Rotating the Replica: Twisted Rényi Coefficients from CFT Operator Counting

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We show that twisting vacuum Rényi entropies by a spatial rotation reveals conformal field theory data governing the asymptotic counting of spinning operators. For a ball in four dimensions, the logarithmic coefficients of these angular-momentum-twisted Rényi entropies are determined by rotating-sphere thermal data to all orders in the local derivative expansion, including the Weyl-anomaly contribution. We test this relation in free theories and in strongly coupled, large-N $\mathcal{N}=4$ Yang–Mills theory using holography.

arXiv:2611.12133v1 [hep-th] 9 Nov 2026

INTRODUCTION

Conformal field theories (CFTs) are important in the study of critical phenomena and quantum gravity, particularly through the AdS/CFT correspondence [1]. Two important aspects of a CFT are its spectrum of local operators and the entanglement structure of its vacuum. The former characterizes the spectrum of states on the sphere through the state–operator correspondence, whereas the latter probes how the degrees of freedom are entangled in a spatial subregion. Whether information about one can be extracted from the other is not immediately clear. It has been shown that the high-temperature partition function of rotating CFTs is systematically organized by a local thermal effective field theory (EFT) [2–4]. More recently, this framework has been used to study the asymptotic spectrum of CFT operators, including large-spin limits and subleading thermal corrections [5–8]. For more recent developments along these lines, see [9–15].

For a spherical region, the Casini–Huerta–Myers (CHM) map relates the vacuum reduced density matrix to the thermal density matrix on the Euclidean geometry $S^1 \times \mathbb{H}^{d-1}$ [16]. Using this conformal map, the vacuum entanglement and Rényi entropies across spherical entangling surfaces have been computed from the thermal partition function on the hyperbolic cylinder in a variety of theories, including free fields, gauge and higher-spin theories, and holographic CFTs [17–22]. More recently, it has been combined with the thermal EFT to show that the leading small-n behavior of the untwisted spherical Rényi entropy is universal and fixed by the equation of state [23].

These developments raise a natural question: can universal information about entanglement be extracted from the high-temperature data that govern the asymptotic operator spectrum? We show that this is the case for Rényi entropies twisted by a spatial rotation. In the local thermal-EFT sector, the twist produces a leading small-n logarithmic contribution whose coefficients probe higher-derivative rotating-sphere thermal data, beyond the equation of state.

Angular-momentum-twisted Rényi partition function– We consider a four-dimensional CFT in its vacuum state, defined on flat spacetime with Euclidean metric

$$ds^2 = dt_E^2 + dr^2 + r^2 d\Omega_2^2. \tag{1}$$

We choose a ball-shaped spatial subregion A of radius R, whose entangling surface is the two-sphere $\partial A = S^2$ at $t_E = 0$ and $r = R$. The corresponding vacuum reduced density matrix is $\rho_A = \mathrm{Tr}_{\bar{A}} |0\rangle\langle 0|$. Here $\bar{A}$ denotes the region outside of the ball-shaped region.

We wish to probe the angular-momentum dependence of the entanglement spectrum by introducing the rotation restricted to A, $U_\theta = e^{i\theta J_A}$, about an axis passing through the center of the ball, where J_A is the corresponding angular-momentum generator. Since both the vacuum and the spherical subregion are invariant under this rotation, the reduced density matrix commutes with J_A, namely $[\rho_A, J_A] = 0$. Consequently, ρ_A and J_A admit a common eigenbasis,

$$\rho_A|\lambda, j\rangle = \lambda|\lambda, j\rangle, \qquad J_A|\lambda, j\rangle = j|\lambda, j\rangle. \tag{2}$$

We therefore define the angular-momentum-twisted Rényi partition function as

$$\mathcal{Z}_n(\theta) \equiv \mathrm{Tr}\,(\rho_A^n U_\theta) = \sum_{\lambda, j} \lambda^n e^{i\theta j}. \tag{3}$$

The eigenvalues of the reduced density matrix are thus weighted by phases determined by their angular momentum quantum numbers. In this way, the twisted partition function encodes the angular momentum dependence of the Rényi partition function.

Since we are considering the vacuum reduced density matrix of a ball-shaped region, we can employ the conformal map of Casini, Huerta, and Myers (CHM) [16], which maps the causal development of the ball to the hyperbolic cylinder $S^1 \times \mathbb{H}^3$. In Euclidean signature, the conformal transformation is given by

$$t_E = \frac{R \sin\tau}{\cosh u + \cos\tau}, \qquad r = \frac{R \sinh u}{\cosh u + \cos\tau}. \tag{4}$$

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Under this transformation, the flat metric (1) takes the form

$$\begin{aligned} ds^2_{\text{flat}} &= \Omega^{-2}\left(d\tau^2 + du^2 + \sinh^2 u\, d\Omega_2^2\right) \\ &= \Omega^{-2} ds^2_{S^1\times\mathbb{H}^3}, \end{aligned} \tag{5}$$

where the conformal factor is

$$\Omega = \frac{\cosh u + \cos\tau}{R}. \tag{6}$$

The conformal transformation leaves the angular coordinates (ϑ, ϕ) of S^2 unchanged. Consequently, J_A maps to the spatial rotation generator $J = -i\partial_\phi$ on $\mathbb{H}^3$. Since the reduced density matrix is mapped to a thermal density matrix on $\mathbb{H}^3$, the twisted Rényi partition function becomes

$$\mathcal{Z}_n(\theta) = \frac{\text{Tr}_{\mathbb{H}^3}\left(e^{-2\pi n H} U_\theta\right)}{\left[\text{Tr}_{\mathbb{H}^3}\left(e^{-2\pi H}\right)\right]^n}, \tag{7}$$

where H generates hyperbolic time translations. The insertion of U_θ is implemented by the twisted identification of $(\tau, \phi) \sim (\tau + 2\pi n, \phi + \theta)$. In the co-rotating angle $\phi - \theta\tau/2\pi n$, it becomes an untwisted identification with an off-diagonal, rotating-frame metric.

The ordinary Rényi entropy and a normalized angular-momentum-twisted Rényi generating functional are defined by, respectively,

$$\begin{aligned} S_n(A) &= \frac{1}{1-n}\log \mathcal{Z}_n(0), \\ S_n(\theta) &= \frac{1}{1-n}\log\left[\frac{\mathcal{Z}_n(\theta)}{\mathcal{Z}_1(\theta)^n}\right]. \end{aligned} \tag{8}$$

PRINCIPAL RESULTS

Local logarithmic Rényi coefficients.— We find that, in the local thermal-EFT regime $2\pi n \ll |\theta| \ll 1$, the logarithmic contribution of the twisted Rényi entropy is governed by high-temperature CFT data. In the leading small-n limit, the local contribution takes the following form

$$\lim_{n\to 0} n S_n(\theta)\Big|_{\text{loc,pert}} = \left[\frac{c'}{\theta^2} + 2\sum_{q\geq 1} s_q^{\text{phys,loc}}\theta^{2q-2}\right]\log\frac{R}{\epsilon}. \tag{9}$$

Here, R and ϵ are the radius of the ball and the UV cutoff, respectively, and "loc, pert" denotes the local thermal-EFT contribution, extracted by first taking the leading small-n limit and then expanding in θ. The coefficients $s_q^{\text{phys,loc}}$ are obtained from the partition function on $S^1 \times \mathbb{H}^3$, including its finite Weyl-anomaly contribution. The coefficient c' is determined by the flat-space equation of state, $\log Z_{\mathbb{R}^3\times S^1_\beta} \sim c' V/\beta^3$.

Our main result is that these coefficients are fixed by the high-temperature data of the rotating sphere,

$$s_q^{\text{phys,loc}} = -\frac{a^{\text{trace}}_{0,q;0}}{4\pi^2}, \qquad q \geq 1. \tag{10}$$

Here $a^{\text{trace}}_{m,k;p}$ are the coefficients of the high-temperature expansion of the physical thermal trace on the rotating S^3, including the Weyl-anomaly term [8, 24]. We first establish this relation for the Weyl-invariant local thermal actions and then show that the finite Wess–Zumino contributions agree under the same leading small-n projection. Thus, the local logarithmic coefficients of twisted Rényi entropies are fixed by the same high-temperature CFT data that govern asymptotic operator counting, without requiring an independent entanglement calculation.

