverse sign after u/o steps. The assertion extends to all integer ℓ, because $\beta_{k+M} = -\beta_k$ and M/u is odd. Removing the coordinate u, we find that

$$\gamma_\ell := (-1)^{\lfloor (j+o\ell)/v \rfloor + \lfloor (j+o\ell)/w \rfloor}$$

has period $p = u/o$.

Set

$$r = \gcd(o, v), \quad s = \gcd(o, w), \quad q = v/r, \quad q' = w/s.$$

For the first factor, put $J = \lfloor j/r \rfloor$ and $a = o/r$. Then $\gcd(a, q) = 1$, and

$$(-1)^{\lfloor (j+o\ell)/v \rfloor} = (-1)^{\lfloor (J+a\ell)/q \rfloor} = (-1)^J (-1)^\ell (-1)^{(J+a\ell) \bmod q}.$$

The analogous identity for w shows that γ is a constant sign times a product $A_\ell A'_\ell$ of the two sequences in Lemma 20.2. Since this product has period p, their quotients under shift p agree. If these quotients are constant, then $q, q' \mid p$. It follows that $v = rq \mid op = u$ and $w = sq' \mid u$, contradicting $M = \operatorname{lcm}(u, v, w) > u$.

Otherwise both quotient sequences are nonconstant. Their least periods are q and q', so the equality of the quotients forces $q = q' = h$. Thus $v = rh$ and $w = sh$. Primitivity, together with $o \mid u$, gives

$$\gcd(r, s) = 1, \qquad \gcd(h, o) = 1.$$

Put $g = \gcd(u, h)$. If $h \mid u$, both v, w divide u and the same contradiction follows. Hence $g < h$. Since $o \mid u$ and $\gcd(h, o) = 1$, also $g < u$. The three pairwise gcds of the highest odd parts are exactly

$$\gcd(u, v) = rg, \qquad \gcd(u, w) = sg, \qquad \gcd(v, w) = h,$$

and their common gcd is $g < \min(u, v, w)$. As $r \neq s$, at most one of these pairwise gcds equals g. But Corollary 15.5 requires exactly two pairwise gcds to equal the common gcd. This final contradiction completes the classification. □

21 Completion at twice odd length

21.1 When the smallest coordinate is highest

Theorem 21.1 (A smallest highest coordinate at twice odd length)**.** *Let P be cyclotomic Littlewood of length $2M$, with M odd, and with at most four active coordinates other than 1. If its highest layer has two members and contains its smallest normalized active coordinate, then P is standard.*

Proof. Choose a least-length counterexample within the stated class. Common-divisor and common-multiple extraction preserve the ordering and the two-member highest layer unless a coordinate disappears or becomes 1; either event leaves at most three coordinates after normalization. Thus the support is primitive. Write its highest coordinates as $a < d$, and the remaining coordinates as $b < c$. Here a is even, b, c are odd, and $a < b$. If $h = \gcd(a, d) < a$, the highest-pair trace at h is $4h$, whereas the actual trace before the first coordinate is zero. Consequently $a \mid d$; write $d = qa$ with $q \geq 3$ odd.

Define the high-chain series

$$\gamma_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/d \rfloor}, \qquad G(x) = \sum_{n \geq 0} \gamma_n x^n, \qquad B(x) = (1-x)G(x) = \frac{R_a(x)}{R_d(x)}.$$

Since a, d are even, B is even. For a formal series $F(0) = 1$, write $\sigma_k(F) = -k[x^k] \log((1-x)F(x))$. The highest-layer identity requires $S_k = \sigma_k(G)$ at every positive even interior index.

Put $k_0 = a\lceil b/a \rceil$. If $d \nmid k_0$, set $k = k_0$; otherwise set $k = k_0 + a$. Then $b < k < 2b$: for the second case, $b > k_0 - a \geq (q-1)a \geq 2a$, so $k < b + 2a < 2b$. If $c > k$, the calculation of Lemma 17.7, with $j = b$, gives a trace discrepancy of absolute value $4k$. Its proof uses only $b < k < 2b$ and remains valid whether $b < d$ or $b > d$. Therefore $c \leq k$; equality is impossible because c is odd and k is even. In particular

$$b < c < k < b + 2a, \qquad c < 2b.$$

The high-chain signs are constant from b through c: the first noncancelled high boundary after b is exactly k. Write that common sign as $\sigma \in \{-1, 1\}$.

Let $D = (1-x)P = D_{\mathrm{e}} + D_{\mathrm{o}}$ be its even and odd parts. Up to degree $b + c < 3b$, its odd part is

$$D_{\mathrm{o}} = -2\sigma x^b + 2\sigma x^c + O(x^{3b}).$$

The even difference $E = D_{\mathrm{e}} - B$ begins at degree $2b$. Through degree $b + c$, its support consists only of $2b$ and multiples of a: the second reversal at b is the only even lower reversal before $b + c < 2c$, and every high boundary is an a-multiple. All series identities here are understood below the length, whenever a tested index is interior.

Set $n = b + c$. This is an interior index, since b, c are distinct proper divisors of N and hence $b + c < N$. The even part of the logarithm, through degree $n < 3b$, gives

$$[x^n](\log D - \log B) = [x^n] \left(\frac{E}{B} - \frac{D_{\mathrm{o}}^2}{2B^2} \right).$$

If neither $a \mid n$ nor $a \mid (c - b)$, the first term is zero, because B^{-1} is supported on $a\mathbb{Z}$. In the second term the x^{2b} contribution also vanishes for the same reason, while the cross term contributes 4. This contradicts equality of the even traces. Therefore

$$a \mid b + c \quad \text{or} \quad c = b + a,$$

where the second alternative uses $0 < c - b < 2a$. Either alternative and primitivity imply $\gcd(a, b) = 1$.

Suppose first that $c \neq b + a$. Put $m_1 = b + a$ and $m_2 = b + 2a$. Both are odd and less than $3b$. The odd part of $\log D$ at these indices is $D_{\mathrm{o}}/D_{\mathrm{e}}$, because cubic odd terms begin at $3b$. Since E begins at $2b > 2a$, the required reciprocal coefficients equal those of B^{-1}, namely two at a and $2a$. No c-term contributes: $c < b + 2a$ and $c \neq b + a$. Thus both traces are nonzero. They are interior since the additive obstruction gives $N > 3b$. Trace reduction forces $m_1, m_2 \mid N$: a proper gcd of either odd index is at most one third of it, thus below b, and odd, where the trace is zero. The two divisors are coprime to each other and to ab, so

$$m_1 m_2 \mid \operatorname{lcm}(q, c), \qquad m_1 m_2 \leq qc < q m_2.$$

Consequently $q > b + a$. In particular $d = qa > b + c$. There is no cancelled high boundary before c, so b, c lie in one open a-cell. We already know $a \mid b + c$; hence $n = b + c$ is the unique odd multiple of a between $2b$ and $2c$. Since $n < d$, its baseline boundary coefficient is -2. The lower signs reverse it to 2, so $E_n = 4$. There are no earlier a-multiples in $[2b, n)$, and $a \nmid (c - b)$, so

$$[x^n](\log D - \log B) = 4 + 4 = 8,$$

again contradicting equality of even traces.

It remains that $c = b + a$. Nearness now forces k_0 to be a cancelled boundary, hence a positive multiple of d. Thus $b > (q - 1)a$. Set

$$u = b + qa, \qquad v = b + (q + 1)a.$$

These odd indices are less than $3b$. The reciprocal B^{-1} has coefficients two at $(q - 1)a$, zero at qa, and minus two at $(q + 1)a$. The two odd boundary terms have opposite signs. The odd logarithmic coefficients at u, v are consequently both 4σ, and their traces are nonzero. Their indices are interior because $N > 3b$, and their proper gcd indices would be odd and below b. Thus $u, v \mid N$.

For completeness, the resulting arithmetic is impossible. Since $\gcd(a, b) = 1$, prime valuations in $u = b + qa \mid \operatorname{lcm}(qa, b, c)$ remove both a and b, giving $u \mid \operatorname{lcm}(q, c)$. As $\gcd(u, c) \mid q - 1$, one has $u \mid q(q - 1)$. Similarly $v = c + qa$ divides $\operatorname{lcm}(q, b)$, and $\gcd(v, b) \mid q + 1$, so $v \mid q(q + 1)$. The positive integers

$$A = \frac{q(q - 1)}{u}, \qquad C = \frac{q(q + 1)}{u + a}$$

satisfy $2q = (C - A)u + Ca$. But $u > qa \geq 2q$ and $u + a = b + (q + 1)a > (q + 1)a$, so $Ca < q$. A positive, zero, or negative value of $C - A$ makes the right side respectively greater than $2q$, less than q, or negative. Every possibility is contradictory. This completes the proof. □

21.2 Comparability of the highest pair

Proposition 21.2 (Highest-pair comparability at twice odd length)**.** *Let P be cyclotomic Littlewood of length $2M$, with M odd, and at most four active coordinates other than 1. If its highest layer has exactly two members $e < c$, then $e \mid c$.*

Proof. If the remaining lower coordinates form a chain, the assertion is Lemma 14.2. Otherwise there are exactly two lower coordinates $a < b$, both odd, with $a \nmid b$. Normalize away coordinate 1 and suppose $h = \gcd(e, c) < e$. All of h, e, c have dyadic valuation one, so

$$e \geq 3h.$$

The highest-layer trace formula has first nonzero even trace $4h$ at h and zero even traces at every index strictly between h and $3h$.

Before the first highest coordinate e, the coefficient word is that of the two odd coordinates a, b. Write $b = qa + r$, where $q \geq 1$ and $0 < r < a$. Let $f(x) = G_a(x)/(1 + x^a)$ and $f_b(x) = \sum_{n\geq 0}(-1)^{\lfloor (b+n)/a \rfloor} x^n$. The shifted square-wave identity gives

$$\frac{f_b}{f} = (-1)^q \frac{1 - 2x^{a-r} + x^a}{1 - x^a}.$$

Consequently, below degree $2b$, the boundary logarithm of the word with coordinates a, b equals

$$\log R_a - 2(-1)^q x^b \frac{1 - 2x^{a-r} + x^a}{1 - x^a}.$$

Since a is odd, $\log R_a$ has no even coefficients. The first two nonzero even traces in this expansion are

$$S_{k_1} = 4k_1, \qquad S_{k_2} = -4k_2,$$

where

$$(k_1, k_2) = \begin{cases} ((q+1)a,\ b+a), & q \text{ odd}, \\ (b+a,\ (q+2)a), & q \text{ even}. \end{cases}$$

In both cases $k_1 < k_2 < 2k_1$ and $k_2 < 2b$. These assertions follow directly by expanding $(1 - x^a)^{-1}$: the two displayed progressions have opposite signs, and their first even members interlace in the stated order.

If $e \leq k_1$, then $h < e \leq k_1$ and the actual trace at h is zero, a contradiction. Thus $e > k_1$. Comparison of the first nonzero even actual and required traces below e forces $h = k_1$. But then

$$k_2 < 2h < 3h \leq e,$$

so its actual trace is $-4k_2$, while the highest-layer formula requires zero. This contradiction proves $e \mid c$. □

21.3 The complete twice-odd classification

Theorem 21.3 (Four active coordinates at twice odd length)**.** *Every cyclotomic Littlewood polynomial of length $2M$, with M odd, and at most four active divisor coordinates other than 1 is standard.*

Proof. Choose a least-length counterexample within the stated class. The three-coordinate theorem and common block extractions reduce it to primitive support with exactly four coordinates. An odd-length quotient, or an extraction leaving at most three coordinates after normalization, is already classified. Proposition 16.1 leaves two or three highest coordinates. Theorem 20.3 excludes the three-member layer. Write the two highest coordinates as $e < c$, and the two lower coordinates as $a < b$. The lower coordinates are odd and the highest coordinates have dyadic valuation one. Proposition 21.2 gives $e \mid c$, so $c \geq 3e$. Theorem 21.1 excludes $e < a$. Hence $a < e$.

Suppose first that $a < b < e$. The first two coordinates cannot be comparable, by Theorem 17.10, so write $b = qa + r$ with $0 < r < a$. Put $n = a + b$. The additive trace identity gives

$$S_n = -n(2b_b + b_n), \qquad b_b = -2(-1)^q.$$

Here $n < 2b < 2e < c$, and n is divisible by neither a nor b. If $n \neq e$, no active coordinate divides n, so $b_n = 0$. The actual trace is then $4n(-1)^q$, while the comparable-highest formula gives zero: $n < 2e$ and $n \neq e$. If $n = e$, its boundary comes only from the high reversal. The lower signs immediately after that index, without the high reversal, have sign

$$(-1)^{\lfloor n/a \rfloor + \lfloor n/b \rfloor} = (-1)^{q+2} = (-1)^q.$$

Consequently $b_n = b_b$, so $S_n = 6n(-1)^q$, whereas the highest formula requires $2n$. Both alternatives are impossible. The tested index is interior because a, b are distinct proper divisors of N and hence $a + b < N$.

It remains that $a < e < b$. The first two coordinates are incomparable, so write

$$e = qa + r, \qquad 0 < r < a.$$

At e, only the lower coordinate a has already entered. Its boundary R_a has no even traces, so the new reversal gives $S_e = 2e(-1)^q$. The highest formula requires $S_e = 2e$. Thus q is even and at least two; in particular $e > 2a$. The lower coordinates cannot form a chain, by Theorem 18.2, so $a \nmid b$.

Put $k = (q + 2)a$. Then k is even and

$$e < k < e + 2a < 2e < c.$$

Define

$$f(x) = \frac{G_a(x)}{1 + x^a}, \qquad f_s(x) = \sum_{n \geq 0} (-1)^{\lfloor (s+n)/a \rfloor} x^n.$$

For $s = ta + u$ with $0 < u < a$, the shifted-series identity is

$$\frac{f_s}{f} = (-1)^t \frac{1 - 2x^{a-u} + x^a}{1 - x^a}.$$

If $b > k$, then through degree $k < 2e$,

$$\log((1 - x)P) = \log R_a - 2x^e \frac{f_e}{f}.$$

The coefficient at k is 4, so its even trace is nonzero, contrary to the highest formula because $e < k < 2e$. Thus $b < k$, since b is odd and k is even.

Both first reversals now occur before $2e$, while their second reversals and the coordinate c do not. Therefore

$$\log((1 - x)P) = \log R_a - 2x^e \frac{f_e}{f} + 2x^b \frac{f_b}{f} + O(x^{2e}).$$

If $(q + 1)a < b < (q + 2)a$, then $\lfloor b/a \rfloor = q + 1$ is odd. At k the two displayed correction terms each contribute 4, again contradicting the zero required trace. The equality $b = (q + 1)a$ is excluded by $a \nmid b$. Consequently

$$qa < e < b < (q + 1)a.$$

Finally set $n = b + a$. This is even and lies strictly between e and $2e$. In the same logarithmic expansion, the b-term contributes 4 at n. The e-term contributes zero:

$$n - e = a + (b - e), \qquad 0 < b - e < a - r,$$

so this degree is neither an a-multiple nor congruent to $-r$ modulo a, the only two possible supports of f_e/f. The baseline $\log R_a$ has no even coefficients. Hence $S_n = -4n \neq 0$, whereas the highest-layer trace is zero for $e < n < 2e < c$. This final contradiction proves the theorem. □

22 Two lower coordinates after a highest coordinate

The next argument uses a parity identity for representations by two odd integers. It supplies coprimality at every dyadic height. A short run of forced additive divisors then gives enough separation for period extraction. With one odd and one even lower coordinate, the same separation follows under the stated coprimality hypothesis.

Lemma 22.1 (Incompatible adjacent translated divisors)**.** *Let $a > 1$, $q > 1$, and $j > 0$ be integers with $\gcd(a, j) = 1$. Put $c = j + a$ and $N = \operatorname{lcm}(qa, j, c)$. The two integers $j + qa$ and $j + (q + 1)a$ cannot both divide N.*

Proof. Put $x = j + qa$, $y = x + a = c + qa$. Suppose both divide N. We first remove the translated summand from each lcm condition:

$$x \mid \operatorname{lcm}(q, c), \qquad y \mid \operatorname{lcm}(q, j).$$

For example, $\gcd(x, a) = 1$. At a prime p, if the valuation of $x = j + qa$ exceeds that of q, the valuation of j equals that of q, so c must supply the full valuation of x in N. Otherwise q already supplies it. This proves the first divisibility, and the second follows by interchanging j, c. Since $c = x - (q - 1)a$ and $j = y - (q + 1)a$, these relations give

$$x \mid q(q - 1), \qquad y \mid q(q + 1).$$

Here coprimality with a removes its factor.