GEOMETRIC ORIGIN OF THE RÉNYI LOGARITHM

To evaluate the thermal EFT on the twisted hyperbolic cylinder, we introduce cylindrical coordinates adapted to the rotational fixed axis:

$$\sinh r = \sinh u \sin\vartheta, \qquad \tanh\eta = \tanh u\cos\vartheta. \tag{11}$$

The Euclidean hyperbolic metric in (5) and thermal identification vector become

$$\begin{aligned} ds^2 &= d\tau^2 + dr^2 + \cosh^2 r\, d\eta^2 + \sinh^2 r\, d\phi^2, \\ v &= (2\pi n)\,\partial_\tau + \theta\,\partial_\phi, \end{aligned} \tag{12}$$

At leading order, the local thermal EFT is determined entirely by the flat-space high-temperature coefficient c'. Since the proper length of the local thermal cycle is $\sqrt{v^2}$, its leading Weyl-invariant contribution is

$$\log Z^{(0)}_{\text{loc}} = c'\int d^4x\,\frac{\sqrt{g}}{(v^2)^2}. \tag{13}$$

Since $\mathbb{H}^3$ is noncompact, the spatial integral requires a regulator. For a nonzero angular twist, the transverse integral is convergent, while the integration along the rotational fixed geodesic remains divergent. We regulate this divergence using the UV cutoff ϵ of the original spherical region. Under the CHM map, the corresponding hyperbolic cutoff is $u_{\max} = \log(2R/\epsilon) + \mathcal{O}(\epsilon/R)$. Since $u = |\eta|$ along the fixed geodesic $r = 0$, its regulated length is

$$L = \int_{-u_{\max}}^{u_{\max}} d\eta = 2\log\frac{R}{\epsilon} + \mathcal{O}(1). \tag{14}$$

We can now evaluate Eq. (13) by integrating over the thermal cycle, the angular coordinate ϕ, and the trans-

verse radial direction. At logarithmic order, we obtain

$$\log Z^{(0)}_{\rm loc} \simeq (2\pi)(2\pi n)c'L\int_0^\infty dr\,\frac{\sinh r\cosh r}{[(2\pi n)^2+\theta^2\sinh^2 r]^2} = \frac{c'L}{2n\theta^2}. \tag{15}$$

Since the Weyl-invariant integrand depends only on r, the inherited CHM cutoff restricts the η-interval to length $L-2\log\cosh r+\cdots$. At fixed n and θ, the second term contributes only to the cutoff-independent part of the convergent transverse integral. It therefore does not affect the coefficient of $\log(R/\epsilon)$, allowing us to factor out L from the local partition function when extracting the logarithmic Rényi coefficient. The radial integral is dominated by a thin tube $r\sim 2\pi n/|\theta|$ around the fixed axis, where the local thermal length $\sqrt{v^2}\sim 2\pi n$ is small and the thermal EFT is reliable; the EFT fails only at $|\theta|\sinh r\gtrsim 1$, a region whose contribution is subleading at small n. The twist thus replaces the $1/n^3$ volume scaling of the untwisted case [23] by $c'(n/\theta)^2/n^3\propto 1/(n\theta^2)$, and at leading order in n the divergence is logarithmic rather than area-law. Thus, the logarithmic dependence on the UV cutoff arises from the regulated length of the fixed geodesic, while its coefficient is determined by the local thermal EFT. Using Eq. (14), we reproduce the leading term of Eq. (9).

ALL-ORDER RÉNYI COEFFICIENT RELATIONS WITH THERMAL CFT DATA

We now establish the relation between the logarithmic Rényi coefficients and the high-temperature operator-counting data of the CFT. The key point here is that both $S^1\times S^3$ and $S^1\times\mathbb{H}^3$ are locally conformally flat. Consequently, the Weyl-invariant local thermal effective actions on these backgrounds are built from the same geometric invariants, evaluated with different thermal identifications.

The correspondence can be understood geometrically. The spherical entangling surface has two fixed points, the north and south poles, under a spatial rotation about the axis passing through them. A conformal transformation can map the entangling sphere to a plane by sending either of its two poles to infinity. In this description, we can identify two orthogonal planes of rotation of S^3 (embedded in $\mathbb{R}^4$): one generates the replica rotation, while the other generates the angular momentum rotation that leaves the poles fixed.

The combined identification is therefore generated by

$$v = 2\pi n\,J_\perp + \theta\,J_\parallel, \tag{16}$$

with angular displacements $(2\pi n,\theta)$. To make the geometric correspondence explicit, consider the rotating-sphere cylinder with metric

$$ds_S^2 = d\eta^2+d\psi^2+\sin^2\psi\,d\tau^2+\cos^2\psi\,d\phi^2, \tag{17}$$

where η is the thermal coordinate and (τ,ϕ) are the two angular coordinates. On the formal $\beta_S=0$ slice of the rotating-sphere thermal EFT, the identification vector is $v_S=(2\pi n)\partial_\tau+\theta\partial_\phi$.

Under the coordinate transformation

$$\sin\psi=\frac{1}{\cosh r},\qquad \cos\psi=\tanh r, \tag{18}$$

the rotating-sphere and hyperbolic-cylinder metric–thermal-vector pairs are locally related by

$$g_S=\frac{1}{\cosh^2 r}\,g_H,\qquad v_S=v_H. \tag{19}$$

Consequently, the local Weyl-invariant thermal-EFT densities considered here agree on the two geometries, independently of derivative order. The relative normalization of the integrated actions follows from their different integration ranges and periods, giving the factor nL, where $L=2\log(R/\epsilon)+O(1)$. The explicit matching of the thermal-EFT densities and integration measures is given in Appendix B.

It was shown in Refs. [8] that on conformally flat backgrounds, the local Weyl-invariant thermal effective action can be expressed in terms of a finite set of geometric invariants. Since the twisted hyperbolic cylinder is also conformally flat, and the thermal identification is generated by a Killing vector v^μ, the same classification applies here. In four dimensions, the relevant scalar invariants are the vorticity scalar $\mathrm{Tr}\,\omega^2$, the Weyl-covariant Schouten scalar $v^2\mathcal{S}$, and the contraction $v^2\mathcal{B}_\mu(\omega^2)^{\mu\nu}\mathcal{B}_\nu$, where $\omega_{\mu\nu}$ is the vorticity tensor, $\mathcal{S}_{\mu\nu}$ is the Weyl-covariant Schouten tensor, and $\mathcal{B}_\mu=\mathcal{S}_{\mu\nu}v^\nu$. We use this basis to evaluate the thermal effective action on the hyperbolic cylinder and the rotating sphere. The conventions and explicit calculations are given in Appendix A.

The local thermal action on $S^1\times\mathbb{H}^3$ is

$$\log Z_{\rm loc}[g,v]=\int d^4x\,\frac{\sqrt{g}}{(v^2)^2}\left[c'+\sum_{j\geq 1}\mathcal{I}_{2j}[g,v]\right], \tag{20}$$

where $\mathcal{I}_{2j}$ denotes the Weyl-invariant local terms at derivative order $2j$. Their coefficients are intrinsic thermal data of the CFT and are independent of the background on which the action is evaluated.

Evaluating Eq. (20) on the twisted hyperbolic cylinder, we obtain

$$\log Z^{\rm loc}_{\mathbb{H}^3}=2\pi(2\pi n)L\int_0^\infty dr\,\frac{\sinh r\cosh r\,\mathcal{I}_{\mathbb{H}^3}(r)}{[(2\pi n)^2+\theta^2\sinh^2 r]^2}, \tag{21}$$

where $\mathcal{I}_{\mathbb{H}^3}$ denotes the complete local derivative expansion, including c', the leading coefficient. Let us evaluate

the Weyl-invariant thermal action on the twisted hyperbolic cylinder. Introducing the radial variable

$$z = \frac{(2\pi n)^2\theta^2\cosh^2 r}{(2\pi n)^2 + \theta^2\sinh^2 r}, \tag{22}$$

we find that the three Weyl-invariant scalars introduced above reduce to

$$\begin{aligned} \frac{1}{8}\,\mathrm{Tr}\,\omega^2 &= -z, \\ v^2\mathcal{S} &= 3z - (2\pi n)^2 - \theta^2, \\ v^2\mathcal{B}_\mu(\omega^2)^{\mu\nu}\mathcal{B}_\nu &= 4z[z-(2\pi n)^2](z-\theta^2). \end{aligned} \tag{23}$$

Moreover, the change of variables in Eq. (22) gives the integral measure

$$\frac{\sinh r\cosh r\,dr}{\left[(2\pi n)^2 + \theta^2\sinh^2 r\right]^2} = \frac{dz}{2(2\pi n)^2\theta^2[(2\pi n)^2-\theta^2]}. \tag{24}$$

Therefore, the local thermal effective action on the twisted hyperbolic cylinder takes the form

$$\log Z^{\rm loc}_{\mathbb{H}^3} = \frac{\pi(2\pi n)L}{(2\pi n)^2\theta^2[(2\pi n)^2-\theta^2]}\int_{\theta^2}^{(2\pi n)^2} dz\,\mathcal{I}_{\mathbb{H}^3}(z). \tag{25}$$

Since the Weyl-invariant scalars are polynomials in z, $(2\pi n)^2$, and θ^2, the integral at each derivative order reduces to a homogeneous polynomial in $(2\pi n)^2$ and θ^2. Consequently, the complete local derivative expansion can be organized as

$$\log Z^{\rm loc}_{\mathbb{H}^3} = -\frac{L}{4\pi^2 n\theta^2}\sum_{m,k\geq 0}\widehat{a}_{m,k}(4\pi^2n^2)^m\theta^{2k}, \tag{26}$$

where $\widehat{a}_{m,k}$ are combinations of the theory-dependent thermal-EFT Wilson coefficients.

We now identify these combinations with the high-temperature rotating-S^3 data. The Weyl-invariant rotating-sphere partition function admits the expansion [8, 24]

$$\begin{aligned} \frac{\log Z^{\rm rot}_{S^3}}{\beta} &= -\frac{1}{y_1y_2}\sum_{m,k,p\geq 0} a^{\rm rot}_{m,k;p}\,y_1^m y_2^k\beta^{2p}, \\ y_i &= \beta^2+\alpha_i^2, \end{aligned} \tag{27}$$

where β is the spherical thermal period and $\alpha_{1,2}$ are angular displacements along its two commuting rotations.

By explicitly matching the Weyl-invariant generators and their reduced integration measures on the two geometries, we find that under the formal continuation

$$\beta = 0, \qquad \alpha_1 = 2\pi n, \qquad \alpha_2 = \theta, \tag{28}$$

the coefficients obtained from the hyperbolic action are precisely those of the rotating-sphere EFT:

$$\widehat{a}_{m,k} = a^{\rm rot}_{m,k;0}. \tag{29}$$

In other words, the local partition functions are related by

$$\log Z^{\rm loc}_{\mathbb{H}^3}(n,\theta) = nL\,\frac{\log Z^{\rm rot}_{S^3}(\beta,\alpha_1,\alpha_2)}{\beta}\bigg|_{\substack{\beta=0\\ \alpha_1=2\pi n,\ \alpha_2=\theta}}. \tag{30}$$

The matching holds at every order in the Weyl-invariant local derivative expansion, independently of the underlying CFT. The explicit derivation, including the relative normalization, is given in Appendix B.

The leading small-n contribution to Eq. (26) comes from $m=0$. Using $L = 2\log(R/\epsilon)+\mathcal{O}(1)$, we obtain

$$\lim_{n\to 0} nS_n(\theta)\Big|^{\rm WI}_{\rm loc,pert} = -\frac{\log(R/\epsilon)}{2\pi^2}\sum_{k\geq 0} a^{\rm rot}_{0,k;0}\theta^{2k-2}. \tag{31}$$

Equation (31) establishes the Weyl-invariant coefficient relation, with $a^{\rm rot}_{0,0;0} = -2\pi^2 c'$, at every order in the local thermal EFT.

Weyl anomaly and physical coefficients.— The relation (31) shows that the angular-momentum-twisted Rényi entropy, expanded first in the small replica index and then in the small rotation angle, is directly related to the high-temperature CFT data encoded in the rotating sphere partition function. However, the physical trace on the rotating-sphere differs from the Weyl-invariant result by a finite Weyl anomaly, $\Delta_S = \log Z_{\rm trace} - \log Z_{\rm rot}$. In the conventions of Ref. [8],

$$\frac{\Delta_S}{\beta_S} = \frac{a_{\rm W}}{12}\left(\frac{y_1}{y_2}+\frac{y_2}{y_1}+7\right), \qquad y_i = \beta_S^2+\alpha_i^2. \tag{32}$$

Similarly, on the hyperbolic cylinder, the physical partition function differs from the Weyl-invariant one by a Wess–Zumino contribution, $\Delta_H = \log Z_H^{\rm phys} - \log Z_H^{\rm WI}$. We work in the $b=0$ curvature-counterterm scheme on both sides, including Eq. (32). Adding a finite $\int\sqrt{g}\,R^2$ counterterm changes the split between the Weyl-invariant and anomalous pieces (including $s_2^{\rm WI}$ in the unit-thermal frame), but not their consistently combined physical-frame result. Although the two Weyl transformations are different, we find that their contributions agree under the same projection as in Eq. (30). In particular, in the leading small-n limit,

$$\begin{aligned} \lim_{n\to 0}\frac{n}{L}\Delta_H\Big|_{\rm loc,pert} &= \lim_{n\to 0} n^2\,\frac{\Delta_S}{\beta_S}\bigg|_{\substack{\beta_S=0\\ \alpha_1=2\pi n,\ \alpha_2=\theta}} \\ &= \frac{a_{\rm W}\theta^2}{48\pi^2}. \end{aligned} \tag{33}$$

The hyperbolic Wess–Zumino contribution is computed independently in Appendix C. Therefore, after including the anomaly on both sides, the correspondence extends to the physical coefficients in Eq. (10). This relation holds for the local perturbative coefficients in the leading small-n expansion.

TESTS OF THE RÉNYI COEFFICIENT RELATIONS

Holographic test.— We first provide a strong-coupling check using five-dimensional Einstein gravity dual to large-N $\mathcal{N}=4$ Yang–Mills theory. We evaluate the local action of a Euclidean rotating topological AdS_5 solution [25, 26], using standard holographic renormalization [27–29]. Writing r_+ for its horizon coordinate and α for the rotation parameter, its radial function is $\Delta(\hat{r}) = (\hat{r}^2-\alpha^2)(\hat{r}^2-1) - 2m$. Smoothness determines the physical cycles,

$$\begin{aligned} \beta_H &= \frac{2\pi(r_+^2-\alpha^2)}{r_+(2r_+^2-\alpha^2-1)}, \\ \theta &= -\frac{2\pi\alpha(r_+^2-1)}{r_+(2r_+^2-\alpha^2-1)}, \end{aligned} \tag{34}$$

where $\beta_H = 2\pi n$. We obtain the local renormalized gravitational action by combining the on-shell Einstein–Hilbert action, the Gibbons–Hawking–York boundary term, and the holographic counterterms. Since the induced boundary metric is conformally related to the physical hyperbolic cylinder,

$$g_b = D^{-1} g_H, \qquad D = 1 + \alpha^2 \sinh^2 r, \tag{35}$$

we must also include the corresponding finite Wess–Zumino contribution to account for the Weyl anomaly and the associated conformal-frame dependence of the partition function.

In the leading small-n sector, let $K_S^{\rm pert}(\theta)$ denote the coefficient defined by $\lim_{n\to 0} nS_n|_{\rm loc,pert} = LK_S^{\rm pert}(\theta)$. Combining the renormalized gravitational on-shell action with the finite Wess–Zumino contribution, we obtain

$$\begin{aligned} K_{S,\mathcal{N}=4}^{\rm pert}(\theta) &= N^2 \frac{(1-q^2)^3}{4q^2(2-q^2)^2} \\ &= N^2\left[\frac{\pi^2}{16\theta^2} - \frac{1}{16} + \frac{\theta^2}{32\pi^2} - \frac{\theta^4}{64\pi^4} + \mathcal{O}(\theta^6)\right], \end{aligned} \tag{36}$$

where $q = \alpha/r_+$ is related to the physical angular twist by $\theta = -2\pi q/(2-q^2)$ in the leading small-n limit. The first line determines the complete leading local high-temperature function, rather than only its small-angle coefficients.

The Wess–Zumino contribution here relates the natural black-hole boundary frame to g_H, rather than the unit-thermal-length frame used in Appendix C. The distinct frame transformations explain its nonzero θ^4 term, as checked explicitly in that appendix.

We independently verify the coefficients using the renormalized on-shell action of singly rotating Kerr–AdS_5, with the conformal boundary chosen to be the round $S^1 \times S^3$ [8, 30]. By symmetry under exchange of the two rotation planes, $a_{m,0;0} = a_{0,m;0}$, so the leading small-n projection selects the one-spin series $\sum_{m\geq 0} a_{0,m;0}\theta^{2m-2}$. Normalizing by the sphere's fixed-circle length 2π, rather than the hyperbolic length L, gives agreement through $\mathcal{O}(\theta^4)$. The details are provided in the Supplemental Material [31].

Free CFT checks.— We verify the perturbative coefficient relation for free scalar and Dirac fields using hyperbolic heat kernels and spin characters. We obtain

$$\begin{aligned} K_{S,\rm sc}^{\rm pert}(\theta) &= \frac{\pi^2}{180\theta^2} + \frac{1}{144} - \frac{\theta^2}{960\pi^2}, \\ K_{S,\rm D}^{\rm pert}(\theta) &= \frac{7\pi^2}{360\theta^2} - \frac{1}{144} - \frac{7\theta^2}{1920\pi^2}, \end{aligned} \tag{37}$$

The θ^2 terms agree with the corresponding $-a_{0,2;0}^{\rm trace}/(4\pi^2)$ from the physical rotating-sphere traces, without adding the anomaly shift a second time. It will be interesting to test this phenomenon in exotic free fields [12] including gauge theories where edge modes require separate treatment [32–34].

DISCUSSION

In this Letter, we have related an information-theoretic quantity, the angular-momentum-twisted Rényi entropy across a spherical entangling surface, to the high-temperature CFT data on S^3 that govern the asymptotic counting of local operators. The origin of this connection lies in the locality of the thermal effective action: the same local Weyl-invariant densities determine both quantities, while their integration measures establish the coefficient correspondence. Thus, although the individual coefficients depend on the underlying CFT, the relation between them is universal. The leading Rényi coefficient is determined by the high-temperature equation of state, whereas the subleading coefficients probe higher-derivative thermal data. We have independently tested this relation in free field theories and in strongly coupled, large-N $\mathcal{N}=4$ Yang–Mills theory using a local gravitational calculation in AdS_5. The agreement provides further evidence for the role of locality in organizing universal properties of CFT entanglement.

An interesting extension is to include global symmetries and study Rényi entropies twisted by both angular momentum and conserved charges, with connections to charged and symmetry-resolved entanglement [35–37]. In this case, the thermal effective action also depends on background gauge fields and chemical potentials [3, 4]. It would be interesting to determine how the additional thermal data and possible anomalies enter the Rényi coefficients.

Another direction is to investigate parity-odd contributions to the stationary thermal effective action, including anomaly-induced terms, and understand how they modify both the high-temperature rotating-sphere data

and the corresponding twisted Rényi coefficients. Finally, locality imposes nontrivial relations among high-temperature CFT coefficients [8]. By Eq. (10), any such relations involving $a^{\text{trace}}_{0,q;0}$ already imply corresponding relations among the local Rényi coefficients. A systematic classification of these constraints remains of interest.

We have focused on the local thermal-EFT contributions to twisted Rényi entropies. A systematic treatment of additional contributions, including rotational winding sectors [38] and gauge zero and edge modes [32, 33], is an interesting direction for future work.

Acknowledgements

The author thanks Christopher Herzog for comments on the manuscript. Claude Opus 5.5 (Anthropic) was used to assist with scientific discussions, manuscript drafting, literature searches, and verification of formulae. The author takes full responsibility for the scientific content and accuracy of the manuscript.