The positive integers

$$R = \frac{q(q - 1)}{x}, \qquad S = \frac{q(q + 1)}{x + a}$$

therefore satisfy

$$2q = (S - R)x + Sa.$$

But $x > qa \geq 2q$ and

$$0 < Sa = \frac{q(q + 1)a}{x + a} < q.$$

If $S - R$ is positive, the right side exceeds $2q$; if negative, it is negative; and if zero, it is smaller than $2q$. All three possibilities contradict the identity. □

Lemma 22.2 (Parity of two midpoint representation counts)**.** *Let $b < c$ be positive odd integers and let $T \geq 2$ be a power of two. For $n > 0$, let $f(n)$ count positive integer solutions of $ib + jc = n$. Then*

$$f\left(\frac{T}{2}(b + c)\right) + f\left(\frac{T}{2}(c - b)\right) \equiv 1 \pmod 2.$$

Proof. Divide by $h = \gcd(b, c)$, so the two coefficients become coprime odd integers $p < r$. Put $n_+ = T(p + r)/2$ and $n_- = T(r - p)/2$. Every positive solution of $ip + jr = n_+$ has $j < T$. No such solution has $i = T$: that would imply $r \mid (T/2)p$, impossible because $r > 1$ is odd and coprime to p. Solutions with $i > T$ correspond bijectively to positive solutions at n_- by replacing i with $i - T$. The remaining solutions lie in $1 \leq i, j < T$ and are paired by $(i, j) \mapsto (T - i, T - j)$. The unique fixed solution is $(T/2, T/2)$. Thus their number is odd, proving the congruence. □

Theorem 22.3 (Coprimality with two odd lower coordinates)**.** *Let P be a normalized cyclotomic Littlewood polynomial with primitive four-coordinate support*

$$\mathcal{A}(P) = \{a, qa, b, c\}, \qquad a \mid qa,$$

where a, qa are its two highest coordinates, $b < c$ are odd, and $T = 2^{v_2(a)} \geq 4$. Then

$$\gcd(a, b) = 1.$$

Proof. Suppose first that $g = \gcd(a, b) > 1$. Primitivity implies $\gcd(g, c) = 1$; also g is odd. Set

$$n_{\pm} = \frac{T}{2}(c \pm b).$$

Both indices are positive multiples of T, strictly below $Tc \leq N$, and neither is divisible by g. The preceding lemma supplies one of these indices, denoted n, at which the number $f(n)$ of positive representations $ib + jc = n$ is odd.

Write $D = (1 - x)P = 1 + 2F$. In the ring of rational numbers with odd denominator, the formal logarithm satisfies

$$\log(1 + 2F) \equiv 2F - 2F^2 + 4F^4 \pmod 8.$$

We claim that

$$F_n = F_{n/2} = F_{n/4} = 0.$$

None of these indices is divisible by g, so none is a multiple of a, qa, or b. If $n/2^s = \ell c$, for $s = 0, 1, 2$, then oddness of c and $T \mid n$ imply $T/2^s \mid \ell$. But $n < Tc$ implies $\ell < T/2^s$, a contradiction. There is therefore no active-coordinate reversal at these indices.

Consider unordered pairs of positive indices $u < v$ with $u + v = n$ and $F_u F_v$ odd. The support of F consists of reversals at multiples of the four active coordinates. Pairs drawn from a, qa, b are impossible because both indices would be divisible by g. A pair of c-multiples would make n a multiple of c, already excluded. A pair consisting of an a-multiple and a c-multiple would force the latter to be a positive multiple of Tc, exceeding n. Thus the only possible pairs are $u = ib, v = jc$, in either order. Such indices cannot simultaneously be multiples of another active coordinate: that would make n a b- or c-multiple, or create the excluded a-and-c pair. Consequently each representation contributes exactly one unordered pair, with odd product. There is no diagonal contribution because $F_{n/2} = 0$. Hence

$$[x^n]F^2 \equiv 2f(n) \equiv 2 \pmod 4.$$

Frobenius modulo two also gives $[x^n]F^4 \equiv F_{n/4} = 0 \pmod 2$. The logarithmic congruence therefore yields

$$[x^n] \log D \equiv 4 \pmod 8,$$

so its trace is nonzero.

The comparable highest-pair identity, however, supports all highest traces on multiples of a. Since $g \nmid n$, this trace must be zero. This contradiction proves $\gcd(a, b) = 1$. □

Corollary 22.4 (Coprimality at both ends of a close odd pair)**.** *Under the preceding hypotheses, suppose additionally $c < 2b$. Then $\gcd(a, c) = 1$ as well.*

Proof. If $g = \gcd(a, c) > 1$, primitivity gives $\gcd(g, b) = 1$. The same midpoint indices $n_\pm = T(c \pm b)/2$ are off g. For $s = 0, 1, 2$, an equality $n_+/2^s = \ell b$ would require ℓ to be a multiple of $T/2^s$, strictly between one and one-and-a-half such multiples. For n_- the multiplier would be smaller than half of $T/2^s$. Thus no such equality is possible; the multiples of a, qa, c are excluded by g. Hence $F_{n_\pm} = F_{n_\pm/2} = F_{n_\pm/4} = 0$.

At n_-, a pair formed from an a-multiple and a b-multiple is impossible because $n_- < Tb$. At n_+, its only possible b-multiple is Tb, but the complementary index is n_-, which is off g and so cannot be an a-multiple. All pairs from a, qa, c are off-g impossible, and a pair of b-multiples would make the tested index a b-multiple, already excluded. Thus only mixed b, c pairs remain, exactly as before. The odd representation count again gives logarithmic coefficient four modulo eight at one midpoint, contradicting the highest trace identity. □

Theorem 22.5 (A smallest highest coordinate with two odd lower coordinates)**.** *Let P be cyclotomic Littlewood of length $N = TM$, where M is odd and $T \geq 4$ is a power of two, with at most four active coordinates other than 1. Suppose its highest layer has two members, includes its smallest normalized active coordinate, and both other coordinates are odd. Then P is standard.*

Proof. The three-coordinate theorem and common block extractions reduce to primitive support $\{a, H, b, c\}$, with $a < b < c$, where a, H are highest. The gcd of the highest pair cannot lie before a, since its required nonzero trace would precede the first coefficient change. Thus $H = qa$ for an odd integer $q \geq 3$. In particular $a \geq T \geq 4$.

Let γ be the word on a, qa and $B = R_a/R_{qa}$ its boundary. The first noncancelled a-grid boundary k after b satisfies $b < k < 2b$: use $k = a\lceil b/a \rceil$, skipping it once if it is an qa-multiple. In that exceptional case $b > (q-1)a \geq 2a$. If $c > k$, the one-reversal calculation at k gives a nonzero highest-trace discrepancy. Therefore

$$b < c < k < b + 2a, \qquad c < 2b,$$

and γ is constant throughout $[b, c]$. Since b, c are odd divisors of M with $c > b$, the odd integer M/b is at least three. Hence $N \geq 3Tb \geq 12b$. All short indices used below are interior. The uniform coprimality theorem gives $\gcd(a, b) = 1$.

Suppose first $c \neq b+a$. As in the odd logarithmic comparison, the even difference of $D = (1-x)P$ from B starts at $2b$, while its odd boundary terms below $3b$ occur only at b, c. The reciprocal coefficients of B at a and $2a$ are both two. Nearness and $c \neq b + a$ therefore give nonzero traces at

$$m_1 = b + a, \qquad m_2 = b + 2a.$$

Each is odd and below $3b$. A proper gcd index would be odd and below b, where its trace is zero. Thus $m_1, m_2 \mid N$. They are coprime to each other and to ab, so

$$(b + a)(b + 2a) \mid \operatorname{lcm}(q, c), \qquad q \geq \frac{(b + a)(b + 2a)}{c} > b + a.$$

Consequently $qa > a(b + a) > b + c$ and $qa > 3b$. There is no cancelled high boundary before c, and b, c lie in the same open a-cell.

We now force enough separation for period extraction, rather than an additional highest-trace calculation. If $b < 2a$, the common cell is $(a, 2a)$ and $c < 2a$. Hence

$$qa - a > \frac{(b + a)(b + 2a)}{2} - a > 2ab > bc;$$

the middle inequality follows from $b^2 - ab + 2a^2 - 2a > 0$. If $b \geq 2a$, put $m_3 = b + 3a < 3b$. The same odd logarithmic comparison applies: $qa > 3b$, its relevant reciprocal coefficient is two, and $c - b \in (0, a)$ cannot contribute. Thus $m_3 \mid N$. The three m_i are pairwise coprime; their gcd with b is at most three, and they are coprime to a. It follows that

$$(b + a)(b + 2a)(b + 3a) \leq 3qc.$$

Using $c < b + a$ gives

$$qa - a > \frac{a}{3}(b + 2a)(b + 3a) - a > b(b + a) > bc.$$

For $a \geq 4$, the second inequality follows by expansion: its difference is $((a - 3)b^2 + a(5a - 3)b + 6a^3 - 3a)/3 > 0$.

The lower word on b, c has period $\ell = \operatorname{lcm}(b, c)$: both ℓ/b and ℓ/c are odd, so their two sign reversals cancel. We have $qa - a > bc \geq \ell$. The comparable-highest-pair period criterion therefore extracts an outer block of length qa. Its initial quotient has at most three active coordinates and is standard, as is the geometric outer block.

It remains that $c = b + a$. Nearness forces the next a-boundary after b to be cancelled, so $b > (q - 1)a$. The odd logarithmic coefficients at $u = b + qa$ and $v = b + (q + 1)a$ are both $4\gamma_b$; the reciprocal of B has coefficients two, zero, and minus two at $(q - 1)a, qa, (q + 1)a$. These odd indices are below $3b$. Their traces are nonzero, and a proper gcd index would be odd and below b. Thus $u, v \mid N$, contradicting Lemma 22.1. □

Theorem 22.6 (Separation from one odd lower coordinate)**.** *Let P be normalized cyclotomic Littlewood with primitive four-coordinate support $\{a, qa, j, h\}$, where a is the smallest coordinate, $\{a, qa\}$ is its two-member highest layer, $v_2(a) \geq 2$, j is odd, and h is even of lower dyadic valuation. If $\gcd(a, j) = 1$, then P is standard.*

Proof. Here $q \geq 3$ is odd. The usual first noncancelled high-boundary comparison puts the two lower coordinates before the first such boundary after their smaller member. Thus $|h - j| < 2a$, and the high-chain signs are constant between them.

Write $D = (1 - x)P = D_{\mathrm{e}} + D_{\mathrm{o}}$. The first odd boundary is at j, with coefficient $O_j \in \{-2, 2\}$. Below $3j$, it is the only odd boundary. Let $C_k = [x^k]D_{\mathrm{e}}^{-1}$. Since $D_{\mathrm{e}} \equiv 1 \pmod 2$,

$$C_k \equiv -D_k \pmod 4 \qquad (k > 0 \text{ even}).$$

The coordinate formula for sign changes shows $C_a = 2$ and $C_{2a} \equiv 2 \pmod 4$: neither j nor h divides $2a$. Consequently the odd traces at $j + a, j + 2a$ are nonzero. Their proper gcd indices are odd and below j, where all traces vanish. Hence both indices divide N. They are pairwise coprime and coprime to aj, so

$$(j + a)(j + 2a) \mid \operatorname{lcm}(q, h).$$

Since $h < j + 2a$, this gives $q > j + a$. In particular $qa > h, j$, no cancelled high boundary lies between the lower coordinates, and h, j lie in one open a-cell. Thus $h < j + a$.

We next force $j + 3a \mid N$. Because $q > j + a > 3$, the coordinate formula gives $C_{3a} \equiv 2 \pmod 4$ unless $h = 3a/2$. Indeed the only lower coordinate other than a that could divide $3a$ is that even coordinate: coprimality excludes j. In the exceptional case both lower coordinates belong to the cell $(a, 2a)$, and direct coefficient comparison gives

$$D_a = -2, \quad D_{2a} = 2, \quad D_h = \pm 2, \quad D_{3a} = 0, \qquad C_{3a} = 4.$$

Thus C_{3a} is always nonzero. At $j+3a$, a cubic odd logarithmic term or the boundary at $3j$ would require a reciprocal shift $3a-2j<a$. That shift is either negative or is a nonzero index below a; it cannot be zero because j is odd and $4 \mid a$. Those contributions vanish. Hence the odd logarithmic coefficient is $O_jC_{3a} \neq 0$.

This index is interior: $j \mid N$, j is odd, and $4 \mid N$, so $N \geq 4j > j+3a$. A proper gcd index is below $(j+3a)/3 < j+a$. In this range the only possible nonzero odd trace is the one at j; but $j \mid j+3a$ would imply $j \mid 3$, contrary to $j > a \geq 4$. Trace reduction therefore gives $j+3a \mid N$.

Put $P_r = \prod_{s=1}^{r}(j+sa)$. The first three additive divisors are pairwise coprime, and their gcd with j is at most three. It follows that $P_3 \leq 3qh$. If $j < 2a$, the common cell gives $h < 2a$, so

$$qa - a > \frac{P_3}{6} - a > 4aj > 2jh.$$

The middle inequality follows by expanding the cubic; every coefficient of $P_3/6 - a - 4aj$ is positive when $a \geq 4$.

If $j > 2a$, also force $j+4a \mid N$. Its index is below $3j$. The reciprocal coefficient C_{4a} is two modulo four: no even lower coordinate $h > a$ with $v_2(h) < v_2(a)$ can divide $4a$, and coprimality again excludes j. Thus the odd trace is nonzero, and its proper gcd index is below j. Among the four additive divisors the only possible common factor between a pair is three, from the first and fourth. Their lcm is at least $P_4/3$, and its gcd with j is at most three. Consequently $P_4 \leq 9qh$, and

$$qa - a > \frac{a(j+2a)(j+3a)(j+4a)}{9} - a > 2j(j+a) > 2jh.$$

The middle inequality is immediate after expansion for $a \geq 4$. The equality case $j = 2a$ is impossible because j is odd.

The lower-coordinate word has period $\ell = 2\,\mathrm{lcm}(j,h) \leq 2jh$, with $\ell \mid N$. The resulting separation $qa - a > \ell$ lets the geometric divisor of the comparable highest pair invoke the period criterion. It extracts an outer block of length qa whose initial quotient has at most three active coordinates. That quotient and the outer block are standard, proving the assertion. □

23 One odd coordinate and two highest coordinates

Theorem 23.1 (A rotated divisor on three odd periods)**.** *Let P be a normalized primitive cyclotomic Littlewood polynomial with active coordinates*

$$\{o, a, Tu, Tv\}, \qquad N = TM, \qquad T = 2^t, \quad o > 1 \text{ odd}, \quad a = Sz, \quad S = 2^s, \quad 1 \leq s < t,$$

where z, u, v are odd and $u < v$. Then

$$h := \gcd(o,u) = \gcd(o,v).$$

Put

$$U = u/h, \quad V = v/h, \quad G = \gcd(U,V), \quad q = o/h, \quad r = \gcd(o,z), \quad Z = z/r.$$

One has $\gcd(q, UV) = 1$ and $\gcd(h,z) = 1$. For every even b with $0 \leq b < T$, put

$$j_b \equiv -(b+1)T^{-1} \pmod{o}, \qquad 0 \leq j_b < o,$$

and define

$$D_b(y) = \sum_{\ell=0}^{M/o-1} (-1)^{\lfloor (T(j_b+o\ell)+b)/a \rfloor + \lfloor (j_b+o\ell)/u \rfloor + \lfloor (j_b+o\ell)/v \rfloor} y^{\ell}.$$

Then the squarefree polynomial

$$K_0(y) = \frac{(1+y^U)(1+y^V)}{(1+y^G)^2}$$

divides every D_b. In particular, D_0 is, up to a constant sign, a product of three permuted alternating words of periods Z, U, V, and its length is $\operatorname{lcm}(Z,U,V)$.

Proof. For each $d > 1$ dividing M, let

$$s_d = \mathbf{1}_{d|u} + \mathbf{1}_{d|v} \pmod 2, \qquad K(x) = \prod_{s_d=1} \Phi_{2d}(x).$$

The saturated highest factors imply $K(x^T) \mid P(x)$. Write $P(x) = \sum_{b=0}^{T-1} x^b P_b(x^T)$. For even $b < T$, the other three coordinates give the same signs at $Tk+b$ and $Tk+b+1$. Therefore

$$P_{b+1}(x) - P_b(x) = \varepsilon_b\, 2C_b(x), \quad C_b(x) = \sum_{k \equiv j_b \pmod o} \beta_{b,k} x^k, \quad \varepsilon_b \in \{-1, 1\},$$

where each $\beta_{b,k}$ is a sign and $0 \le k < M$. The multiplier is constant on the residue class because T is even. Thus $K \mid C_b$. The zero set of C_b is closed under multiplication by oth roots of unity, and $C_b(-1)$ is a nonzero odd integer, since it is a sum of M/o signs.

If a selected conductor d divided o, a primitive $2d$th root of K could be rotated to -1 by an oth root, a contradiction. Consequently $s_d = 0$ for every $d \mid o$. This says precisely that u, v have the same gcd h with o. Primitivity gives $\gcd(h, a) = 1$, hence $\gcd(h, z) = 1$. The equality of gcds gives $\gcd(q, U) = \gcd(q, V) = 1$.

With $g = \gcd(u, v) = hG$, one has

$$K(x) = \frac{(1+x^u)(1+x^v)}{(1+x^g)^2}.$$

The gcd of each of u, v, g with o is h. Hence

$$\prod_{\xi^o=1} K(\xi x) = \left[\frac{(1+x^{oU})(1+x^{oV})}{(1+x^{oG})^2}\right]^h = K_0(x^o)^h.$$

Both K and K_0 are squarefree: their cyclotomic roots are the symmetric differences of two divisor sets. It follows that the union of all oth-root rotations of the roots of K is exactly the root set of $K_0(x^o)$. Since C_b vanishes at this union, $K_0(x^o) \mid C_b(x) = x^{j_b} D_b(x^o)$. Its constant term is one, so $K_0 \mid D_b$.

For the final assertion let $b = 0$, $j = j_0$, and put

$$J = \lfloor j/h \rfloor, \qquad J_0 = \left\lfloor \frac{(T/S)j}{r} \right\rfloor, \qquad c = (T/S)(o/r).$$

Then c is even, $\gcd(c, Z) = 1$, and the coefficient of y^ℓ in D_0, up to the constant sign $(-1)^{J_0}$, is

$$(-1)^{(J_0 + c\ell) \bmod Z} (-1)^{(J+q\ell) \bmod U} (-1)^{(J+q\ell) \bmod V}.$$

The two factors $(-1)^\ell$ from the odd slope q cancel; the lower slope c is even. Finally

$$M/o = \operatorname{lcm}(z/\gcd(o,z),\ u/\gcd(o,u),\ v/\gcd(o,v)) = \operatorname{lcm}(Z,U,V),$$

by the primitive full-lcm hypothesis. □

Theorem 23.2 (A common odd coordinate in a comparable highest pair)**.** *Let P be cyclotomic Littlewood with at most four nontrivial active coordinates and exactly one odd active coordinate o. Suppose its highest pair is comparable and both highest coordinates are divisible by o. Then P is standard.*

Proof. Take a least-length counterexample in this class. Common-divisor and common-multiple extraction either preserve the class or leave at most three coordinates after normalization. Write its primitive support as $\{o, a, ToU, ToV\}$, with $U \mid V$ and $V/U \geq 3$ odd. Primitivity gives $\gcd(o, a) = 1$. The odd coordinate precedes both highest coordinates. If the even lower coordinate a did not precede the first highest coordinate, the first two active coordinates would be comparable, and Theorem 17.10 would apply. Thus o, a are the first two coordinates. Their additive obstruction gives $o + a \mid N$.

Write $a = Sz$ as above. Since $o + a$ is odd and coprime to aTo, its prime powers must all come from V in $N = \operatorname{lcm}(a, ToV)$. Thus $o + a \mid V$, so $V > 2z$. The previous reduction has $h = o$, $q = 1$, $Z = z$, and

$$D_0(y) = \sum_{\ell=0}^{L-1} A_\ell (-1)^{\lfloor \ell/U \rfloor + \lfloor \ell/V \rfloor} y^\ell, \quad L = \operatorname{lcm}(z, V),$$

where $A_\ell = (-1)^{(J_0 + c\ell) \bmod z}$ has exact period z. The selected divisor is $G_{V/U}(-y^U)$. Because $U \leq V/3$, one has $V - U \geq 2V/3 > z$. The period constraint theorem therefore forces A to have period U, and its exact period gives $z \mid U$. It follows that o, a, ToU all divide the last active coordinate ToV, contradicting the primitive full-lcm condition. □

Remark 23.3 (The unresolved part)*.* The rotated-divisor reduction does not assert that h is a unitary divisor of o, or that o/h divides a. The odd coordinate itself can supply missing prime powers in the length. In particular these additional arithmetic conclusions cannot be used in the remaining case. The reduction alone does not classify three arbitrary permuted alternating words with the displayed cyclotomic divisor.

24 Completion at length four times odd

At the next dyadic level, fourth-order traces distinguish the possible orderings of the two lower coordinates. Throughout this section the length is $4M$, with M odd. The first comparisons treat two odd lower coordinates; the subsequent arguments classify mixed lower parity and complete every ordering.

Lemma 24.1 (The first fourth-order traces of two odd coordinates)**.** *Let $1 < a < b$ be odd and $a \nmid b$. Let W be the infinite sign series with coordinates a, b, and define $S_k = -k[x^k] \log((1 - x)W)$. At its first nonzero trace whose index is divisible by four, one of the following holds: the trace is negative; its value divided by the index belongs to $\{8, 16, 20\}$; or its value is four times the index h, and another nonzero trace occurs at an index divisible by four strictly between h and $2h$.*

Proof. Write $b = qa + r$ with $0 < r < a$. Below degree $2b$, the shifted-series identity gives

$$\log((1 - x)W) = \log R_a - 2(-1)^q x^b \frac{1 - 2x^{a-r} + x^a}{1 - x^a}.$$

The first two candidate progressions of indices divisible by four are

$$M + 4a\mathbb{Z}_{\geq 0}, \qquad H + 4a\mathbb{Z}_{\geq 0},$$

where

$$M = \begin{cases} b+a, & 4 \mid b+a, \\ b+3a, & 4 \nmid b+a, \end{cases} \qquad H = 4a\left\lceil \frac{b}{4a} \right\rceil.$$

Their trace-to-index ratios are respectively $4(-1)^q$ and $-4(-1)^q$. The progressions are disjoint, since equality would imply $a \mid b$.

If $q \geq 4$, both $M, H < 2b$, and their difference has absolute value less than $4a < b$. Thus a positive first trace is followed by the opposite-sign trace before twice its index. If $q = 3$, then $H = 4a < M < 2b$; the first ratio is four and the next is minus four, with $M < 2H$. If $q = 2$ and $4 \mid a + b$, then $M = b + a < H = 4a < 2b$, with ratios four and minus four. If $q = 2$ and $4 \nmid a + b$, the first trace is at $H = 4a < 2b$ and is negative. If $q = 1$ and $4 \mid a + b$, the first trace is at $M = b + a < 2b$ and is negative.

It remains that $a < b < 2a$ and $a \equiv b \pmod 4$. Every boundary index is a positive multiple of a or b. A logarithmic monomial at an index divisible by four therefore has index $ua + vb$, where $u, v \geq 0$ and $u + v \equiv 0 \pmod 4$. No such positive index is below $4a$. At $4a$, only $u = 4, v = 0$ is possible. Thus only the boundary coefficients $d_j = [x^{ja}](1 - x)W$ for $1 \leq j \leq 4$ contribute. They satisfy

$$d_1 = d_2 = -2, \qquad d_3 = \begin{cases} -2, & b < 3a/2, \\ 2, & b > 3a/2, \end{cases} \qquad d_4 = \begin{cases} -2, & b < 4a/3, \\ 2, & b > 4a/3. \end{cases}$$

The excluded equalities are impossible for odd a, b. The logarithmic coefficient is

$$d_4 - d_1 d_3 - \frac{d_2^2}{2} + d_1^2 d_2 - \frac{d_1^4}{4} = d_4 + 2d_3 - 14.$$

It equals $-8, -16, -20$ in the three respective intervals $b > 3a/2$, $4a/3 < b < 3a/2$, and $b < 4a/3$. This proves the stated alternatives. □

Proposition 24.2 (Two odd lower coordinates at dyadic height two)**.** *Let P be cyclotomic Littlewood of length $4M$, with M odd, and with at most four active coordinates other than 1. If its highest layer has two members $e < c$ and its other coordinates are odd, then $e \mid c$.*

Proof. A lower chain is already covered by Lemma 14.2. Otherwise the lower coordinates are odd $a < b$, with $a \nmid b$. Suppose $g = \gcd(e, c) < e$. The highest-layer formula has first nonzero trace on the grid $4\mathbb{Z}$ at g, of value $4g$, and no other such trace strictly between g and $3g$. Moreover $e \geq 3g$.

Let h be the first nonzero fourth-order trace index of the word on a, b. If $e \leq h$, the actual trace at $g < e \leq h$ is zero, a contradiction. Thus $e > h$, and comparison of first nonzero traces forces $g = h$ and $S_h = 4h$. The preceding lemma then supplies another nonzero trace at $h < k < 2h < 3h \leq e$, contradicting the required zero trace there. Hence $e \mid c$. □

Lemma 24.3 (A fourth-order mixed trace detects a common divisor)**.** *Let $4 \mid a$, let $q \geq 3$ be odd, and let $b < c$ be odd integers with $a < b$, $c < 2b$. Put*

$$\gamma_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/(qa) \rfloor}, \qquad B = (1 - x)\sum_{n \geq 0} \gamma_n x^n = R_a / R_{qa}.$$

Suppose γ is constant on $[b, c]$, and let D be the boundary series of the coefficient word with coordinates a, qa, b, c. If $g = \gcd(a, b) > 1$ and $\gcd(g, c) = 1$, some index in $4\mathbb{Z}$ not exceeding $3b + c$ has a trace different from B.

Proof. Write $D = B + E + O$, where O is odd and E is even. With $\sigma = \gamma_b$, the first odd coefficients are

$$O_b = -2\sigma, \qquad O_c = 2\sigma, \qquad O_{3b} \in \{-2, 2\}.$$

The even difference E starts at $2b$. Up to degree $3b + c < 5b$, its support lies on $a\mathbb{Z}$ and at $2b, 2c, 4b$. In particular $E_{2b} = -2\gamma_{2b} \in \{-2, 2\}$. These facts follow directly from the first four reversals of the two lower coordinates. The condition $c < 2b$ ensures that both first reversals have occurred before the second reversal at b.

If $b + c \equiv 0 \pmod 4$, the quadratic calculation at $b + c$ gives coefficient 4 in $\log D - \log B$: the common divisor g excludes both $a \mid b + c$ and $a \mid c - b$, leaving only the cross term $-O_b O_c = 4$.

Otherwise b, c have the same odd residue modulo four. Put $n = 3b + c$. Since g divides a, b and is coprime to c, none of

$$b + c, \quad 3b - c, \quad c - b, \quad 3b + c$$

is divisible by a. Also $a \nmid 2b$, because $4 \mid a$ and b is odd. Through degree $n < 5b$, the even logarithm is

$$\log D - \log B = \frac{E}{B} - \frac{E^2}{2B^2} - \frac{O^2}{2B^2} + \frac{EO^2}{B^3} - \frac{O^4}{4B^4} \quad \text{at even degrees.}$$

All powers of B^{-1} are supported on $a\mathbb{Z}$. The displayed divisibility exclusions remove every contribution at n except, respectively, zero, zero, $-O_c O_{3b}$, $2E_{2b} O_b O_c = -8E_{2b}$, and $-O_b^3 O_c = 16$. Hence

$$[x^n](\log D - \log B) = 16 - 8E_{2b} - O_c O_{3b} \in \{-4, 4, 28, 36\},$$

which is nonzero. □

Lemma 24.4 (Two odd reversals in one cell at fourth dyadic height)**.** *Let $4 \mid a$, and let odd $b < c$ belong to one open a-cell:*

$$ma < b < c < (m + 1)a, \qquad m \geq 1.$$

Let D be the boundary series of the sign word with coordinates a, b, c, and put $B = R_a$. There is an index in $4\mathbb{Z}$, no larger than $3b + c$, at which its logarithmic trace differs from that of B.

Proof. Write $b = ma + r$, $c = ma + s$, where $0 < r < s < a$. If $b + c \equiv 0 \pmod 4$, the quadratic mixed calculation at $b + c$ gives coefficient four in $\log D - \log B$ when $a \nmid b + c$, and eight when $a \mid b + c$. In the latter case $b + c$ is the odd a-multiple between $2b$ and $2c$; its boundary difference is four. Thus assume b, c have the same odd residue modulo four, and set $n = 3b + c$.

Let D_{e}, O be the even and odd parts of D, and put $A_n = [x^n](\log D_{\mathrm{e}} - \log B)$. The same even-log expansion as in the preceding lemma gives

$$[x^n](\log D - \log B) = A_n + 16 - 8E_{2b} - O_c O_{3b}, \qquad E_{2b} = -2(-1)^{\lfloor 2r/a \rfloor}.$$

Here $O_c O_{3b} \in \{-4, 4\}$. If $a \nmid n$, then $A_n = 0$: the even off-grid terms occur only at $2b, 2c, 4b$; neither their linear contributions nor their possible square at $4b$ can reach n after a shift by a multiple of a. The resulting coefficient is one of $\{-4, 4, 28, 36\}$.

It remains that $a \mid n$. Write $3r + s = ka$, where $k \in \{1, 2, 3\}$. The possible grid contributions are recorded below:

k	r/a	A_n	$16 - 8E_{2b}$	$-O_c O_{3b}$
1	$(0, 1/4)$	$\{-4, 0, 4, 12, 20\}$	32	$\{-4, 4\}$
2	$(1/3, 1/2)$	$\{-12, -4, 4\}$	32	-4
3	$(2/3, 3/4)$	4	0	4

Each row has a strictly positive total.

For clarity, the table is an elementary grid count, not an enumeration of coefficient words. Before $n < 5b$, negative lower intervals are $[2b, 2c)$, $[3b, 3c)$, and $[4b, 4c)$, with overlaps counted modulo two. The first contains only the possible grid point $(2m+1)a$, with boundary difference four; the second contains only $(3m+1)a, (3m+2)a$; and the third contains only the final grid point n. A double overlap can occur only at $m = 1, k = 1, n = 5a$. In the logarithm, the only quadratic grid contribution is the square of the difference at $(2m+1)a$, and it occurs only when $k = 2$, with value minus eight. Substituting $s = ka - 3r$ gives exactly the three displayed sets. All endpoint equalities are excluded by oddness of r, s, divisibility of a by four, or the strict ordering $r < s$. The final cross-term sign is positive for $k = 3$ and negative for $k = 2$, directly from the third reversal at b. This proves nonvanishing in every case. □

Theorem 24.5 (A smallest highest coordinate with two odd lower coordinates)**.** *Let P be cyclotomic Littlewood of length $4M$, M odd, with at most four active coordinates other than 1. If its highest layer has two members, contains its smallest normalized active coordinate, and its other two coordinates are odd, then P is standard.*

Proof. The three-coordinate theorem and common block extractions reduce to primitive support $\{a, qa, b, c\}$, with $a < b < c$, a the smallest coordinate, and a, qa the highest pair. Indeed, an incomparable highest pair would have a nonzero gcd trace before a. Here $v_2(a) = 2$, $q \geq 3$ is odd, and b, c are odd.

Let γ, B be the high-chain word and boundary from Lemma 24.3. Put $k_0 = a\lceil b/a \rceil$ and let $k = k_0$ unless $qa \mid k_0$, in which case take $k = k_0 + a$. The first noncancelled high boundary after b is k. If $c > k$, the one-reversal comparison produces a trace discrepancy of absolute value $4k$ at this multiple of four. Here $b < k < 2b$: in the skipped-boundary case $b > (q-1)a \geq 2a$. Hence

$$b < c < k < b + 2a, \qquad c < 2b,$$

and γ is constant throughout $[b, c]$.