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APPENDIX A: WEYL-INVARIANT REDUCTION ON THE HYPERBOLIC CYLINDER

In this appendix, we explain why the invariant densities entering the local correspondence have the same functional form on the two geometries. The thermal identification on the hyperbolic cylinder is generated by $v = 2\pi n\,\partial_\tau + \theta\,\partial_\phi$. Its norm, which determines the local proper length of the thermal circle, is

$$v^2 = (2\pi n)^2 + \theta^2 \sinh^2 r. \tag{38}$$

The all-order correspondence has a simple geometric origin. Let us consider the metric on the rotating-sphere cylinder, with thermal identification vector evaluated on the formal $\beta_S = 0$ slice,

$$\begin{aligned} ds_S^2 &= d\eta^2 + d\psi^2 + \sin^2\psi\, d\tau^2 + \cos^2\psi\, d\phi^2, \\ v_S &= (2\pi n)\partial_\tau + \theta\partial_\phi. \end{aligned} \tag{39}$$

Under the coordinate transformation

$$\sin\psi = \frac{1}{\cosh r}, \qquad \cos\psi = \tanh r, \tag{40}$$

the rotating-sphere and hyperbolic-cylinder metrics are locally related by

$$g_S = \frac{1}{\cosh^2 r}\, g_H, \qquad v_S = v_H. \tag{41}$$

Thus, the two metric–thermal-vector pairs are locally Weyl equivalent. Consequently, every Weyl-invariant local thermal density agrees on the two geometries, independently of its derivative order. In particular, the measure $\sqrt{g}/(v^2)^2$ is Weyl invariant in four dimensions. This establishes the all-order matching of the local Weyl-invariant bulk sector without requiring an explicit classification of independent scalar invariants.

The conformal equivalence is local. The relative normalization of the integrated actions is established in Appendix B, while anomalous Weyl contributions are treated separately in Appendix C.

To describe the thermal circle as a fibration, we choose a unit-period coordinate $t = \tau/(2\pi n)$ and $\varphi = \phi - \theta t$. The Killing vector becomes ∂_t, and the metric can be written as

$$\begin{aligned} ds^2 &= v^2(dt + \mathcal{A})^2 + h_{ij}dx^i dx^j, \\ \mathcal{A} &= \frac{\theta\sinh^2 r}{v^2}\, d\varphi, \\ h &= dr^2 + \cosh^2 r\, d\eta^2 + \frac{(2\pi n)^2\sinh^2 r}{v^2} d\varphi^2. \end{aligned} \tag{42}$$

Under a Weyl transformation, h and v^2 have the same weight. Hence the quotient metric h/v^2 and the one-form $\mathcal{A}$ encode Weyl-invariant local data. In particular, their curvature and field strength, written using the variable z in Eq. (22), obey

$$\mathcal{R}[h/v^2] = 20z - 6[(2\pi n)^2 + \theta^2], \qquad (d\mathcal{A})^2 = 8z. \tag{43}$$

Here the last contraction uses the metric h/v^2. The Weyl-covariant combinations used in the main text consequently take the polynomial form displayed in Eq. (23). For instance, the two independent scalars at two-derivative order reduce to $-z$ and $3z - (2\pi n)^2 - \theta^2$.

These expressions illustrate the reduction of local thermal invariants to functions of the radial variable z and the angular displacements. More generally, the matching of higher-derivative Weyl-invariant densities follows directly from the local conformal equivalence in Eq. (41), without assuming that the displayed invariants exhaust the independent scalar structures at every derivative order.

APPENDIX B: REDUCED MEASURES AND THE COEFFICIENT RELATION

We now verify the normalization in Eq. (30) directly. For a fixed local thermal density, let $\mathcal{I}(z)$ denote its value on the hyperbolic cylinder after inserting the invariants in Eq. (23). Its radial integral is Eq. (21). Differentiating Eq. (22), we obtain

$$\frac{\sinh r\cosh r\, dr}{(v^2)^2} = \frac{dz}{2(2\pi n)^2\theta^2[(2\pi n)^2 - \theta^2]}. \tag{44}$$

At $r = 0$ and $r = \infty$, respectively, $z = \theta^2$ and $z = (2\pi n)^2$. Performing the integrals over the twisted circle and ϕ gives

$$\log Z^{\rm loc}_{\mathbb{H}^3} = \frac{\pi(2\pi n)L}{(2\pi n)^2\theta^2[(2\pi n)^2 - \theta^2]} \times \int_{\theta^2}^{(2\pi n)^2} dz\, \mathcal{I}(z). \tag{45}$$

For the rotating sphere, write the unit-round metric as $ds^2_{S^3} = d\psi^2 + \sin^2\psi\, d\phi_1^2 + \cos^2\psi\, d\phi_2^2$. On the formal $\beta = 0$ slice, the thermal vector has angular displacements $2\pi n$ and θ. Its squared norm is $w_S = (2\pi n)^2\sin^2\psi + \theta^2\cos^2\psi$. The variable $z_S = (2\pi n)^2\theta^2/w_S$ gives exactly the same invariant density $\mathcal{I}(z_S)$ by conformal covariance. Integrating over both sphere angles and substituting z_S yields

$$\begin{aligned} &\frac{\log Z^{\rm rot}_{S^3}}{\beta}\bigg|_{\substack{\beta=0 \\ \alpha_1=2\pi n,\, \alpha_2=\theta}} \\ &\qquad = \frac{2\pi^2}{(2\pi n)^2\theta^2[(2\pi n)^2 - \theta^2]} \int_{\theta^2}^{(2\pi n)^2} dz\, \mathcal{I}(z). \end{aligned} \tag{46}$$

The ratio of the prefactors in Eqs. (45) and (46) is nL: this proves Eq. (30) without using any particular thermal-EFT coefficient. The equality is understood order by order under the indicated formal continuation.

At derivative order $2j$, homogeneity implies $\mathcal{I}_j(z;0,\theta^2) = \theta^{2j}\mathcal{I}_j(z/\theta^2;0,1)$. Taking $n \to 0$ in Eq. (45), we obtain

$$\log Z^{{\rm loc},(2j)}_{\mathbb{H}^3}\Big|_{L/n} = \frac{L}{2n}\,\theta^{2j-2}\int_0^1 dt\, \mathcal{I}_j(t;0,1). \tag{47}$$

Thus, the local twisted Rényi coefficients are determined by the corresponding rotating-sphere thermal-EFT couplings. Anomalous Weyl contributions and nonuniform rotational windings are treated separately.

APPENDIX C: FINITE WEYL ANOMALY IN THE LOCAL THERMAL FRAME

We derive the anomalous correction appearing in Eq. (33) directly on the hyperbolic cylinder. The Weyl-invariant thermal effective action is naturally evaluated in the unit-thermal-length frame, $\widehat{g}_H = g_H/v^2$, where $v^2 = \beta_H^2 + \theta^2 \sinh^2 r$ and $\beta_H = 2\pi n$. The corresponding Weyl factor, up to a constant rescaling, is

$$\sigma(r) = \frac{1}{2} \log \left(1 + \frac{\theta^2}{\beta_H^2} \sinh^2 r \right). \tag{48}$$

In the $b = 0$ counterterm scheme, both frames are conformally flat, so only the type-A anomaly contributes. We use the finite Wess–Zumino functional [42], whose explicit form and evaluation are given in the Supplemental Material [31].

In the leading small-n limit, the local contribution is concentrated near the rotational fixed axis, $r \sim \beta_H/|\theta|$. Introducing $w = \theta^2 r^2/\beta_H^2$, the transverse integral of the finite Wess–Zumino action reduces to

$$\begin{aligned} \Delta_H &= -\frac{a_{\mathrm{W}}}{16\pi^2} \frac{2\pi L \theta^2}{\beta_H} \int_0^\infty dw \, \frac{w(w-4)}{(1+w)^4} + o(L/\beta_H) \\ &= \frac{L}{n} \frac{a_{\mathrm{W}} \theta^2}{48\pi^2} + o(L/n). \end{aligned} \tag{49}$$

This reproduces the leading small-n projection of the rotating-sphere anomaly in Eq. (32), establishing the correspondence for the physical thermal coefficients in Eq. (10).

The holographic calculation involves a different Weyl transformation, relating the natural black-hole boundary metric $g_b = D^{-1} g_H$, with $D = 1 + \alpha^2 \sinh^2 r$, to g_H. In the leading small-n limit, the horizon relations give

$$\theta = -\frac{2\pi q}{2 - q^2}, \qquad \alpha = -\frac{\theta}{\beta_H}(1 - q^2), \qquad q = \frac{\alpha}{r_+}. \tag{50}$$

The same finite Wess–Zumino functional, now evaluated with the Weyl factor determined by D, gives

$$\begin{aligned} \Delta_{b \to H} \log Z|_{L/n} &= \frac{L}{n} \frac{a_{\mathrm{W}}}{48\pi^2} \theta^2 (1 - q^2)^2 \\ &= \frac{L}{n} N^2 \left[\frac{\theta^2}{192\pi^2} - \frac{\theta^4}{96\pi^4} + \mathcal{O}(\theta^6) \right], \end{aligned} \tag{51}$$

where $a_{\mathrm{W}} = N^2/4$. Thus, the additional θ^4 term in the holographic calculation arises because the two Wess–Zumino contributions connect different conformal frames.

SUPPLEMENTAL MATERIAL

In this supplemental material, we provide the details of the independent holographic and free-field checks of the local perturbative coefficient relation described in the Letter.