Both odd lower coordinates divide M, so M/b is an odd integer at least three (otherwise $c \leq M = b$). Consequently $N \geq 12b$; all indices below $5b$ used here are interior. If $\gcd(a, b) > 1$, primitivity makes c coprime to that gcd, and Lemma 24.3 contradicts the highest trace formula. Thus $\gcd(a, b) = 1$.

Suppose $c \neq b + a$. Write $D = D_{\mathrm{e}} + D_{\mathrm{o}}$. The odd part of $\log D$ below $3b$ is $D_{\mathrm{o}}/D_{\mathrm{e}}$; the even difference from B starts at $2b$. The reciprocal coefficients of B at a and $2a$ are both two. The only odd boundaries below $3b$ that can contribute at $b + a, b + 2a$ are the one at b and possibly the one at c. The latter contributes to neither index, by nearness and $c \neq b + a$. Both traces are therefore nonzero. Their proper gcd indices would be odd and below b, hence have zero trace. Thus

$$b + a \mid N, \qquad b + 2a \mid N.$$

These two integers are coprime to each other and to ab. They consequently have product dividing $\mathrm{lcm}(q, c)$, so

$$(b + a)(b + 2a) \leq qc < q(b + 2a), \qquad q > b + a.$$

It follows that $qa > 3b + c$. No cancelled high boundary lies before c, and b, c lie in one open a-cell. Lemma 24.4 now contradicts the highest trace formula.

Finally suppose $c = b + a$. Nearness forces the next a-boundary after b to be cancelled, so $b > (q-1)a$. The odd logarithmic coefficients at

$$u = b + qa, \qquad v = b + (q+1)a$$

are both $4\gamma_b$: the reciprocal of B has coefficients two, zero, and minus two at $(q-1)a, qa, (q+1)a$, respectively. Both indices are odd and below $3b$. The same gcd argument forces $u, v \mid N$. This contradicts Lemma 22.1, applied with $j = b$. □

Lemma 24.6 (One odd coordinate before the first highest coordinate)**.** *There is no primitive cyclotomic Littlewood polynomial of length $4M$, M odd, with active support $\{a, b, e, c\}$ satisfying*

$$a, b \text{ odd}, \qquad a < e < b, \qquad e \mid c, \qquad v_2(e) = v_2(c) = 2.$$

Proof. The comparable-first theorem excludes $a \mid e$, and the lower-chain classification through dyadic height two excludes $a \mid b$. Write $e = qa + r$, with $0 < r < a$. At the first highest coordinate, $S_e = 2e(-1)^q$, while the highest-layer formula requires $2e$. Thus $q \geq 2$ is even, and r is even. Also $c \geq 3e$.

Let k be the least multiple of $4a$ strictly greater than e. If $q \equiv 0 \pmod 4$, then $k = (q+4)a$; if $q \equiv 2 \pmod 4$, then $k = (q+2)a$. In both cases

$$e < k < 2e < c.$$

Put $f = G_a/(1+x^a)$ and, for $s \geq 0$, let f_s denote the shifted square-wave series. For $s = ta + u$, $0 < u < a$,

$$H_s := \frac{f_s}{f} = (-1)^t \frac{1 - 2x^{a-u} + x^a}{1 - x^a}.$$

If $b > k$, the coefficient at k of $\log R_a - 2x^e H_e$ is 4, contradicting the required zero trace. Hence $b < k$, since b is odd. Before degree $2e$,

$$\log((1-x)P) = \log R_a - 2x^e H_e + 2x^b H_b.$$

At k, the two corrections have coefficients 4 and $-4(-1)^t$, where $t = \lfloor b/a \rfloor$. Thus t is even. Write $b = ta + s$ with $0 < s < a$; then s is odd. The possible cells are $t = q$, and additionally $t = q + 2$ when $q \equiv 0 \pmod 4$.

Let

$$n = \begin{cases} b + a, & 4 \mid a + b, \\ b + 3a, & 4 \nmid a + b. \end{cases}$$

The b-correction has coefficient 4 at n. The e-correction has coefficient zero: its possible residues modulo a are 0 and $-r$, whereas $n - e \equiv s - r$; equality would require respectively $s = r$ or $s = 0$, both impossible. The baseline $\log R_a$ has no coefficients at indices divisible by four. Therefore $n < 2e$ immediately gives a contradiction.

The bounds on q, t leave only two possibilities with $n \geq 2e$:

q	t	n
2	2	$b + 3a$
4	6	$b + 3a$.

In both cases $e < b < 2e < 2b$, and $n < 3e \leq c$. Through degree n, the successive lower/high reversals therefore give

$$\frac{P}{f} = 1 - 2x^e H_e + 2x^b H_b - 2x^{2e} H_{2e} + 2x^{2b} H_{2b}.$$

Terms above degree n may be omitted. Its logarithm, through that degree, is

$$\log((1-x)P) = \log R_a - 2x^e H_e + 2x^b H_b - 2x^{2e}(H_{2e} + H_e^2) + 2x^{2b}(H_{2b} - H_b^2) + 4x^{e+b} H_e H_b.$$

There are no cubic correction terms because their degrees are at least $3e$.

For $q = 4, t = 6$, put $d = n - 2e = a + s - 2r$. If $d < 0$, the second-high contribution is absent. Otherwise

$$[x^d](H_{2e} + H_e^2) = 0.$$

To verify this directly, the possible residues of H_e^2 are $0, -r, -2r$ modulo a, and those of H_{2e} are $0, -2r$. Since r is even and s is odd, the only possible coincidence within $0 \le d < 2a$ is $d = 0$, equivalently $s = 2r - a$. In that event $\lfloor 2e/a \rfloor$ is odd, so the two constant coefficients are -1 and 1 and cancel. The $2b$ and $e + b$ terms exceed n in this case. Hence the total coefficient at n remains 4.

For $q = t = 2$, one has $r < s$ and

$$n - 2e = a + s - 2r, \qquad n - 2b = a - s, \qquad n - e - b = a - r.$$

The first of these again gives zero from $H_{2e} + H_e^2$. At the other two degrees, direct use of the displayed shifted series gives

$$[x^{a-s}]H_{2b} = 0, \quad [x^{a-s}]H_b^2 = -4, \quad [x^{a-r}]H_e H_b = -2.$$

Thus the total coefficient at n is

$$4 + 0 + 2(0 - (-4)) + 4(-2) = 4.$$

In every case $S_n = -4n \neq 0$, whereas $e < n < 3e \le c$ and $4 \mid n$ require zero, including the case $n = 2e$. This final contradiction proves the lemma. □

Lemma 24.7 (Two odd coordinates before a comparable highest pair)**.** *There is no primitive normalized cyclotomic Littlewood polynomial of dyadic length valuation two with active support*

$$\{a, b, e, c\}, \qquad a < b < e < c,$$

where a, b are odd and the highest coordinates satisfy $e \mid c$.

Proof. The comparable-first-coordinate theorem excludes $a \mid b$; otherwise standardness would make the primitive support a chain. Write $c = Qe$, with $Q \ge 3$ odd. Set

$$\gamma_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/b \rfloor}.$$

For every interior index divisible by four, the highest-layer trace formula requires

$$S_k = 2e\mathbf{1}_{e|k,\ k/e \text{ odd}} - 2c\mathbf{1}_{c|k,\ k/c \text{ odd}}.$$

Suppose first that $4 \mid a + b$. Put $n = a + b$ and $q = \lfloor b/a \rfloor$. The additive trace identity gives

$$S_n = -n(2b_b + b_n), \qquad b_b = -2(-1)^q.$$

Since $n < 2b < 2e < c$, its boundary coefficient is zero unless $n = e$. If $n \neq e$, the actual trace is $4n(-1)^q$, whereas the required trace is zero. If $n = e$, the boundary comes only from the high reversal and $b_n = b_b$, giving $6n(-1)^q$ instead of $2n$. Thus this case is impossible.

Henceforth $a \equiv b \pmod 4$. If $b > 3a$, put $n = b + 3a$. This is divisible by four and lies below $2b$. Before b, the reciprocal boundary is

$$R_a^{-1} = \frac{1 + x^a}{1 - x^a},$$

whose coefficient at $3a$ is two. At degree n, the off-support logarithm is linear, since $n < 2b$. A high boundary could contribute only at $e = b + a$, $b + 2a$, or $b + 3a$; the first two are not divisible by four. Thus only $e = n$ is possible. As in the preceding argument, its boundary is $b_n = b_b$, and the actual trace is either $4n(-1)^{\lfloor b/a \rfloor}$ or $6n(-1)^{\lfloor b/a \rfloor}$. Neither agrees with the required zero or $2n$. This excludes $b > 3a$.

We use a simple support observation for the remaining ranges. Every boundary index is a multiple of a, b, or e, since c is a multiple of e. In a logarithmic monomial, write the total index as

$$ra + sb + te, \qquad r, s, t \geq 0.$$

If it is divisible by four, then $r + s \equiv 0 \pmod 4$. A nonzero lower contribution therefore has $r + s \geq 4$ and degree at least $4a$. In particular, the two-odd-coordinate word has no nonzero trace at a multiple of four below $4a$. At $4a$, its only possible logarithmic monomials use a-boundaries alone. For boundary coefficients $d_j = [x^{ja}]D$ with $d_1 = -2$, their contribution at degree $4a$ is

$$d_4 + 2d_3 - \frac{d_2^2}{2} + 4d_2 - 4. \tag{24.1}$$

Suppose $2a < b < 3a$. Without either highest coordinate, the boundary coefficients at $a, 2a, 3a, 4a$ are $-2, 2, 2, -2$, respectively. Thus its logarithmic coefficient at $4a$ is 4. If $e > 4a$, this already contradicts the required zero trace. If $e = 4a$, adding its reversal changes that logarithmic coefficient by at most two, and cannot produce the required coefficient -2. Consequently $b < e < 4a$. At e, the low word has zero trace, so the new reversal gives $S_e = 2e\gamma_e$. Thus $\gamma_e = 1$. Since $b < e < 4a < 2b$, this forces $3a < e < 4a$. The boundary coefficients at $a, 2a, 3a, 4a$ are now $-2, 2, 2, 2$. There is no pure high monomial at $4a$, because $e \nmid 4a$, and no mixed high–lower monomial by the support observation. Expression (24.1) is therefore 8, whereas the highest-layer formula requires zero. This excludes the range.

It remains that $a < b < 2a$ and $a \equiv b \pmod 4$. For the low word alone, the logarithmic coefficient at $4a$ is

$$\begin{cases} -8, & 3a/2 < b < 2a, \\ -16, & 4a/3 < b < 3a/2, \\ -20, & a < b < 4a/3. \end{cases}$$

Indeed its boundary coefficients satisfy $d_1 = d_2 = -2$; d_3 is 2 or -2 according as $b > 3a/2$ or $b < 3a/2$; and d_4 is 2 or -2 according as $b > 4a/3$ or $b < 4a/3$. The stated values follow from (24.1). The endpoints cannot occur because a, b are odd. As before, these values exclude $e \geq 4a$, including equality, since a reversal at that index changes the coefficient by at most two. Thus $b < e < 4a$ and $\gamma_e = 1$. If $2e < 4a$, the support observation applied at $2e$ gives

$$[x^{2e}] \log D = 2\gamma_{2e} - 2.$$

The required trace there is zero, so $\gamma_{2e} = 1$ as well.

We claim that $e < 3a/2$. To see this, suppose otherwise. The sign conditions just obtained leave precisely the ranges in the table below. In every row $c > 4a$, $e \nmid 4a$, and the support observation

reduces the logarithmic coefficient at $4a$ to (24.1).

range of e	necessary range of b	(d_2, d_3, d_4)	$[x^{4a}] \log D$
$3b/2 < e < 2a$	$b < 4a/3$	$(2, 2, -2)$	4
$2b < e < 3a$	$b < 3a/2$	$(-2, 2, \pm 2)$	-8 or -12
$3a < e < 2b$	$b > 3a/2$	$(-2, 2, -2)$	-12
$3b < e < 4a$	$b < 4a/3$	$(-2, -2, 2)$	-16

For the first row, $\gamma_{2e} = 1$ requires $e > 3b/2$. For $e > 2a$, the other three rows are exactly the intervals on which $\lfloor e/a \rfloor$ and $\lfloor e/b \rfloor$ have the same parity. Each displayed coefficient is nonzero, while the required trace is zero. The claim follows.

We are left with $a < b < e < 3a/2$. Put $n = b + 3a$. This index is divisible by four. A mixed high–lower logarithmic monomial cannot occur at n, because its lower part has degree at least $4a$, whereas $n - e < 3a$. A pure lower monomial must have $r + s = 4$, and the equality $ra + sb = b + 3a$ then gives $r = 3, s = 1$. The relevant boundary coefficients are

$$[x^a]D = -2, \quad [x^{2a}]D = 2, \quad [x^{3a}]D = -2, \quad [x^b]D = 2.$$

The coefficient of degree $3a$ in the reciprocal of their a-part is $8 - 8 + 2 = 2$. Thus the pure lower contribution to $[x^n] \log D$ is 4.

The inequalities $e > b$ and $e < 3a/2$ show that the only possible positive integer value of n/e is three. If $n \neq 3e$, there is no pure high monomial, so $S_n = -4n$, contradicting the required zero. If $n = 3e$, the previous sign conditions give boundary coefficients $-2, 2$ at $e, 2e$. Moreover

$$\gamma_n = (-1)^{\lfloor n/a \rfloor + \lfloor n/b \rfloor} = (-1)^{4+3} = -1.$$

If $c > n$, the coefficient at $3e$ is 2, so the pure high logarithmic contribution is $2 + 4 - 8/3 = 10/3$. Adding the lower contribution gives $S_n = -22e$, instead of $2e$. If $c = n = 3e$, the two high reversals cancel there and its boundary coefficient is zero. The pure high contribution is then $4/3$, giving $S_n = -16e$ instead of $2e - 2c = -4e$. These contradictions exhaust the final range. All tested indices are interior. Since c is a proper divisor of N with the same dyadic valuation, N/c is odd and at least three. Thus $N \geq 3c \geq 9e > 9b$, while every index used above is at most $b + 3a < 4b$ or $4a < 4b$. □

Theorem 24.8 (Two odd lower coordinates at length four times odd)**.** *Let P be cyclotomic Littlewood of length $4M$, with M odd, and with at most four active coordinates other than 1. If its highest layer has exactly two members and all its other active coordinates are odd, then P is standard.*

Proof. A least-length counterexample reduces to primitive four-coordinate support by the three-coordinate theorem and the common block extractions. An extraction producing coordinate 1 or an endpoint leaves at most three coordinates after normalization. Write the lower coordinates as $a < b$ and the highest coordinates as $e < c$. Proposition 24.2 gives $e \mid c$. Theorem 24.5 excludes $e < a$. Lemma 24.6 excludes $a < e < b$. Lemma 24.7 excludes $a < b < e$. These exhaust the possible orderings, proving the theorem. □

24.1 Comparability with mixed lower valuations

The same highest-pair conclusion holds when one lower coordinate is odd and the other has dyadic valuation one. This comparison alone does not classify that remaining case.

Lemma 24.9 (Fourth-order traces for mixed lower parity)**.** *Let $o > 1$ be odd and let a have dyadic valuation one, with $o \nmid a$. For the infinite sign word with active coordinates o, a, its first nonzero trace on $4\mathbb{Z}$ is either negative, or has value $4h$ at its index h and is followed by another nonzero trace on that grid at an index strictly between h and $3h$.*

Proof. Suppose first that $o < a$, and write $a = qo + r$, $0 < r < o$. Below degree $2a$, the shifted-series expansion relative to R_o gives the two fourth-order progressions

$$M + 4o\mathbb{Z}_{\geq 0}, \qquad H + 4o\mathbb{Z}_{\geq 0}, \qquad M = a + 2o, \quad H = 4o\left\lceil \frac{a}{4o} \right\rceil.$$

Their trace-to-index ratios are respectively $4(-1)^q$ and $-4(-1)^q$. If $q \geq 4$, both initial indices are below $2a$ and their distance is less than $4o < a$, so a positive first trace has an opposite-sign successor before twice its index. If $q = 3$, $H = 4o < M < 2a$ gives a positive trace followed by a negative one, with $M < 2H$. If $q = 2$, the first trace at $H = 4o < 2a$ is negative.