The angular-momentum twisted Rényi partition function of a conformal field theory across a spherical entangling surface can be computed using Casini-Huerta-Myers map [1]

$$\begin{aligned} Z_n(\theta) &= \mathrm{Tr}_{\mathbb{H}^3}\left(e^{-\beta H} e^{i\theta J}\right), & \beta &= 2\pi n, \\ ds_H^2 &= d\tau^2 + dr^2 + \cosh^2 r\, d\eta^2 + \sinh^2 r\, d\phi^2, & (\tau,\phi) &\sim (\tau+\beta, \phi+\theta). \end{aligned} \tag{S1}$$

Introducing a UV cutoff ϵ near the spherical entangling surface in the original flat-space geometry induces the following cutoff on hyperbolic space:

$$\cosh r \cosh \eta \le \cosh U, \qquad U = \log\frac{2R}{\epsilon} + \mathcal{O}(\epsilon/R), \qquad L = 2U = 2\log\frac{R}{\epsilon} + \mathcal{O}(1). \tag{S2}$$

We develop a thermal effective action on the hyperbolic cylinder and the local expansion of the effective field theory is organized for $\beta = 2\pi n \ll |\theta| \ll 1$. We define $K^{\mathrm{pert}}(\theta)$ by extracting the coefficient of L/n from the local thermal effective action, or, in the free-field winding representation, by the corresponding termwise Mellin/ζ-function coefficient prescription:

$$\begin{aligned} \log Z_n(\theta)|_{\mathrm{loc,log,pert}} &= \frac{L}{n} K^{\mathrm{pert}}(\theta) + \text{subleading local terms in } n, \\ n S_n(\theta)|_{\mathrm{loc,log,pert}} &= L K^{\mathrm{pert}}(\theta) + \text{subleading local terms in } n. \end{aligned} \tag{S3}$$

Here $S_n(\theta) = (1-n)^{-1}\log[\mathcal{Z}_n(\theta)/\mathcal{Z}_1(\theta)^n]$, with $\mathcal{Z}_n(\theta) = Z_n(\theta)/Z_1(0)^n$. The normalization terms do not affect the leading L/n coefficient. We work throughout in the local thermal-EFT regime $2\pi n \ll |\theta| \ll 1$, where the perturbative expansion is organised by first taking the small-n limit of the local contribution and subsequently expanding in small θ.

All-order correspondence with the rotating-sphere EFT

We write the metric on $S^1 \times \mathbb{H}^3$ in cylindrical coordinates as

$$ds^2 = d\tau^2 + dr^2 + \cosh^2 r\, d\eta^2 + \sinh^2 r\, d\phi^2, \qquad (\tau,\eta,\phi) \sim (\tau+\beta, \eta, \phi+\theta). \tag{S4}$$

The thermal vector and its norm are

$$v = \beta\partial_\tau + \theta\partial_\phi, \qquad v^2 = \beta^2 + \theta^2 \sinh^2 r. \tag{S5}$$

We introduce the variable

$$z = \frac{\beta^2\theta^2\cosh^2 r}{\beta^2 + \theta^2 \sinh^2 r}. \tag{S6}$$

Using the unit-thermal-norm Weyl frame $\widehat{g} = g/v^2$, the vorticity and Weyl-covariant Schouten tensors give

$$\begin{aligned} \frac{1}{8}\mathrm{Tr}\,\omega^2 &= -z, \\ v^2\mathcal{S} &= 3z - \beta^2 - \theta^2, \\ v^2 \mathcal{B}_\mu (\omega^2)^{\mu\nu} \mathcal{B}_\nu &= 4z(z-\beta^2)(z-\theta^2), \qquad \mathcal{B}_\mu = \mathcal{S}_{\mu\nu} v^\nu. \end{aligned} \tag{S7}$$

The first relation follows from $\omega = d(v^\flat/v^2)$ in the unit-norm frame, the second from its scalar curvature, and the third from the conformal-Killing identity $\omega_\mu{}^\nu \mathcal{B}_\nu = -v^2 D_\mu \mathcal{S}/3$.

We now consider the local thermal-EFT integral. The η integral along the CHM-regulated fixed geodesic gives the logarithmic factor L in Eq. (S2). The remaining transverse integral can be written in terms of z using

$$\frac{\sinh r \cosh r\, dr}{(v^2)^2} = \frac{dz}{2\beta^2\theta^2(\beta^2-\theta^2)}. \tag{S8}$$

The integration limits are $z = \theta^2$ and $z = \beta^2$, with the orientation understood when $\beta^2 < \theta^2$.

We next compare this integral with the rotating-sphere EFT on $S^1 \times S^3$. We write the round three-sphere metric as

$$ds^2_{S^3} = d\psi^2 + \cos^2\psi\, d\phi_1^2 + \sin^2\psi\, d\phi_2^2, \qquad 0 \le \psi \le \frac{\pi}{2}. \tag{S9}$$

Taking the two rotational displacements to be (β, θ), the rotational vector and its norm, are

$$\begin{aligned} v_S &= \beta\partial_{\phi_1} + \theta\partial_{\phi_2}, \\ v_S^2 &= \beta^2\cos^2\psi + \theta^2\sin^2\psi. \end{aligned} \tag{S10}$$

Introducing, the variable z_S

$$z_S = \frac{\beta^2\theta^2}{v_S^2}, \tag{S11}$$

we find that the three scalar quantities in Eq. (S7) take exactly the same form on the rotating sphere, with z replaced by z_S. In particular, if the sphere basis is written using $\zeta = \mathrm{Tr}\,\omega^2/8$, our convention gives $\zeta = -z_S$, fixing the relative sign.

The sphere integral also takes the same form, since

$$\frac{\sin\psi\cos\psi\, d\psi}{(v_S^2)^2} = \frac{dz_S}{2\beta^2\theta^2(\beta^2 - \theta^2)}. \tag{S12}$$

The integration limits are again θ^2 and β^2. In fact, the change of coordinates

$$\tan\psi = \sinh r \tag{S13}$$

identifies $z_S = z$ and the two reduced measures.

Since every relevant parity-even Weyl-invariant bulk density can be expressed as a polynomial in these three scalars, up to total derivatives [2], the two backgrounds give the same reduced local integrals. The transverse angular integrals give 2π on the hyperbolic cylinder and $(2\pi)^2$ on the sphere. We therefore obtain the all-order relation

$$\log Z_H^{\mathrm{WI,loc}}(n,\theta) = nL\, \frac{\log Z_{S^3}^{\mathrm{WI,rot}}(\beta_S, \alpha_1, \alpha_2)}{\beta_S}\bigg|_{\substack{\beta_S=0 \\ \alpha_1=2\pi n,\ \alpha_2=\theta}}. \tag{S14}$$

Here the formal $\beta_S = 0$ specialization and the $n \to 0$ limit are understood as coefficient extractions in the local derivative expansion. Neither defines a pointwise angular thermal trace.

Wess–Zumino contribution for a general four-dimensional CFT

The Weyl-invariant thermal effective action is naturally evaluated in the frame where the thermal vector has unit norm. On the hyperbolic cylinder, this frame is defined by

$$\widehat{g}_H = \frac{g_H}{v^2}, \tag{S15}$$

where the thermal vector and its norm are given in Eq. (S5).

In even dimensions, the partition function is not invariant under a Weyl transformation because of the Weyl anomaly. Under $g_{ij} \to e^{2\omega} g_{ij}$, its variation in $d = 4$ dimension is given by [3, 4]

$$\delta_\omega \log Z[g] = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\omega\left(cW_{ijkl}W^{ijkl} - a_{\mathrm{W}}E_4 + b\Box R\right), \tag{S16}$$

where W_{ijkl} is the Weyl tensor and E_4 is the four-dimensional Euler density,

$$E_4 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2. \tag{S17}$$

The coefficient b is scheme dependent and can be shifted by a finite local R^2 counterterm. We work in the scheme $b = 0$ [3, 5].

Since the hyperbolic cylinder is conformally flat, the Weyl tensor vanishes in both frames. Therefore, only the Euler anomaly contributes. We write the Weyl transformation as

$$\widehat{g}_H = \beta^{-2} e^{-2\sigma} g_H, \qquad \sigma(r) = \frac{1}{2}\log\left(1 + \frac{\theta^2}{\beta^2}\sinh^2 r\right). \tag{S18}$$

The constant rescaling by β^{-2} does not contribute to the leading local L/β term in the chosen scheme.

The finite Wess–Zumino contribution associated with this transformation is [4]

$$\Delta_{\mathrm{th}\to\mathrm{H}} \log Z = -\frac{a_{\mathrm{W}}}{16\pi^2}\int d^4x\sqrt{g_H}\left[\sigma E_4[g_H] + 4G_H^{ij}\partial_i\sigma\partial_j\sigma - 4(\nabla\sigma)^2\nabla^2\sigma + 2(\nabla\sigma)^4\right], \tag{S19}$$

where $G_H^{ij} = R_H^{ij} - g_H^{ij}R_H/2$ is the Einstein tensor of g_H.