For $q = 1$, one has $o < a < 2o$. Every boundary index is a multiple of o or a. As o is odd and $a \equiv 2 \pmod 4$, the first possible positive fourth-order monomial indices are $2a$ and $M = a + 2o$, followed by $4o$; here $2a < M < 4o$. Let $f = G_o/(1 + x^o)$ and $H_s = f_s/f$ be its shifted quotient. The exact expansion through degree $M < 3a$ is

$$\log((1 - x)W) = \log R_o - 2x^a H_a + 2x^{2a}(H_{2a} - H_a^2).$$

It gives

$$S_{2a} = \begin{cases} 0, & a < 3o/2, \\ 4(2a), & a > 3o/2, \end{cases} \qquad S_M = -12M.$$

Indeed $H_a = -\frac{1-2x^{o-r}+x^o}{1-x^o}$, so its coefficient at $2o$ is -2, $[x^{o-r}]H_a^2 = -4$, and $[x^{o-r}]H_{2a} = 0$. At degree $2a$, the coefficient is $2((-1)^{\lfloor 2a/o \rfloor} - 1)$. These identities prove the assertion in this case as well.

Now suppose $a < o$ and write $o = qa + r$, $0 < r < a$. The baseline R_a has no traces on $4\mathbb{Z}$. The unshifted progression from the first reversal at o has odd indices and does not meet this grid. Its shifted progression has first grid index

$$H = 2a\left\lceil \frac{o}{2a} \right\rceil < 2o, \qquad S_H = -4(-1)^q H.$$

For even q this is negative. For odd $q \geq 3$, the next grid index $H + 2a < 2o$ has the same nonzero ratio, and $H < H + 2a < 2H$.

It remains that $a < o < 2a$, so $H = 2a$ and $S_H = 4H$. We show $S_{4a} \neq 0$. A logarithmic monomial at $4a$ has index $ua + vo$. Its odd-coordinate count v must be even. Since $o > a$, one has $v \leq 2$, and $v = 2$ is possible only when $o = 3a/2$ and $u = 1$. Away from that exceptional equality, only the a-boundary coefficients contribute. They satisfy

$$d_1 = d_2 = -2, \quad d_3 = \begin{cases} -2, & o < 3a/2, \\ 2, & o > 3a/2, \end{cases} \quad d_4 = \begin{cases} -2, & o < 4a/3, \\ 2, & o > 4a/3. \end{cases}$$

Thus $[x^{4a}] \log D = d_4 + 2d_3 - 14$ is $-8, -16$, or -20, all nonzero. The equality $o = 4a/3$ is impossible because o is odd and $v_2(a) = 1$. If $o = 3a/2$, the simultaneous reversals at $3a = 2o$ give $d_3 = 0$, while $d_4 = 2$. The pure a contribution is then -12. The only additional monomial is the cubic product using one boundary at a and two at o, with contribution $(-2) \cdot 2^2 = -8$. Hence the logarithmic coefficient is -20 in this case too. The nonzero trace at $4a = 2H$ lies strictly before $3H$, completing the proof. □

Proposition 24.10 (Mixed lower parity forces highest comparability)**.** *Let P be cyclotomic Littlewood of length $4M$, with M odd, and with at most four active coordinates other than 1. Suppose its highest layer has exactly two members $e < c$ and the remaining two coordinates consist of an odd coordinate o and a coordinate a of dyadic valuation one. Then $e \mid c$.*

Proof. If $o \mid a$, the lower-chain theorem already gives comparability. Otherwise apply the preceding lemma to the lower word. If $g = \gcd(e, c) < e$, the highest-layer formula has first nonzero fourth-order trace $4g$ at g, and zero on that grid strictly between g and $3g$. Also $e \geq 3g$. Let h be the first nonzero fourth-order trace of the lower word. The case $e \leq h$ contradicts the nonzero required trace at $g < e$. Otherwise the first-trace comparison gives $h = g$ and $S_h = 4h$. The lemma supplies a nonzero trace at $h < k < 3h \leq e$, again contradicting the highest-layer identity. Therefore $e \mid c$. □

24.2 The odd coordinate first

With mixed lower valuations, the following comparisons settle both orderings in which the odd lower coordinate is the first active coordinate.

Theorem 24.11 (An odd smallest coordinate below a mixed lower layer)**.** *There is no primitive four-coordinate cyclotomic Littlewood polynomial with active support $\{o, a, e, c\}$ satisfying*

$$o > 1 \text{ odd}, \qquad v_2(a) = 1, \qquad o < a < e < c, \qquad v_2(e) = v_2(c) = 2, \qquad e \mid c.$$

Proof. Write $a = 2z$ with z odd. If $o \mid a$, the first two coordinates are comparable, and the established classification applies. Thus assume $o \nmid a$. Put $n = 2o + a$, a multiple of four. Since $e > a > o$ and $c \geq 3e$, one has $n < 3e \leq c$. The highest-layer identity therefore requires

$$S_n = \begin{cases} 2n, & e = n, \\ 0, & e \neq n. \end{cases} \tag{24.2}$$

Indeed the only possible odd multiple of e below $3e$ is e itself; its even multiple $2e$ has zero required trace.

Let $\gamma_j = (-1)^{\lfloor j/o \rfloor + \lfloor j/a \rfloor}$, $F(x) = \sum_{j \geq 0} \gamma_j x^j$, and $F^{[h]}(x) = \sum_{j \geq 0} \gamma_{h+j} x^j$. Write $Q_h = F^{[h]}/F$ and $\sigma_k(F) = -k[x^k] \log((1-x)F)$. The first highest reversal, and its possible second occurrence, give through degree n

$$S_n = \sigma_n(F) + 2n[x^{n-e}]Q_e - 2n[x^{n-2e}]Q_{2e} + 2n[x^{n-2e}]Q_e^2, \tag{24.3}$$

where negative-degree coefficients are zero. Terms of third order begin at $3e > n$.

First suppose $a > 2o$. For the one-coordinate series $f = G_o(x)/(1 + x^o)$, write $a = qo + r$, with $0 < r < o$. The shifted ratio is

$$\frac{f^{[a]}}{f} = (-1)^q \frac{1 - 2x^{o-r} + x^o}{1 - x^o}.$$

Since $n = a + 2o < 2a$, the reversal at a gives $\sigma_n(F) = 4n(-1)^q$. If $a < e < n$, put $d = n - e < 2o$. The reciprocal boundary of F is $1 + 2x^o + O(x^{2o})$. Therefore $[x^d]Q_e$ is zero: the only relevant shifted boundaries are at n and $n - o = o + a$, which are multiples of neither lower coordinate. A possible shifted constant at $d = o$ would require $e = o + a$, an odd integer. Also $2e > n$. If $e > n$, no correction occurs at all. Consequently (24.2) forces $e = n$. At this index the correction is just $2n\gamma_n$, with $\gamma_n = -(-1)^q$, so equality additionally requires q even. Only the relation $e = a + 2o$ will be needed below.

It remains that $o < a < 2o$; equality $a = 2o$ was excluded at the start. In this interval a direct two-reversal calculation gives

$$\sigma_n(F) = -12n. \tag{24.4}$$

For completeness, use the one-coordinate f above. The first correction at a contributes 4 to $[x^n]\log(F/f)$. Put $d_0 = 2o - a$, with $0 < d_0 < o$. The ratio $f^{[a]}/f$ has constant term -1 and coefficient 2 at d_0; its square has coefficient -4 there. The ratio $f^{[2a]}/f$ has zero coefficient at d_0. Thus the second-reversal and quadratic terms contribute 0 and 8, respectively. The logarithmic coefficient is 12, which proves (24.4).

If $e > n$, equation (24.2) is already contradicted. If $e = n$, then $\gamma_n = -1$ and (24.3) gives $S_n = -14n$, also impossible. Suppose therefore that $a < e < n$. Below degree $2o$,

$$\frac{1}{(1-x)F} = 1 + 2x^o - 2x^a + O(x^{2o}).$$

For $d = n - e < 2o$, this gives

$$[x^d]Q_e = \begin{cases} 4, & e < 2o, \\ 0, & e > 2o. \end{cases}$$

Indeed the boundaries at n and $n - o$ vanish, whereas the boundary at $n - a = 2o$ equals -2, and its multiplier is -2. The two possible shifted constants would require $e = o + a$ or $e = 2o$, which have dyadic valuations zero and one, respectively.

If $2e < n$, set $d' = n - 2e$. Then $0 < d' < 2o - a < o$. The shifted boundary of $F^{[2e]}$ at this degree is the vanishing boundary at n, and the reciprocal boundary of F has no positive term below o. Thus $[x^{d'}]Q_{2e} = 0$. Also $d' < 2o - e$, the first boundary after the shift e; hence $[x^{d'}]Q_e^2 = 0$. It follows that, unless $n = 2e$,

$$S_n = \begin{cases} -4n, & e < 2o, \\ -12n, & e > 2o, \end{cases}$$

contradicting (24.2). At $n = 2e$, the constant terms are $Q_e(0) = 1$ and $Q_{2e}(0) = \gamma_n = -1$, and the corrections sum to $12n$. This is the sole remaining possibility, with $e = o + z$.

In both ranges the only surviving relation is therefore $e = 2(o + z)$ or $e = o + z$. Primitivity forces $\gcd(o, z) = 1$, since a common divisor would divide a, e, c as well. The first additive divisor $m = o + a = o + 2z$ is coprime to oa and to e. Since $m \mid N = \operatorname{lcm}(o, a, c)$, it follows that $m \mid c/e$. The lower-coordinate word has period $\ell = 4oz$, while

$$c - e \geq e(m - 1) \geq (o + z)(o + 2z - 1) > 4oz.$$

The strict inequality follows from

$$(o + z)(o + 2z - 1) - 4oz = (o - z)^2 + (o - 1)(z - 1) + z^2 - 1 > 0.$$

The highest-pair period criterion now extracts an outer block of length c, leaving at most three active coordinates. Its quotient is standard, and so is P, contradicting its nonchain support. □

Lemma 24.12 (An odd lower coordinate before the highest pair)**.** *There is no primitive nonstandard cyclotomic Littlewood polynomial of length $4M$, M odd, with active support $\{o, b, e, c\}$, where*

$$o < e < b, \qquad o \text{ odd}, \quad v_2(b) = 1, \quad e \mid c, \quad v_2(e) = v_2(c) = 2.$$

Proof. The comparable-first theorem excludes $o \mid e$, and the lower-chain theorem through dyadic height two excludes $o \mid b$. Write $e = qo+r$, with $0 < r < o$. Its first highest trace gives $2e(-1)^q = 2e$, so $q \geq 2$ is even and r is even. Let k be the next multiple of $4o$ after e. Then $k = (q+2)o$ or $(q+4)o$ according as $q \equiv 2$ or $0 \pmod 4$, and $e < k < 2e < c$.

Put $f = G_o/(1+x^o)$ and let $H_s = f_s/f$ denote its shifted quotient. For $s = to + u$, $0 < u < o$,

$$H_s = (-1)^t \frac{1 - 2x^{o-u} + x^o}{1 - x^o}.$$

If $b > k$, the coefficient at k in $\log R_o - 2x^e H_e$ is four, contradicting the required zero trace. Thus $b < k$. Before degree $2e$ the full logarithm is

$$\log((1-x)P) = \log R_o - 2x^e H_e + 2x^b H_b.$$

At k, the corrections have coefficients 4 and $-4(-1)^t$, where $t = \lfloor b/o \rfloor$. Hence t is even. Write $b = to + s$, $0 < s < o$; then s is even. The possibilities are $t = q$, and additionally $t = q+2$ when $q \equiv 0 \pmod 4$.

Set $n = b + 2o$, which is divisible by four. The b-term contributes four at n. The e-term can contribute only when $s = r$: its possible residues modulo o are $0, -r$, while $n - e \equiv s - r$. The other possibility $s = 0$ is excluded. Since $b > e$, the equality $s = r$ gives

$$b = e + 2o, \qquad t = q + 2, \qquad q \equiv 0 \pmod 4.$$

In this resonance, primitivity forces $\gcd(o, e) = 1$. The additive divisor $m = e + o$ is coprime to o, e, b, so from $m \mid N = \operatorname{lcm}(o, b, c)$, with $c = ve$, it follows that $m \mid v$. Consequently

$$c - e \geq e(e + o - 1) > 2ob,$$

since $e > 4o$ and $b = e + 2o$. The lower word has period $2ob$, so the comparable-highest period criterion extracts a quotient with at most three active coordinates. This gives standardness, a contradiction.

Assume now that $s \neq r$. If $n < 2e$, its logarithmic coefficient is four, already contradicting the zero required trace. Only the cases

$$(q, t) = (2, 2) \quad \text{or} \quad (q, t) = (4, 6)$$

can have $n \geq 2e$. Here $2b > n$, $e + b > n$, and $n < 3e < c$. The second reversal at e is the only additional correction, giving

$$\log((1-x)P) = \log R_o - 2x^e H_e + 2x^b H_b - 2x^{2e}(H_{2e} + H_e^2) \quad \text{through degree } n.$$

In both cases $n - 2e = s - 2r$. If this number is positive, then $2r < s < o$, and it is smaller than the first positive degree $o - 2r$ in H_{2e} and the first positive degree $o - r$ in H_e^2. Thus the second-high correction vanishes. If it is negative the correction is absent. Either case leaves the impossible coefficient four. The only surviving possibility is

$$n = 2e, \qquad b = 2e - 2o.$$

At zero degree $H_{2e} + H_e^2 = 2$, because $2r = s < o$, so this is precisely the case in which the added coefficient is minus four.

The new relation again makes $\gcd(o,e)=1$ by primitivity. Now $m=e+o$ is coprime to o,e and also to $b=2e-2o$, since m is odd and $\gcd(m,4o)=1$. Therefore $m \mid v=c/e$ and

$$c-e \geq e(e+o-1) > 2ob.$$

For the last inequality write $e=2o+d$, $d>0$; its difference is

$$e^2-3eo+4o^2-e = 2o(o-1)+d(o-1)+d^2 > 0.$$

The same period extraction and three-coordinate classification now complete the contradiction. □

24.3 An even lower coordinate first

The next comparison treats the tight placement of the first highest coordinate after both lower coordinates.

Lemma 24.13 (The tight even-first lower layer)**.** *There is no cyclotomic Littlewood polynomial whose active coordinates are exactly $\{a,o,e,c\}$ with*

$$v_2(a)=1, \qquad o \text{ odd}, \qquad a<o<e\leq 2a, \qquad v_2(e)=v_2(c)=2, \qquad e \mid c.$$

Proof. Put $r=o-a$, $s=e-a$, and $n=a+2o=3a+2r$. Thus $0<r<s\leq a$, and n is divisible by four. Also $e<n<3e\leq c$. The highest-layer identity therefore requires $S_n=0$, since the only possible multiple of e in this interval is its even multiple $2e$.

Let $\gamma_j=(-1)^{\lfloor j/a\rfloor+\lfloor j/o\rfloor}$, $F=\sum_{j\geq 0}\gamma_j x^j$, and $Q_h=\sum_{j\geq 0}\gamma_{h+j}x^j/F$. As $3e>n$, the highest reversal gives the exact identity

$$\frac{S_n}{n}=\frac{\sigma_n(F)}{n}+2[x^{n-e}]Q_e-2[x^{n-2e}]Q_{2e}+2[x^{n-2e}]Q_e^2, \tag{24.5}$$

where $\sigma_k(F)=-k[x^k]\log((1-x)F)$ and a negative-degree coefficient is zero. We compute its four terms.

First suppose $2r\neq a$, and put $B=1$ if $2r<a$, and $B=-1$ if $2r>a$. The lower trace is

$$\frac{\sigma_n(F)}{n}=8-4B.$$

To verify it, start from $f=G_a(x)/(1+x^a)$. The shifted ratio is

$$\frac{f^{[o]}}{f}=-\frac{1-2x^{a-r}+x^a}{1-x^a}.$$

Before $3o>n$, only the first two reversals at o occur. In the logarithm at n, their linear and quadratic contributions are 0, $4B$, and -8, respectively. The baseline R_a has zero trace at n. Thus the logarithmic coefficient is $4B-8$.

For the highest correction note that, below degree $a+o$,

$$\frac{1}{(1-x)F}=1+2x^a-2x^o+6x^{2a}+O(x^{a+o}).$$

The boundary at n is zero; the boundary at $n-a=2o$ is $2B$; the boundary at $n-o=a+o$ is zero; and the boundary at $n-2a=2o-a$ is zero. The only possible shifted constant occurs when

$n - e = 2a$, that is, when $s = 2r$; its contribution is six. The other two possible constants would require $e = 2o$ or $e = a + o$, whose dyadic valuations are one and zero. Consequently, with

$$I = \mathbf{1}_{s=2r}, \qquad J = \mathbf{1}_{n=2e},$$

we have

$$[x^{n-e}]Q_e = 4B + 6I, \qquad [x^{n-2e}]Q_{2e} = -BJ, \qquad [x^{n-2e}]Q_e^2 = J - 4I.$$

Here is the coefficient check for the last two formulas. If $n - 2e > 0$, then it is less than a, so the reciprocal boundary contributes only its constant term. The boundary of $F^{[2e]}$ is the zero boundary at n. In Q_e^2, a nonzero contribution below a can occur only at the first shifted boundary $2a - e$ or twice that boundary. The first equality $n - 2e = 2a - e$ is exactly $s = 2r$, where the contribution is $2\gamma_e(-2) = -4$. The second equality would give $2r = a$, excluded in this case. If $n = 2e$, the constants are $\gamma_n = -B$ and $\gamma_e^2 = 1$. These observations also cover negative degrees.

Substitution in (24.5) gives

$$\frac{S_n}{n} = 8 + 4B + 4I + 2(B + 1)J > 0.$$

Indeed it is four when $B = -1$, and is at least twelve when $B = 1$. This contradicts the required zero trace.

It remains that $2r = a$. Then $o = 3a/2$ and $n = 4a$. The same lower calculation gives $\sigma_n(F) = 20n$: the exceptional linear coefficient contributes -4, the second linear contribution is zero, and the quadratic contribution is -16 to the logarithm. If $e < 2a$, the reciprocal-boundary calculation gives

$$[x^{n-e}]Q_e = -10, \qquad [x^{n-2e}]Q_{2e} = 2, \qquad [x^{n-2e}]Q_e^2 = 4.$$

For the first value, the boundary at n is 2, the boundary at $2o = 3a$ is zero, and the boundary at $n - 2a = 2a$ is -2. For the latter two, $n - 2e = 2(2a - e) < a$; this is twice the first shifted boundary, which has value -2. Thus $S_n/n = 20 - 20 - 4 + 8 = 4$. If $e = 2a$, the three coefficients are $-4, 1, 1$, and $S_n/n = 20 - 8 - 2 + 2 = 12$. Both values again contradict $S_n = 0$. This exhausts the interval. □

Lemma 24.14 (An even lower coordinate first)**.** *There is no cyclotomic Littlewood polynomial with active support* $\{a, o, e, c\}$ *satisfying*

$$v_2(a) = 1, \quad o \text{ odd}, \quad a < o < e < c, \quad v_2(e) = v_2(c) = 2, \quad e \mid c.$$

Proof. Write $o = qa + r$ with $q \geq 1$ and $0 < r < a$; the remainder is nonzero because o is odd and a is even. Let $H = 2a\lceil o/(2a)\rceil$. The word on the two lower coordinates has first nonzero trace on $4\mathbb{Z}$ at H, with value $-4(-1)^q H$. This follows from the shifted-series calculation in Lemma 24.9; in every case $H < 2o$. If $e > H$, its nonzero actual trace contradicts the required zero. Hence $e \leq H$.

At $e = H$, the low word has sign -1, since H/a is even and $\lfloor H/o \rfloor = 1$. The new high reversal changes its trace by $-2H$. Thus $e = H$ is impossible for even q, giving $-6H$ instead of $2H$; for odd q, it gives the required $2H$.

Suppose $e < H$. The lower word has zero trace at e, so the first high trace forces

$$(-1)^{\lfloor e/a \rfloor + 1} = 1.$$

Consequently $t = \lfloor e/a \rfloor$ is odd. Let $f = G_a/(1 + x^a)$ and write $H_s = f_s/f$ for the shifted quotient. Before degree $2o$,

$$\log((1 - x)P) = \log R_a - 2x^o H_o + 2x^e H_e.$$

At H, its two correction coefficients are $4(-1)^q$ and 4. If q is even, their sum is eight, again contradicting the zero required trace at $e < H < 2e$. Thus q is odd.

For odd $q \geq 3$, put $k = e + 2a$. Since $e \leq H = (q+1)a$,

$$e < k \leq (q+3)a < 2o < 2e < c.$$

If $e < H$, it lies in the same a-cell as o, with odd cell number. The index k is not an a-multiple, so the odd-coordinate correction has zero coefficient there: its even support consists only of a-multiples. The high correction contributes minus four, because $[x^{2a}]H_e = -2$. If $e = H$, the quotient H_e is identically one, and the next term of the odd-coordinate correction instead contributes minus four at $k = H + 2a$. In both cases $S_k = 4k \neq 0$, whereas the highest-layer formula requires zero for $e < k < 2e < c$.

The only remaining possibility is $q = 1$ and $e \leq H = 2a$. This is precisely the tight configuration excluded by Lemma 24.13. All tested indices are interior: $H < 2o < 2e < c$ and $k < 2e < c < N$. □

Lemma 24.15 (An even lower coordinate before the highest pair)**.** *There is no primitive non-standard cyclotomic Littlewood polynomial of length $4M$, M odd, with active support $\{a, o, e, c\}$ satisfying*

$$a < e < o, \qquad v_2(a) = 1, \quad o \text{ odd}, \quad e \mid c, \quad v_2(e) = v_2(c) = 2.$$

Proof. The comparable-first theorem excludes $a \mid e$. Write $e = qa + r$, $0 < r < a$. The first highest trace gives $2e(-1)^q = 2e$, so $q \geq 2$ is even. Set $k = (q+2)a$. Then $4 \mid k$ and $e < k < 2e < c$. With $f = G_a/(1 + x^a)$ and $H_s = f_s/f$, one has

$$H_s = (-1)^t \frac{1 - 2x^{a-u} + x^a}{1 - x^a} \quad \text{if } s = ta + u,\ 0 < u < a.$$

If $o > k$, the coefficient at k of $\log R_a - 2x^e H_e$ is four, contradicting the zero required trace. Thus $o < k$, since o is odd. Before degree $2e$ the logarithm is

$$\log((1-x)P) = \log R_a - 2x^e H_e + 2x^o H_o.$$

At k the two correction coefficients are 4 and $-4(-1)^{\lfloor o/a \rfloor}$. The required zero trace forces $\lfloor o/a \rfloor = q$, hence

$$e = qa + r, \qquad o = qa + s, \qquad 0 < r < s < a.$$

We claim $\gcd(a, o) = 1$. Suppose $g = \gcd(a, o) > 1$. Since o is odd, $g \geq 3$ is odd. Primitivity and $e \mid c$ give $\gcd(g, e) = 1$. Every boundary index below c is a multiple of a, o, or e. At degree $2e < c$, a logarithmic monomial containing a lower boundary has index $L + te$, where $g \mid L$ and $0 \leq t \leq 2$. The equality $L + te = 2e$ and coprimality force $t = 2$, $L = 0$. Thus only the boundaries at $e, 2e$ contribute. One has $b_e = -2$ and

$$b_{2e} = -2(-1)^{\lfloor 2r/a \rfloor},$$

because the odd coordinate has reversed once, while the highest coordinate has reversed twice. Neither boundary coincides with a lower boundary, by $\gcd(g, e) = 1$. The zero required trace at $2e$ therefore gives

$$0 = [x^{2e}] \log D = b_{2e} - \frac{b_e^2}{2}, \qquad r > a/2.$$

In particular $s > r > a/2$.

Put $a = gu$, $o = gv$, where $\gcd(u, v) = 1$, u is even and v is odd. The case $u = 2$ would make the fractional part of $o/a = v/2$ equal one half, giving $s = a/2$, impossible. Hence $u > 2$. Set $n = a + 2o$. Then $4 \mid n$, $g \mid n$, and

$$2e < n < 3e \le c.$$

Indeed $3e - n = (q - 1)a + 3r - 2s > 0$. The same residue argument now excludes every high-boundary monomial at n: its total high multiplicity is at most two and must be divisible by $g \ge 3$. A lower-only monomial has index $Ra + So = n$. Since $n < 3o$, one has $S \le 2$. Coprimality of u, v and $u > 2$ exclude $S = 0, 1$, leaving only $R = 1, S = 2$.

The relevant boundary coefficients are

$$b_a = -2, \qquad b_o = 2, \qquad b_{2o} = -2.$$

For the last one, $\lfloor 2o/a \rfloor = 2q + 1$ and $2e < 2o < 3e$, while the odd coordinate has reversed twice. There are no coincident boundaries at $2o$, since $u > 2$. Consequently

$$[x^n] \log D = -b_a b_{2o} + b_a b_o^2 = -4 - 8 = -12.$$

This contradicts the zero required trace for $2e < n < 3e \le c$. The claim $\gcd(a, o) = 1$ follows.

Finally, before the odd coordinate enters, all boundary indices are even. The odd part of the logarithm below $3o$ is linear in its odd part, whose first term is $b_o x^o$. Since $e > 2a$, the reciprocal even boundary has coefficient two at both a and $2a$. Thus the odd traces at $o + a, o + 2a$ are nonzero. They are interior and their proper gcd indices would be odd and below o, where all traces vanish. Hence

$$o + a \mid N, \qquad o + 2a \mid N.$$

These two divisors are coprime to each other and to ao. Since $N = \operatorname{lcm}(a, o, c)$, their product divides c. Therefore

$$c - e \ge \frac{2c}{3} \ge \frac{2(o + a)(o + 2a)}{3} > 2ao.$$

The lower word has period $2ao$, so the comparable-highest period criterion extracts a quotient with at most three active coordinates. The three-coordinate theorem makes that quotient standard, completing the contradiction. □

24.4 Completion at length four times odd

The last ordering has a highest coordinate first. Coprimality with the odd lower coordinate permits the uniform separation argument above.

Lemma 24.16 (Coprimality of an odd lower coordinate at fourth dyadic height)**.** *Let P be normalized cyclotomic Littlewood of length $4M$, M odd, with primitive four-coordinate support $\{a, qa, j, h\}$. Suppose a is the smallest coordinate, $\{a, qa\}$ is its highest layer, j is odd, and $h \equiv 2 \pmod 4$. Then* $\gcd(a, j) = 1$.

Proof. Here $v_2(a) = 2$ and $q \ge 3$ is odd. Let $B = R_a/R_{qa}$ be the boundary of the high-chain word γ. The first noncancelled high boundary after the smaller lower coordinate puts the larger before that boundary. Thus γ is constant between j, h, and

$$\max(j, h) < 2 \min(j, h), \qquad |j - h| < 2a.$$

Suppose $g = \gcd(a, j) > 1$. It is odd, and primitivity gives $\gcd(g, h) = 1$. Put $n = 2j + h \in 4\mathbb{Z}$. This index is interior: if $j < h$, then $n < 4j \le N$; if $h < j$, then $n < 3j < N$. At n the high-chain trace vanishes, because $g \nmid n$.

Write $D = (1-x)P$, with boundary coefficients b_k. Expand $\log D - \log B$ in the lower-coordinate boundary differences. All inverse powers of B are supported on $a\mathbb{Z}$. Any surviving off-grid monomial must contain total h-multiplicity congruent to one modulo g. The cubic mixed terms give exactly

$$b_h(b_j^2 - b_{2j}).$$

Here $b_h, b_j \in \{-2, 2\}$. Apart from the exceptions treated below, $b_{2j} \in \{-2, 2\}$, so this is one of $\{-12, -4, 4, 12\}$.

If $j < h$, the expansion stops before degree $4j$ and no further monomial can contribute. A term involving the even reversal $2j$ and h is the displayed quadratic term; three first reversals give the displayed cubic term. All other residual shifts are either nonzero modulo g or are $j, 2j, h$, whose dyadic valuations are smaller than that of a. Thus the coefficient is nonzero.

Suppose $h < j$. Since $n < 5h$, the only additional possible total h-multiplicity is four, and this requires $g = 3$. Such a contribution can survive only if

$$2j - 3h = ma \qquad (m \geq 0).$$

Nearness gives

$$m < \min\{h/a, 4 - h/a\} \leq 2,$$

so $m = 0$ or $m = 1$. The even boundary differences at grid points start at degree $2h$; they cannot alter this fourth-order contribution, whose remaining shift is smaller than h.

If $m = 0$, then $j = 3h/2$ and the two reversals at $3h = 2j$ cancel, so $b_{3h} = b_{2j} = 0$. The common divisor is exactly three, and primitivity implies that neither $2h$ nor $4h$ is a high grid point. Consequently $b_{2h}, b_{4h} \in \{-2, 2\}$. The pure fourth coefficient is

$$b_{4h} - \frac{b_{2h}^2}{2} + b_h^2 b_{2h} - \frac{b_h^4}{4} = b_{4h} - 6 + 4b_{2h} \in \{4, 0, -12, -16\}.$$

The mixed coefficient is $4b_h \in \{-8, 8\}$. None of the possible sums is zero.

Finally suppose $m = 1$, so $2j = 3h + a$. Nearness first gives $a < h < 3a$. If $h < 2a$, the interval from h to j crosses the noncancelled boundary $2a$, which is impossible. Thus $h > 2a$, and the crossed boundary $3a$ must be cancelled. This forces $q = 3$; the next noncancelled boundary gives $2a < h < 7a/3$ and $j < 4a$. Also $g = 3$. Write $a = 3w$, $j = 3v$; then

$$\gcd(w, v) = 1, \qquad h = 2v - w, \qquad 3 \nmid h.$$

The first odd trace is at j. The usual support-defect argument, or the nonzero odd logarithmic coefficient at $j + a$, forces $j + a = 3(v + w) \mid N$. But

$$N = \operatorname{lcm}(9w, h, 3v), \qquad \gcd(v + w, 9wh\, 3v) = 1.$$

Indeed $\gcd(v+w, wv) = 1$, $\gcd(v+w, h) = \gcd(v+w, 3) = 1$, and $3 \nmid (v+w)$ because $h \equiv -(v+w) \pmod 3$. The forced divisor is therefore impossible. Every case contradicts $g > 1$, proving the lemma. □

Theorem 24.17 (A smallest highest coordinate with mixed lower parity)**.** *Let P be cyclotomic Littlewood of length $4M$, M odd, with at most four active coordinates other than 1. Suppose its two-member highest layer contains its smallest normalized active coordinate, and its other coordinates have different parity. Then P is standard.*

Proof. Common block extraction either reduces to at most three coordinates or preserves the stated class and makes the support primitive. The highest pair is comparable: otherwise its gcd trace would occur before the smallest coordinate. The preceding lemma gives coprimality with the odd lower coordinate. Theorem 22.6 then supplies the required period separation and standard factorization. □

Theorem 24.18 (Complete four-coordinate classification at length four times odd)**.** *Every cyclotomic Littlewood polynomial of length $4M$, with M odd, and with at most four active divisor coordinates other than 1 is standard.*

Proof. Normalize the first two coefficients. The three-coordinate theorem handles any loss of a coordinate. A common inner block with even length reduces to odd or twice-odd length, where the assertion is already proved. A common odd inner block preserves the present dyadic valuation unless an active coordinate disappears. A proper outer block either preserves that valuation, decreases it to an already classified odd or twice-odd length, or deletes a coordinate. Thus a least counterexample has primitive support of exactly four coordinates.

The highest-layer split and the complete three-member-highest theorem leave exactly two highest coordinates $e < c$. If both lower coordinates are odd, Theorem 24.8 applies. They cannot both be even, since that would give a common inner block. The remaining case has one odd lower coordinate o and one lower coordinate a of dyadic valuation one. Proposition 24.10 gives $e \mid c$.