For $S^1 \times \mathbb{H}^3$, we have

$$E_4[g_H] = 0, \qquad G_H^{rr} = 1. \tag{S20}$$

Since σ depends only on r,

$$(\nabla\sigma)^2 = (\sigma')^2, \qquad \nabla^2\sigma = \sigma'' + (\tanh r + \coth r)\sigma'. \tag{S21}$$

We now evaluate the leading contribution in the limit $\beta = 2\pi n \to 0$ at fixed nonzero θ. The dominant local contribution comes from $r \sim \beta/|\theta|$, near the rotational fixed axis. In this region, the metric becomes

$$ds_H^2 \simeq d\tau^2 + d\eta^2 + dr^2 + r^2 d\phi^2. \tag{S22}$$

Introducing the variable

$$w = \frac{\theta^2 r^2}{\beta^2}, \qquad \sigma = \frac{1}{2}\log(1+w), \tag{S23}$$

we obtain

$$(\nabla\sigma)^2 = \frac{\theta^2}{\beta^2}\frac{w}{(1+w)^2}, \qquad \nabla^2\sigma = \frac{2\theta^2}{\beta^2}\frac{1}{(1+w)^2}. \tag{S24}$$

The term involving the Einstein tensor is subleading in the small-β expansion. The leading contribution therefore comes from the last two terms in Eq. (S19), which give

$$-4(\nabla\sigma)^2\nabla^2\sigma + 2(\nabla\sigma)^4 = \frac{2\theta^4}{\beta^4}\frac{w(w-4)}{(1+w)^4}. \tag{S25}$$

Using the transverse measure $d^2x_\perp = r\,dr\,d\phi$, with $r\,dr = \frac{\beta^2}{2\theta^2}\,dw$, we find

$$\int d^2x_\perp\left[-4(\nabla\sigma)^2\nabla^2\sigma + 2(\nabla\sigma)^4\right] = \frac{2\pi\theta^2}{\beta^2}\int_0^\infty dw\,\frac{w(w-4)}{(1+w)^4} = -\frac{2\pi\theta^2}{3\beta^2}. \tag{S26}$$

Including the thermal period β and the regulated length L along the fixed geodesic, Eq. (S19) gives

$$\Delta_{\mathrm{th}\to\mathrm{H}} \log Z = \frac{a_{\mathrm{W}} L\theta^2}{24\pi\beta} + o(L/\beta) = \frac{L}{n}\frac{a_{\mathrm{W}}\theta^2}{48\pi^2} + o(L/n). \tag{S27}$$

Thus, the leading local Weyl-anomaly contribution is determined without expanding in θ. In particular, it contributes only at order θ^2,

$$\Delta K_{\mathrm{th}\to\mathrm{H}}(\theta) = \frac{a_{\mathrm{W}}\theta^2}{48\pi^2}. \tag{S28}$$

There are no additional analytic $\theta^4, \theta^6, \ldots$ contributions to this leading local term.

This result concerns the Weyl transformation between the unit-thermal-norm frame and the physical hyperbolic cylinder. It is distinct from the finite Weyl transformation between the natural conformal boundary of the rotating AdS_5 solution and the hyperbolic cylinder, which is evaluated separately in the holographic calculation.

Agreement with the spherical anomaly and coefficient map

We now compare the Wess–Zumino contribution obtained above with the anomaly contribution in the rotating-sphere EFT. In the conventions of Ref. [2], the finite type-A anomaly shift between the trace and Weyl-invariant frames is

$$\frac{\log Z_{S^3}^{\rm trace} - \log Z_{S^3}^{\rm WI,rot}}{\beta_S} = \frac{a_{\rm W}}{12}\left(\frac{y_1}{y_2} + \frac{y_2}{y_1} + 7\right), \qquad y_i = \beta_S^2 + \alpha_i^2. \tag{S29}$$

Setting $\beta_S = 0$, $\alpha_1 = 2\pi n$ and $\alpha_2 = \theta$, and multiplying by nL, we obtain

$$nL\,\frac{\log Z_{S^3}^{\rm trace} - \log Z_{S^3}^{\rm WI,rot}}{\beta_S}\bigg|_{\substack{\beta_S=0\\ \alpha_1=2\pi n,\,\alpha_2=\theta}} = \frac{a_{\rm W} nL}{12}\left(\frac{4\pi^2 n^2}{\theta^2} + \frac{\theta^2}{4\pi^2 n^2} + 7\right) = \frac{L}{n}\frac{a_{\rm W}\theta^2}{48\pi^2} + \mathcal{O}(nL). \tag{S30}$$

This agrees precisely with the Wess–Zumino contribution obtained on the hyperbolic cylinder in Eq. (S27).

Denoting the operator-counting coefficients in the Weyl-invariant and trace frames by $a_{0,k;0}^{\rm WI}$ and $a_{0,k;0}^{\rm trace}$, respectively, the anomaly shifts the coefficients according to

$$a_{0,k;0}^{\rm trace} - a_{0,k;0}^{\rm WI} = -\frac{a_{\rm W}}{12}\delta_{k,2}, \qquad s_k^{\rm phys,loc} - s_k^{\rm WI} = \frac{a_{\rm W}}{48\pi^2}\delta_{k,2}. \tag{S31}$$

Combining these relations with the Weyl-invariant correspondence in Eq. (S14), we obtain

$$s_k^{\rm phys,loc} = -\frac{a_{0,k;0}^{\rm trace}}{4\pi^2}, \qquad k \geq 1. \tag{S32}$$

This establishes the all-order correspondence between the local logarithmic coefficients of the twisted Rényi entropy and the rotating-sphere EFT coefficients, including the Weyl-anomaly contribution.

Independent holographic test at large N

Exact continued topological Kerr–AdS$_5$ geometry.— We work in five-dimensional Einstein gravity, with the AdS radius set to one,

$$R_{MN} + 4G_{MN} = 0, \qquad G_5^{-1} = \frac{2N^2}{\pi}, \qquad a_{\rm W} = c = \frac{\pi}{8G_5} = \frac{N^2}{4}. \tag{S33}$$

The bulk dual of the thermal CFT on $\mathbb{H}^3$ is the topological AdS$_5$ black hole. We consider its singly rotating generalization [6, 7], analytically continued from the Lorentzian rotation parameter $a_L = i\alpha$. The Euclidean metric takes the form

$$\begin{aligned} ds_5^2 = {} & \frac{\Delta}{PD^2}\left(d\tau - \alpha\sinh^2 r\, d\phi\right)^2 + \frac{P}{\Delta}\,d\hat{r}^2 + \frac{P}{D}\,dr^2 \\ & + \frac{\sinh^2 r}{PD^2}\left[\alpha(1-\hat{r}^2)d\tau - (\hat{r}^2 - \alpha^2)d\phi\right]^2 + \frac{\hat{r}^2\cosh^2 r}{D}\,d\eta^2, \end{aligned} \tag{S34}$$

where

$$D = 1 + \alpha^2\sinh^2 r, \qquad P = \hat{r}^2 - \frac{\alpha^2\cosh^2 r}{D}, \qquad \Delta = (\hat{r}^2 - \alpha^2)(\hat{r}^2 - 1) - 2m, \tag{S35}$$

Here $\hat{r}$ is the radial coordinate in the five-dimensional bulk, while r parametrizes the transverse direction of the boundary hyperbolic space. The horizon is located at $\hat{r} = r_+$, where r_+ is the largest root of $\Delta = 0$, with $r_+ > \max(1, |\alpha|)$. Thus, the location of the horizon can be found from the following equation

$$2m = (r_+^2 - \alpha^2)(r_+^2 - 1). \tag{S36}$$

The determinant of the bulk metric is also given by

$$\sqrt{G} = \frac{\hat{r}P\sinh r\cosh r}{D^2}. \tag{S37}$$

The natural conformal boundary metric is obtained by taking $\hat{r} \to \infty$,

$$g_b = \lim_{\hat{r}\to\infty} \hat{r}^{-2} G\big|_{d\hat{r}=0} = \frac{g_H}{D}, \qquad D = 1 + \alpha^2 \sinh^2 r. \tag{S38}$$

The finite Weyl-anomaly contribution associated with the transformation from g_b to the physical hyperbolic metric g_H must therefore be included in the renormalized on-shell action.

Regularity at the horizon determines the thermal period β and angular twist θ. The horizon angular velocity and the corresponding identifications are

$$\begin{aligned} \Omega_H &= -\frac{\alpha(r_+^2 - 1)}{r_+^2 - \alpha^2}, \\ \beta &= \frac{2\pi(r_+^2 - \alpha^2)}{r_+(2r_+^2 - \alpha^2 - 1)}, \\ \theta &= \beta\Omega_H = -\frac{2\pi\alpha(r_+^2 - 1)}{r_+(2r_+^2 - \alpha^2 - 1)}. \end{aligned} \tag{S39}$$

For the high-temperature expansion, we introduce

$$q = \frac{\alpha}{r_+}, \qquad \kappa = r_+^{-2}. \tag{S40}$$

The relations in Eq. (S39) become

$$\beta r_+ = \frac{2\pi(1 - q^2)}{2 - q^2 - \kappa}, \qquad \theta = -\frac{2\pi q(1 - \kappa)}{2 - q^2 - \kappa}. \tag{S41}$$

We take the high-temperature limit $r_+ \to \infty$ with q fixed, so that $\alpha = q r_+$ and the physical twist θ remains finite. In this limit,

$$\beta r_+ = \frac{2\pi(1 - q^2)}{2 - q^2} + \mathcal{O}(\beta^2), \qquad q = -\frac{\theta}{\pi} + \frac{\theta^3}{2\pi^3} + \mathcal{O}(\theta^5). \tag{S42}$$

We first extract the leading high-temperature contribution at fixed θ and then expand the result at small θ.

The relevant contribution comes from the region $r \sim \alpha^{-1} \sim \beta/|\theta|$ near the rotational axis. We use the exact continued metric to determine the local high-temperature coefficient, without assuming a smooth global completion of the twisted boundary geometry. The corresponding cutoff estimate is given after Eq. (S67).

Einstein, boundary, and counterterm actions

We evaluate the renormalized Euclidean action using standard holographic renormalization [5, 8, 9]. The action consists of the Einstein–Hilbert term, the Gibbons–Hawking boundary term, and the counterterms,

$$I_E = I_{\rm EH} + I_{\rm GH} + I_{\rm ct}, \tag{S43}$$

where

$$\begin{aligned} I_{\rm EH} &= -\frac{1}{16\pi G_5} \int_M d^5x \sqrt{G}\,(R + 12), \\ I_{\rm GH} &= -\frac{1}{8\pi G_5} \int_{\partial M} d^4x \sqrt{h}\,K, \\ I_{\rm ct} &= \frac{1}{8\pi G_5} \int_{\partial M} d^4x \sqrt{h} \left(3 + \frac{1}{4}\mathcal{R}[h]\right) + \frac{\log(\Lambda/\mu)}{64\pi G_5} \int_{\partial M} d^4x \sqrt{h} \left(\mathcal{R}_{ab}[h]\mathcal{R}^{ab}[h] - \frac{1}{3}\mathcal{R}[h]^2\right). \end{aligned} \tag{S44}$$

Here h_{ab} is the induced boundary metric, K is the trace of the extrinsic curvature, and $\mathcal{R}_{ab}[h]$ is the Ricci tensor of h_{ab}.

We introduce a large radial cutoff $\hat{r} = \Lambda$. On this surface, the induced metric is

$$ds_h^2 = \frac{\Delta_\Lambda}{P_\Lambda D^2}\big(d\tau - \alpha \sinh^2 r\, d\phi\big)^2 + \frac{P_\Lambda}{D}\, dr^2 + \frac{\sinh^2 r}{P_\Lambda D^2}\big[\alpha(1 - \Lambda^2)d\tau - (\Lambda^2 - \alpha^2)d\phi\big]^2 + \frac{\Lambda^2 \cosh^2 r}{D}\, d\eta^2, \tag{S45}$$

where $P_\Lambda = P|_{\hat{r}=\Lambda}$ and $\Delta_\Lambda = \Delta|_{\hat{r}=\Lambda}$. We introduce a regulated length L_η along the fixed geodesic to extract the local tube contribution. The relation to the CHM cutoff is discussed below.

Using the on-shell relation $R = -20$ and the bulk determinant in Eq. (S37), the Einstein–Hilbert contribution becomes

$$I_{\rm EH} = \frac{\beta L_\eta}{8G_5\alpha^2}\left[\Lambda^4 - r_+^4 - (\alpha^2+1)(\Lambda^2 - r_+^2)\right]. \tag{S46}$$

The Gibbons–Hawking term evaluated on the cutoff surface gives

$$I_{\rm GH} = -\frac{\beta L_\eta}{4G_5}\left[\frac{2\Lambda^4}{\alpha^2} - \frac{5(\alpha^2+1)\Lambda^2}{4\alpha^2} - \frac{\alpha^4 - 12\alpha^2 r_+^2 + 4\alpha^2 + 12r_+^4 - 12r_+^2 + 1}{12\alpha^2}\right] + \mathcal{O}(\Lambda^{-2}). \tag{S47}$$

Similarly, the power counterterms give

$$I_{\rm ct}^{\rm power} = \frac{\beta L_\eta}{G_5}\left[\frac{3\Lambda^4}{8\alpha^2} - \frac{3(\alpha^2+1)\Lambda^2}{16\alpha^2} - \frac{\alpha^4 - 5\alpha^2 + 24m + 1}{64\alpha^2}\right] + \mathcal{O}(\Lambda^{-2}). \tag{S48}$$

The Λ^4 and Λ^2 divergences cancel upon adding these three contributions. The logarithmic counterterm has vanishing integrated contribution to the leading local term.

Using the horizon relation $2m = (r_+^2 - \alpha^2)(r_+^2 - 1)$, we obtain the finite result

$$I_{\rm ren,tube}^{\rm power} = \frac{\beta L_\eta}{192G_5\alpha^2}\left[\alpha^4 + 12\alpha^2 r_+^2 - 5\alpha^2 - 12r_+^4 + 12r_+^2 + 1\right]. \tag{S49}$$

This result includes the Einstein–Hilbert, Gibbons–Hawking, and power counterterm contributions. We will evaluate the finite Weyl-anomaly contribution associated with the transformation from g_b to g_H separately.

Natural-frame high-temperature coefficient

We now extract the high-temperature coefficients from the renormalized gravitational action given in (S49) and compare them with the rotating-sphere EFT. The gravitational action computed above is in the natural boundary frame g_b, whereas the twisted Rényi partition function is defined in the physical hyperbolic frame g_H. We first obtain the coefficients in g_b and then include the finite Weyl-anomaly contribution below.

Using Eqs. (S39)–(S41) in Eq. (S49), and taking the high-temperature limit $\beta \to 0$ at fixed θ, we obtain

$$\frac{\log Z_b}{L_\eta} = \frac{\pi^2}{48G_5\beta}\frac{(1-q^2)^2(12 - 12q^2 - q^4)}{q^2(2-q^2)^2} + \mathcal{O}(\beta). \tag{S50}$$

Expanding this result at small θ using Eq. (S41), and setting $G_5^{-1} = 2N^2/\pi$ and $\beta = 2\pi n$, we find

$$\left.\frac{n\log Z_b}{L_\eta}\right|_{\beta\to 0} = N^2\left[\frac{\pi^2}{16\theta^2} - \frac{1}{16} + \frac{5\theta^2}{192\pi^2} - \frac{\theta^4}{192\pi^4} - \frac{\theta^6}{256\pi^6} + \mathcal{O}(\theta^8)\right]. \tag{S51}$$

These are the high-temperature coefficients in the natural boundary frame. To obtain the corresponding coefficients in the physical hyperbolic frame, we must include the finite Weyl-anomaly contribution, which we compute next.

Finite Weyl-anomaly correction

The gravitational action computed above is defined in the natural boundary frame g_b as given in (S38). To obtain the partition function in the physical hyperbolic frame g_H, we must include the finite Weyl-anomaly contribution. The two metrics are related by

$$g_b = e^{-2\sigma} g_H, \qquad \sigma(r) = \frac{1}{2}\log(1 + \alpha^2 \sinh^2 r). \tag{S52}$$

Since both metrics are conformally flat, only the type-A anomaly contributes in the scheme $b = 0$. The corresponding finite Wess–Zumino action is given by [4]

$$\Delta_{b\to H}\log Z = -\frac{a_{\rm W}}{16\pi^2}\int d^4x\sqrt{g_H}\left[\sigma E_4[g_H] + 4G_H^{ij}\partial_i\sigma\partial_j\sigma - 4(\nabla\sigma)^2\nabla^2\sigma + 2(\nabla\sigma)^4\right], \tag{S53}$$

where G_H^{ij} is the Einstein tensor of g_H.

For $S^1 \times \mathbb{H}^3$, we have $E_4[g_H] = 0$ and $G_H^{rr} = 1$. In the high-temperature limit, the leading contribution comes from the region $r \sim \alpha^{-1}$ near the rotational axis. Introducing

$$w = \alpha^2 r^2, \qquad \sigma = \frac{1}{2}\log(1+w), \tag{S54}$$

we obtain

$$(\nabla\sigma)^2 = \frac{\alpha^2 w}{(1+w)^2}, \qquad \nabla^2\sigma = \frac{2\alpha^2}{(1+w)^2}. \tag{S55}$$

The Einstein-tensor term is subleading. The leading transverse integral becomes

$$\int d^2x_\perp \left[-4(\nabla\sigma)^2\nabla^2\sigma + 2(\nabla\sigma)^4\right] = 2\pi\alpha^2 \int_0^\infty dw\, \frac{w(w-4)}{(1+w)^4} = -\frac{2\pi\alpha^2}{3}. \tag{S56}$$

Including the thermal period β and the regulated geodesic length L_η, we find

$$\Delta_{b\to H} \log Z = \frac{a_{\mathrm{W}}\beta L_\eta \alpha^2}{24\pi} + o(\beta L_\eta \alpha^2). \tag{S57}$$

Using $\alpha = qr_+$ and the high-temperature relations in Eq. (S41), this gives

$$\frac{\Delta_{b\to H}\log Z}{L_\eta} = \frac{\pi^2}{48 G_5 \beta}\frac{q^2(1-q^2)^2}{(2-q^2)^2} + o(1/\beta). \tag{S58}$$

Expanding in the physical angular twist θ, we obtain

$$\left.\frac{n\Delta_{b\to H}\log Z}{L_\eta}\right|_{\beta\to 0} = N^2\left[\frac{\theta^2}{192\pi^2} - \frac{\theta^4}{96\pi^4} + \frac{\theta^6}{64\pi^6} + \mathcal{O}(\theta^8)\right]. \tag{S59}$$

This correction must be added to the natural-frame result in Eq. (S51) to obtain the physical hyperbolic partition function.

Combining the natural-frame result in Eq. (S50) with the finite Weyl-anomaly contribution in Eq. (S58), we obtain

$$\begin{aligned}\left.\frac{n\log Z_H}{L}\right|_{\beta\to 0} &= N^2\left[\frac{(1-q^2)^2(12-12q^2-q^4)}{48q^2(2-q^2)^2} + \frac{q^2(1-q^2)^2}{48(2-q^2)^2}\right] \\ &= N^2\frac{(1-q^2)^3}{4q^2(2-q^2)^2}.\end{aligned} \tag{S60}$$

This relation holds for the full leading local high-temperature function, rather than only for the first few terms in the small-angle expansion.