If e is the smallest coordinate, Theorem 24.17 applies. Otherwise exactly one of the following orderings holds:

$$o < a < e, \qquad o < e < a, \qquad a < o < e, \qquad a < e < o.$$

These are excluded, respectively, by Theorem 24.11, Lemma 24.12, Lemma 24.14, and Lemma 24.15. This exhausts the primitive support and proves the theorem. □

Corollary 24.19. *Every cyclotomic Littlewood polynomial whose length is not divisible by eight and whose active support has at most four coordinates other than 1 is standard.*

Proof. Combine the odd-length theorem with Theorems 21.3 and 24.18. □

25 A lower chain and an integer-chain trace

Consider the remaining primitive lower-chain configuration $\{a, b, e, c\}$ at length $N = TM$, where $T = 2^t \geq 8$, M is odd, $a \mid b$, $e \mid c$, and e, c are the highest coordinates. The preceding reductions allow us to write $e = Tu$, $c = ev$, with u, v odd, $v \geq 3$, a odd, and $\gcd(a, e) = 1$. The lower-chain short-trace and additive-barrier arguments give

$$2u < a < e/2, \qquad b > Ta, \qquad q = \lfloor e/a \rfloor \text{ even}.$$

Thus $2 \leq q < T/2$, and the fourth coordinate does not affect any trace through Ta. As before, put $R_d = (1 - x^d)/(1 + x^d)$, with the variable indicated by its argument when needed.

Lemma 25.1 (Separation of the two boundary grids)**.** *Let $T \geq 2$ be even, let $a > 1$ be odd with $\gcd(a, T) = 1$, and let $e = Tu$, $c = ev$, where u, v are odd and $v \geq 3$. Assume $\gcd(a, e) = 1$. Let P_∞ be the infinite sign series with active coordinates a, e, c. If its boundary traces agree with those of R_e/R_c at every positive multiple of T strictly below Ta, then*

$$\left\lfloor \frac{\ell e}{a} \right\rfloor \equiv 0 \pmod 2 \quad \textit{whenever } \ell u < a \textit{ and } v \nmid \ell.$$

Proof. Below Ta, boundary indices lie in the disjoint sets $a\mathbb{Z}$ and $T\mathbb{Z}$. A logarithmic monomial of degree $Tk < Ta$ containing any a-grid boundary has total a-grid degree am with $m > 0$. Reduction modulo T forces $T \mid m$, so that its degree is already at least Ta, a contradiction. Thus the traces below Ta depend only on the T-grid boundary series. Trace agreement identifies this series with R_u/R_{uv} through degree $a-1$ in $y = x^T$. At the noncancelled boundary ℓe, the actual coefficient is the high-chain boundary coefficient multiplied by $(-1)^{\lfloor \ell e/a \rfloor}$. Equality forces the displayed parity. □

Proposition 25.2 (The pure-power highest scale)**.** *In the unresolved primitive lower-chain configuration, if the smaller highest coordinate is $e = T$, then a highest-trace discrepancy occurs at a positive multiple of T below Ta. In particular that configuration cannot be cyclotomic.*

Proof. The existing short-trace lemma and additive barrier give $2 < a < T/2$, $q = \lfloor T/a \rfloor$ even, and $b > Ta$. Hence the fourth coordinate b cannot affect this prefix. Put $\theta = T/a - q$. Since T and q are even and a is odd, $\theta = r/a$ for an even integer $r \geq 2$. Suppose no earlier discrepancy exists. The preceding lemma requires $\lfloor \ell\theta \rfloor$ even for $1 \leq \ell < a$, except when $v \mid \ell$. In particular $\ell = 2$ gives $\theta < 1/2$. For $a = 3$ this contradicts $r \geq 2$ directly. For $a \geq 5$, put $\ell_0 = \lceil 1/\theta \rceil$. Then

$$\ell_0 \leq \lceil a/2 \rceil = (a+1)/2, \qquad \ell_0 + 1 \leq a - 1.$$

Both $\lfloor \ell_0\theta \rfloor$ and $\lfloor (\ell_0+1)\theta \rfloor$ equal one, since $\ell_0\theta < 1 + \theta$ and $2\theta < 1$. At least one of two consecutive indices is not divisible by $v \geq 3$, contradicting the required parity. □

Lemma 25.3 (An integer-chain reduction of the local target)**.** *Suppose $u > 1$ in the same configuration, and put $\alpha = e/a = q + \theta$, where q is even and $0 < \theta < 1$. If no highest-trace discrepancy occurs below Ta, then for $1 \leq m \leq T$,*

$$(-1)^{\lfloor ma/e \rfloor + \lfloor ma/c \rfloor} = (-1)^{\lfloor (m-1)/q \rfloor + \lfloor (m-1)/(qv) \rfloor}.$$

Consequently the only remaining discrepancy at Ta is the Tth boundary trace of

$$B_{q,v}(z) = 1 - \frac{2z}{1+z} \frac{R_q(z)}{R_{qv}(z)}.$$

Proof. Put $L = \lfloor a/u \rfloor$. Coprimality and $u > 1$ make a/u nonintegral. The parity lemma requires $\lfloor \ell\theta \rfloor$ even for $1 \leq \ell \leq L$, except when $v \mid \ell$. As $a > 2u$, the index two is available and is not cancelled; therefore $\theta < 1/2$. The first odd value occurs at $\ell_0 = \lceil 1/\theta \rceil$. Its successor also has odd floor, because $\ell_0\theta < 1 + \theta$ and $2\theta < 1$. Thus $\ell_0 \geq L$, and equality is possible only if $v \mid L$. All noncancelled boundaries before T consequently occur at

$$m = \lceil \ell\alpha \rceil = \ell q + 1.$$

There are no extra boundaries of this integer-chain word before T. Indeed $(L+1)\theta < 2$. If $(L+1)q + 1 \leq T$, evenness of T and q would imply $T \geq (L+1)q + 2$, contrary to $T < (L+1)(q+\theta)$. This identifies the two sampled sign sequences through T. At degree Ta, no mixed logarithmic monomial between the two grids is possible either; moreover no T-grid boundary occurs there, since $u \nmid a$. The top-only logarithmic contribution therefore still agrees with the required trace. The remaining contribution is the logarithm of

$$1 + 2 \sum_{m \geq 1} (-1)^m (-1)^{\lfloor (m-1)/q \rfloor + \lfloor (m-1)/(qv) \rfloor} z^m.$$

The chain-boundary formula evaluates this rational series as the stated $B_{q,v}$. □

The proposition excludes a pure power of two as the smaller highest coordinate in this remaining lower-chain class. For $u > 1$, the lemma reduces the local obstruction to the precise assertion

$$-T[z^T]\log\left(1-\frac{2z}{1+z}\frac{R_q(z)}{R_{qv}(z)}\right)\neq 0,\qquad T=2^t,\quad 2\le q<T/2,\quad q\text{ even},\quad v\ge 3\text{ odd}.$$

A uniform proof of this nonvanishing is not supplied here. In particular, the reduction does not establish the full lower-chain classification at arbitrary dyadic height.

26 Large prime powers with an arbitrary fixed cofactor

The local rigidity argument treats two odd prime supports. A different periodicity argument permits any number of prime supports in the cofactor. Its conclusion inserts a prime block between two shorter cyclotomic Littlewood polynomials; it need not place that block at either end.

Theorem 26.1 (Large-prime insertion)**.** *Let $A \ge 1$ and let q be an odd prime satisfying*

$$q > A. \tag{26.1}$$

For every $s \ge 1$, every cyclotomic Littlewood polynomial P of length Aq^s, normalized by $P(0) = 1$, has a factorization

$$P(x) = R(x)\Phi_q(\delta x^h)W(x^{qh}), \tag{26.2}$$

where $h \mid A$, $\delta \in \{-1, 1\}$, and R, W are cyclotomic Littlewood polynomials of lengths h and $(A/h)q^{s-1}$, respectively, both with constant term one.

The decisive observation is that a reciprocal rational function cannot hide its first coefficient discrepancy near the middle. A congruence modulo four first determines a signed period; reversal then turns that congruence into an identity.

Lemma 26.2 (Signed rational interpolation)**.** *Let $U, V \in \mathbb{Z}[x]$, with $U(0) = V(0) = 1$, $\deg U = L$, and $\deg V = L + 1$, where $L \ge 1$. Suppose*

$$V^*(x) = \sigma V(x),\qquad B(1/x) = \kappa x B(x),\qquad B = U/V,$$

where $\sigma, \kappa \in \{-1, 1\}$ and $V^*(x) = x^{L+1}V(1/x)$. Write $B(x) = \sum_{n\ge 0}\beta_n x^n$. If, for a sign δ,

$$\beta_{n+L} \equiv \delta\beta_n \pmod 4 \quad (n \ge 0),\qquad \beta_n \in \{-1, 1\} \quad (0 \le n \le L),$$

then

$$B(x) = \frac{R(x)}{1-\delta x^L},\qquad R(x) = \sum_{n=0}^{L-1}\beta_n x^n.$$

Proof. Write $U^* = x^L U(1/x)$ and $R^* = x^{L-1}R(1/x)$. The assumed reciprocity gives $U^* = \kappa\sigma U$. The signed congruence says that the polynomial

$$F = (1-\delta x^L)U - RV$$

has every coefficient divisible by four. Reversing this congruence at degree $2L$ and using $(1-\delta x^L)^* = -\delta(1-\delta x^L)$ gives

$$R^*V \equiv -\delta\kappa RV \pmod 4.$$

Since $V(0) = 1$, cancellation in $(\mathbb{Z}/4\mathbb{Z})[[x]]$ yields $R^* \equiv -\delta\kappa R \pmod 4$. Both sides have sign coefficients, so they are equal over $\mathbb{Z}$. Consequently $x^{2L}F(1/x) = -\delta\kappa\sigma F(x)$. By construction, F vanishes to order at least L at zero. Reversal leaves only the possibility $F = cx^L$. Its middle coefficient is

$$c = \beta_L - \delta\beta_0.$$

This is a difference of signs divisible by four, hence zero. □

The next finite-prefix lemma gives a second periodicity criterion. It is useful for more general rational functions for which the numerator and denominator degrees do not meet the preceding lemma's hypotheses. Its properness assumption removes exceptional initial indices.

Lemma 26.3 (Periodic signs from a reciprocal finite prefix)**.** *Let B be a proper rational function regular at zero, whose poles lie among the Lth roots of unity and have multiplicity at most m, and suppose $B(1/x) = \kappa x B(x)$ for a sign κ. Suppose its Taylor coefficients are integers and satisfy*

$$\beta_{n+L} \equiv \beta_n \pmod 4 \qquad (n \geq 0).$$

If they also satisfy

$$\beta_n \in \{-1, 1\} \qquad (0 \leq n < Q), \qquad 2Q \geq Lm,$$

then all its Taylor coefficients are signs and $\beta_{n+L} = \beta_n$ for every $n \geq 0$.

Proof. Partial fractions express the coefficients as sums of terms

$$c\binom{n+r-1}{r-1}\zeta^{-n}, \qquad 1 \leq r \leq m, \quad \zeta^L = 1.$$

There is no polynomial part because B is proper. The displayed expression extends β_n to every integer n by treating the binomial coefficient as a polynomial. On each residue class $n = a + Lk$, this extension is a polynomial $f_a(k)$ of degree at most $m - 1$, valid for all integer k.

For $n \geq 0$, the coefficient of x^{-n-1} in the Laurent expansion of B at infinity is $-\beta_{-n-1}$. Indeed, for a single partial fraction this follows from

$$\binom{r-n-2}{r-1} = (-1)^{r-1}\binom{n}{r-1}$$

and expansion of $c(1 - x/\zeta)^{-r}$ in powers of x^{-1}. On the other hand, reciprocity gives $B(x) = \kappa x^{-1}B(1/x)$, so the same coefficient is $\kappa\beta_n$. Hence

$$\beta_{-n-1} = -\kappa\beta_n.$$

The hypothesis therefore supplies signs at every integer $-Q \leq n < Q$.

The congruence hypothesis also extends to all integer indices of the polynomial continuation. Indeed, for a residue representative $0 \leq a < L$, put $g_a(k) = f_a(k+1) - f_a(k)$. This polynomial takes values in $4\mathbb{Z}$ for all integers $k \geq 0$. Its finite-difference expansion is

$$g_a(k) = \sum_{r \geq 0} \Delta^r g_a(0)\binom{k}{r},$$

with only finitely many nonzero terms. Every coefficient $\Delta^r g_a(0)$ is divisible by four, and $\binom{k}{r}$ is an integer for every integer k, including negative k. Hence $g_a(k) \in 4\mathbb{Z}$ for every integer k. All values of f_a at integer arguments are consequently congruent modulo four.

The $2Q$ consecutive signed indices contain at least m representatives of each residue modulo L. In a fixed residue, their signs are congruent modulo four and therefore all equal to some sign η_a. Thus $f_a(k) - \eta_a$, of degree at most $m-1$, has at least m distinct zeros. It vanishes identically, and f_a is the constant η_a. □

Remark 26.4. Without the congruence hypothesis, the same periodic-sign conclusion follows from the stronger bound $2Q \geq L(2m-1)$. The reflected prefix then gives $2m-1$ zeros of $f_a^2 - 1$, whose degree is at most $2m-2$, on each residue. This version does not require integral Taylor coefficients. The modulo-four information is what reduces the required sample count from $2m-1$ to m.

Proof of Theorem 26.1. We isolate the q-free rational factor first. Its degrees and its initial sign coefficients will force a signed period. The remaining factor can then change a whole period only by preserving it or reversing every sign; this is the source of the inserted block.

Write $A = TM$, where $T = 2^{v_2(A)}$ and M is odd, and put $D = (1-x)P$. Let E_n denote the multiplicity of Φ_n in D. To keep constant terms equal to one, write $\Psi_1(x) = 1-x$ and $\Psi_d(x) = \Phi_d(x)$ for $d > 1$. Every conductor divides $2Aq^s$ and is uniquely $q^j d$ with $d \mid 2A$ and $0 \leq j \leq s$.

For $j = 0, 1$, define

$$V_j(y) = \prod_{d \mid 2A} \Psi_d(y)^{E_{q^j d}}.$$

The dyadic budgets for D give, for every $r \mid M$ and $0 \leq j \leq s$,

$$\sum_{a=0}^{v_2(A)+1} \varphi(2^a) E_{2^a q^j r} = T. \tag{26.3}$$

Multiplying by $\varphi(r)$ and summing shows that $\deg V_0 = \deg V_1 = A$. Reduction of the same budgets modulo two gives

$$V_0(x) \equiv V_1(x) \equiv 1 - x^A \pmod 2. \tag{26.4}$$

Indeed, modulo two the factors at conductors $2^a r$ contribute $\Phi_r^{\varphi(2^a)}$; taking the product over $r \mid M$ gives $(1-x^M)^T = 1 - x^A$ in characteristic two.

For $q \nmid d$, the cyclotomic identities are

$$\Psi_{qd}(x) = \frac{\Psi_d(x^q)}{\Psi_d(x)}, \qquad \Phi_{q^j d}(x) = \Phi_{q^{j-1} d}(x^q) \quad (j \geq 2).$$

Consequently

$$P(x) = B(x) C(x^q), \qquad B(x) = \frac{V_0(x)}{(1-x) V_1(x)}, \tag{26.5}$$

where

$$C(y) = V_1(y) \prod_{j=2}^{s} \prod_{d \mid 2A} \Phi_{q^{j-1} d}(y)^{E_{q^j d}}.$$

The empty product is one when $s = 1$. This polynomial has constant term one, only roots of unity as zeros, and degree

$$K = A \sum_{j=0}^{s-1} \varphi(q^j) = A q^{s-1}. \tag{26.6}$$

The numerator and denominator in (26.5) have degrees A and $A+1$, respectively. They are reciprocal up to sign, so $B(1/x) = \kappa x B(x)$ for a sign κ. Its Taylor coefficients β_n are integers, and they agree with those of P for $n < q$. In particular, $\beta_0, \ldots, \beta_A$ are signs because $q > A$.

Put $J = 1 - x^A$ and use (26.4) to write $V_j = J + 2H_j$, with $H_j \in \mathbb{Z}[x]$ and $\deg H_j \le A$. Set $H = H_0 - H_1$. Formal inversion gives

$$B(x) \equiv \frac{1}{1-x} + \frac{2H(x)}{(1-x)(1-x^A)} \pmod 4.$$

In particular every β_n is odd. Multiplying by $1 - x^A$ and comparing coefficients of degree at least A yields

$$\beta_{n+A} - \beta_n \equiv 2H(1) \pmod 4 \qquad (n \ge 0).$$

For $\delta_0 = (-1)^{H(1)}$, oddness makes this equivalent to $\beta_{n+A} \equiv \delta_0 \beta_n \pmod 4$. Apply Lemma 26.2 with $U = V_0$, $V = (1-x)V_1$, and $L = A$. It follows that

$$\beta_n \in \{-1, 1\}, \qquad \beta_{n+A} = \delta_0 \beta_n \quad (n \ge 0). \tag{26.7}$$

Thus β has period $L = 2A$. Extend it periodically to all integers. Although q need not reach L, every interval of $q > A$ consecutive indices meets every residue modulo A; the signed period then determines all residues modulo L. Also $\gcd(q, L) = 1$.

Write $C(y) = \sum_{k=0}^{K} c_k y^k$, where $c_0 = 1$. For $0 \le j \le K$ define a function on the *absolute residue* $n \pmod L$ by

$$F_j(n) = \sum_{k=0}^{j} c_k \beta_{n-kq}. \tag{26.8}$$

The function F_j records the coefficient contributions once the first $j+1$ terms of $C(x^q)$ have entered. Keeping the global index n fixed makes consecutive such functions comparable on the same residues; using an offset within a block would shift that comparison. For $jq \le n < (j+1)q$, equation (26.5) says that $[x^n]P = F_j(n)$. Each F_j inherits the signed period $F_j(n + A) = \delta_0 F_j(n)$. Every such interval contains every residue modulo A, so F_j is sign-valued on all residues modulo L for $0 \le j < K$. Since $\deg P = Kq - 1$, the next block gives $F_K = 0$.

Now $F_0 = \beta$, and the recurrence in these absolute coordinates is

$$F_j(n) - F_{j-1}(n) = c_j \beta_{n-jq}. \tag{26.9}$$

For $1 \le j < K$, either $c_j = 0$ and the functions agree, or their difference is nonzero at every residue. The nonzero difference of two signs has absolute value two. Hence in the latter case $|c_j| = 2$ and $F_j = -F_{j-1}$ pointwise. Inductively, every F_j with $j < K$ is β or $-\beta$. At every nonzero internal coefficient, (26.9) then shows that shifting β by jq gives β or $-\beta$. The terminal equation

$$c_K \beta_{n-Kq} = -F_{K-1}(n)$$

gives the same assertion for Kq, and also gives $|c_K| = 1$.

Consider the subgroup

$$S = \{u \in \mathbb{Z}/L\mathbb{Z} : \ \beta_{n+u} = \beta_n \text{ for all } n, \text{ or } \beta_{n+u} = -\beta_n \text{ for all } n\}.$$

It is a subgroup because composing signed shifts adds their indices and multiplies their signs. Let $h \mid L$ be its least positive generator, taking $h = L$ if $S = \{0\}$. The signed period A already belongs to S, so $h \mid A$. There is a sign δ with $\beta_{n+h} = \delta \beta_n$ for every n. If $c_j \ne 0$, then $jq \in S$, so $h \mid jq$. Since $\gcd(q, L) = 1$, this implies $h \mid j$. Thus the signed symmetry of the baseline forces the entire correction polynomial onto multiples of the same h. We have obtained

$$C(y) = C_0(y^h), \qquad B(x) = \frac{R(x)}{1 - \delta x^h}, \qquad R(x) = \sum_{r=0}^{h-1} \beta_r x^r. \tag{26.10}$$

Thus R is Littlewood of length h.

Put $m = K/h$ and write $C_0(z) = \sum_{k=0}^{m} d_k z^k$. Because q is odd,

$$\beta_{n-qhk} = \delta^k \beta_n.$$

For $0 \le \ell < m$, equation (26.8) at $j = h\ell$ shows that $\sum_{k=0}^{\ell} d_k \delta^k$ is a sign. The equation $F_K = 0$ gives $C_0(\delta) = 0$. Hence

$$C_0(z) = (1 - \delta z)W(z), \qquad [z^\ell]W = \delta^\ell \sum_{k=0}^{\ell} d_k \delta^k \in \{-1, 1\} \quad (0 \le \ell < m).$$

This proves that W is Littlewood of length $m = (A/h)q^{s-1}$.

Both shorter polynomials are cyclotomic. For R, this follows from $R = (1 - \delta x^h)B$: every zero and pole of B, and every zero of the multiplying binomial, is a root of unity. For W, every hth root of a zero of W is a zero of C, hence a root of unity. Finally, substituting (26.10) into (26.5) yields

$$P(x) = R(x)\frac{1 - \delta x^{qh}}{1 - \delta x^h} W(x^{qh}) = R(x)\Phi_q(\delta x^h)W(x^{qh}),$$

where the last equality again uses that q is odd. This proves the theorem. □

Remark 26.5 (A pole-multiplicity alternative). Lemma 26.3 gives a separate proof of periodicity under the sufficient bound $q \ge A \max\{2, T\}$, with $T = 2^{v_2(A)}$. Indeed, B is proper, all its poles are $(2A)$th roots of unity, and each pole has order at most T by (26.3). At 1, the extra factor $1 - x$ in the denominator is offset by $E_1 \ge 1$. Moreover,

$$(1 - x^{2A})B(x) \equiv G_{2A}(x) + 2H(x)G_A(x) \pmod 4,$$

a polynomial of degree less than $2A$. Thus its coefficients are periodic modulo four with period $2A$. Reciprocity and the first q sign coefficients allow Lemma 26.3 with $L = 2A$, $m = T$, and $Q = q$. The signed interpolation argument above uses the numerator and denominator degrees instead of this pole-order bound.

Corollary 26.6 (Inheritance from the cofactor)**.** *Suppose every cyclotomic Littlewood polynomial whose length divides A is standard. If $q \nmid A$ is an odd prime satisfying (26.1), then every cyclotomic Littlewood polynomial of length Aq^s is standard for every $s \ge 1$.*

Proof. Induct on s, simultaneously for all cofactors $a \mid A$. The case $s = 0$ is the hypothesis. Since $q > A \ge a$, Theorem 26.1 applies at length aq^s. Its factor R has length dividing A, and its factor W has length $(a/h)q^{s-1}$. These are standard by the hypothesis and induction, respectively. Inserting the displayed q-block between their standard products proves the claim. □

For example, all divisor lengths of 30 are covered by the established odd-length and length-$2pq$ results. Thus every length $30q^s$ is covered for primes $q > 30$ and all $s \ge 1$. This illustrates an arbitrary-power family with three odd prime supports.

Corollary 26.7 (A necessary balance condition)**.** *If N is the least length of a counterexample to the Borwein–Choi conjecture, then every odd prime $q \mid N$, with $A = N/q^{v_q(N)}$, satisfies*

$$q < A. \tag{26.11}$$

Proof. Since $q \nmid A$, equality $q = A$ is impossible. If the displayed inequality failed, Theorem 26.1 would express the counterexample using a prime block and two cyclotomic Littlewood polynomials of strictly smaller lengths. Both shorter polynomials are standard by minimality, which makes the original polynomial standard as well. □

The bound compares q to its entire q-free cofactor. It does not show that a prime larger merely than the dyadic part can always be removed. Nor does (26.11) bound the number or size of mutually comparable odd prime factors. Thus this insertion theorem gives a uniform reduction for unrestricted cofactors and prime powers, while leaving the balanced cases unresolved.

27 Prime insertion from reduced rational complexity

The size of the cofactor is a uniform bound for two different quantities: the degree of a rational function and the signed period of its coefficients modulo four. Keeping those quantities separate gives a more flexible test.

Lemma 27.1 (Reciprocal interpolation with a prescribed period)**.** *Let $U, V \in \mathbb{Z}[x]$ have constant term one and degrees $d, d+1$, respectively, where $d \geq 0$. Suppose that*

$$V^*(x) = \sigma V(x), \qquad B(1/x) = \kappa x B(x), \qquad B = U/V,$$

with $\sigma, \kappa \in \{-1, 1\}$. Write $B = \sum_{n\geq 0} \beta_n x^n$. Suppose that all β_n are odd and, for some $h \geq 1$ and sign δ,

$$\beta_{n+h} \equiv \delta \beta_n \pmod 4 \qquad (n \geq 0).$$

If the first Q coefficients are signs and $2Q > h + d$, then

$$B(x) = \frac{R(x)}{1 - \delta x^h}, \qquad R(x) = \sum_{n=0}^{h-1} r_n x^n, \qquad r_n \in \{-1, 1\}.$$

Here r_n is the unique sign congruent to β_n modulo four.

Proof. The congruence says that

$$F = (1 - \delta x^h)U - RV \equiv 0 \pmod 4.$$

Reverse this congruence at degree $h + d$. The reciprocity assumptions give $U^* = \kappa\sigma U$, and cancellation of the invertible series V over $\mathbb{Z}/4\mathbb{Z}$ yields

$$R^* \equiv -\delta\kappa R \pmod 4.$$

Both polynomials have sign coefficients, so this is an equality over $\mathbb{Z}$. Consequently

$$x^{h+d} F(1/x) = -\delta\kappa\sigma F(x).$$

The series $R/(1 - \delta x^h)$ consists entirely of signs and is coefficientwise congruent to B modulo four. Their first Q coefficients therefore agree. Hence $x^Q \mid F$, and reversal also annuls the last Q positions among degrees $0, \ldots, h + d$. The inequality $2Q > h + d$ leaves no nonzero coefficient. Thus $F = 0$. □

Theorem 27.2 (An effective prime-insertion test)**.** *Let q be an odd prime not dividing A, and let P be a cyclotomic Littlewood polynomial of length Aq^s, where $s \geq 1$ and $P(0) = 1$. For the multiplicities E_m of $D = (1-x)P$, put*

$$V_j(x) = \prod_{d \mid 2A} \Psi_d(x)^{E_{q^j d}} \quad (j = 0, 1), \qquad G = \gcd(V_0, V_1), \qquad d_0 = A - \deg G,$$

where $\Psi_1 = 1-x$, $\Psi_d = \Phi_d$ for $d > 1$, and $G(0) = 1$. Let β_n be the coefficients of $B = V_0/((1-x)V_1)$, and let r_n be their representatives in $\{-1, 1\}$ modulo four. Extend (r_n) to all integers using its period $2A$. Let h be its least positive signed period: $r_{n+h} = \delta r_n$ for all integers n and some sign δ. Then $h \mid A$. If

$$q > h, \qquad 2q > h + d_0,$$

there are cyclotomic Littlewood polynomials R, W, with constant term one and respective lengths $h, (A/h)q^{s-1}$, such that

$$P(x) = R(x)\Phi_q(\delta x^h)W(x^{qh}).$$

Proof. The multiplicity budgets and the q-adic factorization in the proof of Theorem 26.1 do not require $q > A$. They give

$$\deg V_0 = \deg V_1 = A, \qquad P(x) = B(x)C(x^q), \qquad \deg C = Aq^{s-1}.$$

They also show that every β_n is odd and that $\beta_{n+A} \equiv \delta_0 \beta_n \pmod 4$ for a sign δ_0. Thus the sign representatives have period $2A$ and signed period A. Their signed shifts form a subgroup of $\mathbb{Z}/(2A)\mathbb{Z}$; its least positive generator h therefore divides A.

Cancel G in B and write

$$U = V_0/G, \qquad V = (1-x)V_1/G.$$

These integer polynomials have degrees $d_0, d_0 + 1$ and constant term one. Their cyclotomic factors give the reciprocity hypotheses of Lemma 27.1. The first q coefficients of B agree with those of P, since $C(0) = 1$ and all nonconstant terms of $C(x^q)$ have degree at least q. Apply that lemma with $Q = q$ and the signed period h. It follows that

$$B(x) = R(x)/(1 - \delta x^h), \qquad R(x) = \sum_{n=0}^{h-1} r_n x^n.$$

In particular B is an exact sign series, and h is still its least signed period.

Write $C(y) = \sum_{j=0}^{K} c_j y^j$, $K = Aq^{s-1}$, and set

$$F_j(n) = \sum_{k=0}^{j} c_k \beta_{n-kq}.$$

These functions inherit the signed period h. On each interval $jq \leq n < (j+1)q$, $F_j(n)$ is the coefficient of P at n. Because $q > h$, this interval meets every residue modulo h. Therefore F_j is sign-valued everywhere for $j < K$, and $F_K = 0$. The recurrence

$$F_j - F_{j-1} = c_j \beta_{\cdot - jq}$$

now has precisely the same consequences as in the proof of Theorem 26.1: a nonzero internal c_j has absolute value two and reverses every sign. Hence jq is a signed period whenever $c_j \neq 0$, including the terminal index K. Since $q \nmid A$ and $h \mid A$, this forces $h \mid j$. Thus $C(y) = C_0(y^h)$. Its terminal cancellation gives $C_0(\delta) = 0$, and the signed partial sums show that $C_0(z) = (1 - \delta z)W(z)$ with W Littlewood of length K/h. Substitution gives the displayed prime block. Every zero of R or W is a root of unity, by the same root-lifting argument as in Theorem 26.1. □

The original cofactor bound is recovered from $h \le A$ and $d_0 \le A$. The new test can also apply below that bound, when the rational factors have a large common divisor or the signs modulo four have a shorter signed period. It is a test on the multiplicity array of an individual polynomial; it does not assert that all balanced cofactors satisfy it.

For example, take $P = (1+x)\Phi_5(-x^2)\Phi_3(x^{10})$, with $N = 30$, $q = 5$, and $A = 6$. Its baseline is $B = (1+x)/(1+x^2)$, so $h = 2$ and $d_0 = 2$. The effective test applies although $q < A$. This illustrates a strictly less restrictive test on multiplicities; length 30 is already covered by Theorem 11.1.

28 The remaining assertion

The chain criterion gives a precise form of the unrestricted question: must the active divisor coordinates of every cyclotomic Littlewood polynomial form a chain? Gcd invariance alone is insufficient, as is gcd invariance together with positivity on $(-1, 1)$.

A least counterexample has no common inner block and no proper outer block, so its active coordinates have gcd one and lcm equal to its length. The three-coordinate theorem requires at least four active coordinates after normalizing $a_0 = a_1 = 1$. Prime insertion requires $q < N/q^{v_q(N)}$ at each odd prime $q \mid N$. An odd highest layer must have full lcm. A comparable highest pair $e < c$ must satisfy $c - e < \ell - \gcd(e, \ell)$, with ℓ as in Corollary 15.1.

Theorem 12.1 excludes every length supported on $\{2, 3, 7\}$. Theorem 9.2 requires an additional nonzero binomial exponent in its divisor window at every first-block failure. The effective insertion test also requires, for each odd prime, $q \le h$ or $2q \le h + d_0$, with h, d_0 defined in Theorem 27.2. For four coordinates, Proposition 16.1 leaves only highest layers of size two or three. Theorem 20.3 excludes the latter, so exactly two highest coordinates remain. Theorems 17.10 and 18.4 exclude the stated infinite subfamilies. The first two active coordinates must be incomparable under divisibility. Theorems 21.3 and 24.18 exclude all lengths whose dyadic valuation is at most two. Hence a remaining four-coordinate counterexample has length divisible by eight. Theorem 22.5 excludes a smallest highest coordinate with two odd lower coordinates. If there is instead one odd and one even lower coordinate, such a smallest-highest configuration must fail the coprimality hypothesis of Theorem 22.6. A comparable highest pair cannot both contain the unique odd coordinate, by Theorem 23.2.

A lower-chain candidate is restricted further by Lemma 18.3 and Section 25. Its smaller highest coordinate cannot be a power of two. The remaining local trace is reduced exactly to a rational integer-chain ghost, whose uniform nonvanishing remains unproved. The general two-member highest layer at greater dyadic height remains unclassified. These necessary conditions have not been proved incompatible in general. The intermediate norms need not be assumed Littlewood, and neither a flat terminal norm nor a geometric divisor alone has been shown to force all remaining blocks. The supplement gives exact examples that invalidate stronger local inferences. Its finite four-coordinate classification through length 3000 remains a separate certificate result; no assertion at arbitrary length is inferred from it.

Usage of AI

AI tools assisted with literature searches, algebraic exploration, proof drafting, code generation, and consistency checks. The author is responsible for the mathematical claims, proofs, and final text. This assistance does not constitute independent mathematical validation or formal proof verification.

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