Using the relation between q and the physical twist θ in Eq. (S41), and expanding at small θ, we find

$$K^{\mathrm{pert}}_{\mathcal{N}=4}(\theta) = N^2\left[\frac{\pi^2}{16\theta^2} - \frac{1}{16} + \frac{\theta^2}{32\pi^2} - \frac{\theta^4}{64\pi^4} + \frac{3\theta^6}{256\pi^6} + \mathcal{O}(\theta^8)\right]. \tag{S61}$$

Independent rotating-sphere calculation

We now compute the rotating-S^3 partition function independently using the thermodynamics of the singly rotating Kerr–AdS$_5$ black hole. Here we consider a physical thermal partition function with one nonzero angular twist.

The Kerr-AdS$_5$ thermodynamic quantities are given by [10]

$$\begin{aligned}\Xi &= 1 - a_L^2, \qquad m = \frac{1}{2}(r_+^2 + a_L^2)(1 + r_+^2), \\ T &= \frac{r_+(2r_+^2 + 1 + a_L^2)}{2\pi(r_+^2 + a_L^2)}, \qquad \Omega = \frac{a_L(1+r_+^2)}{r_+^2 + a_L^2}, \\ S &= \frac{\pi^2 r_+(r_+^2 + a_L^2)}{2G_5\Xi}, \\ M_{\mathrm{state}} &= \frac{\pi m(2+\Xi)}{4G_5\Xi^2}, \qquad J = \frac{\pi m a_L}{2G_5\Xi^2}.\end{aligned} \tag{S62}$$

Here r_+ denotes the horizon radius of the spherical black hole. The corresponding grand-canonical partition function is

$$\log Z_{S^3}^{\rm state} = -\beta(M_{\rm state} - TS - \Omega J) = \frac{\pi\beta(r_+^2 + a_L^2)(r_+^2 - 1)}{8G_5\Xi}. \tag{S63}$$

Continuing $a_L = i\alpha$ and taking the high-temperature limit at fixed physical twist θ, we obtain

$$\log Z_{S^3}^{\rm state} = \frac{N^2\pi^2}{\beta}\frac{(1-q^2)^3}{q^2(2-q^2)^2} + \mathcal{O}(\beta), \tag{S64}$$

where $q = \alpha/r_+$ and its leading relation to θ agrees with Eq. (S41).

The rotational fixed circle on the unit sphere has length 2π. Comparing Eq. (S64) with the hyperbolic result in Eq. (S60), we find

$$\left.\frac{n\log Z_{S^3}^{\rm state}}{2\pi}\right|_{\beta\to 0} = \left.\frac{n\log Z_H}{L}\right|_{\beta\to 0} = N^2\frac{(1-q^2)^3}{4q^2(2-q^2)^2}. \tag{S65}$$

Thus, the two independent gravitational calculations agree as functions of q at leading order in the high-temperature expansion, rather than only at a few orders in θ. Expanding at small θ also reproduces Eq. (S61).

The Casimir energy in the round-sphere boundary frame is

$$E_C^{\rm ESU} = \frac{3\pi}{32G_5} = \frac{3N^2}{16}. \tag{S66}$$

Its contribution $-\beta E_C^{\rm ESU}$ is subleading and does not affect the leading high-temperature coefficients.

Cutoff and global regularity

The cutoff inherited from the CHM map gives

$$2\eta_{\max}(r) = 2\,\mathrm{arcosh}\,\frac{\cosh U}{\cosh r} = 2U - 2\log\cosh r + \cdots. \tag{S67}$$

The leading high-temperature contribution comes from the region $r \sim \alpha^{-1}$ near the rotational axis. In this region, replacing the varying geodesic length by $L = 2U$ does not change the leading L/β coefficient. The remaining contributions from the outer region are subleading in the local radial calculation.

The logarithmic counterterm also has no contribution to this leading coefficient. In the high-temperature tube, its transverse integral is proportional to

$$\int_0^\infty du\,\frac{2u-1}{(1+u)^4} = 0, \qquad u = \alpha^2 r^2. \tag{S68}$$

Finally, the continued topological Kerr–AdS$_5$ geometry need not be globally regular for the twisted thermal identification. In this family, the second shrinking cycle at large r requires

$$p\beta = 2\pi, \qquad p\theta + 2\pi k = 0, \qquad p, k \in \mathbb{Z}. \tag{S69}$$

Smoothness at the intersection of the shrinking cycles further requires $|k| = 1$, implying $|\theta| = \beta$, which is incompatible with the regime $\beta \ll |\theta|$.

For our local thermal-EFT calculation, the near-axis analysis is sufficient to extract the leading tube contribution, since the L/β enhancement comes from the short thermal cycle near the rotational axis. The construction of a globally regular solution and the evaluation of its complete on-shell action are left for future work [11].

Conformally coupled scalar: hyperbolic heat kernel

We first consider a conformally coupled scalar on $S^1 \times \mathbb{H}^3$. Using the standard scalar heat kernel on $\mathbb{H}^3$ [12], the spatial heat trace for a nontrivial rotational image is

$$\frac{1}{L}\int_{\mathbb{H}^3} d^3x\, K_{\mathbb{H}^3}(s; x, R_{k\theta}x)e^s = \frac{1}{\sqrt{4\pi s}}\,\frac{1}{4\sin^2(k\theta/2)}. \tag{S70}$$

Here $R_{k\theta}$ denotes a rotation by $k\theta$, and the factor e^s cancels the curvature contribution through the conformal coupling. Combining this with the thermal heat kernel on S^1 gives the scalar partition function.

To regulate the winding sum, we first introduce a transverse cutoff $r \le r_{\max}$. The heat-kernel calculation then gives

$$\left.\frac{\log Z_n^{\rm sc}(\theta; r_{\max})}{L}\right|_{k\neq 0} = \frac{1}{8\pi^2 n}\sum_{k=1}^{\infty}\frac{\Sigma_k^2}{k^2 s_k^2 (B_k^2+\Sigma_k^2)}, \tag{S71}$$

where

$$s_k = \left|\sin\frac{k\theta}{2}\right|, \qquad B_k = \beta k, \qquad \Sigma_k = 2\,\mathrm{arsinh}(s_k \sinh r_{\max}). \tag{S72}$$

This expression is finite and absolutely convergent at every finite cutoff, including the resonant windings with $s_k = 0$. The transverse cutoff is auxiliary and differs from the inherited CHM cutoff in Eq. (S2).

Removing the cutoff at each fixed nonresonant winding before summing gives the formal series

$$K_{\rm sc}^{\rm formal}(\theta) = \frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2\sin^2(k\theta/2)}. \tag{S73}$$

This series is not pointwise convergent. At rational $\theta/(2\pi)$, the exact-return windings are singular, while at irrational angles Dirichlet approximation gives infinitely many summands that do not approach zero. The finite-cutoff expression regulates these contributions, which are related to angular fractals [13].

We extract the local perturbative coefficients by expanding in small θ and using the Mellin/zeta prescription,

$$\begin{aligned} K_{\rm sc}^{\rm pert}(\theta) &= \frac{1}{8\pi^2}\left[\frac{4\zeta(4)}{\theta^2} + \frac{\zeta(2)}{3} + \frac{\zeta(0)\theta^2}{60}\right] \\ &= \frac{\pi^2}{180\theta^2} + \frac{1}{144} - \frac{\theta^2}{960\pi^2}. \end{aligned} \tag{S74}$$

All higher analytic terms vanish in this prescription because $\zeta(-2j) = 0$ for positive integers j.

Direct rotating-sphere scalar character

A separate operator-counting route gives the same scalar coefficients without using the hyperbolic heat kernel. The conformally coupled real scalar one-particle partition function on the unit round S^3, with two angular displacements, is

$$f_{\rm sc}(\beta_S, \alpha_1, \alpha_2) = \frac{\sinh\beta_S}{2(\cosh\beta_S - \cos\alpha_1)(\cosh\beta_S - \cos\alpha_2)}. \tag{S75}$$

The bosonic multiparticle trace is $\log Z_{S^3}^{\rm sc} = \sum_{k=1}^{\infty} k^{-1} f_{\rm sc}(k\beta_S, k\alpha_1, k\alpha_2)$. In the one-spin specialization $\alpha_1 = 0$, $\alpha_2 = \theta$, taking the leading $\beta_S \to 0$ term for *each fixed winding* gives

$$\beta_S \log Z_{S^3}^{\rm sc}(\beta_S, 0, \theta)\big|_{\rm formal} = \frac{1}{2}\sum_{k\ge 1}\frac{1}{k^2\sin^2(k\theta/2)}. \tag{S76}$$

The geometric fixed-circle length is 2π. Combining this length with $\beta = 2\pi n$, the common local coefficient is $K^{\rm pert} = \beta_S \log Z_{S^3}\big|_{\rm formal}/(4\pi^2)$. Thus Eq. (S76) reproduces Eq. (S73) in the same coefficient prescription.

Dirac field: hyperbolic and rotating-sphere characters

For a free Dirac fermion, the spinor character on $\mathbb{H}^3$, together with the antiperiodic thermal boundary condition, gives the following formal local contribution:

$$K_{{\rm D},H}^{\rm formal}(\theta) = \frac{1}{2\pi^2}\sum_{k\ge 1}\frac{(-1)^{k+1}\cos(k\theta/2)}{k^2\sin^2(k\theta/2)}. \tag{S77}$$

Here the cosine factor comes from the spinor trace under spatial rotations, while $(-1)^{k+1}$ accounts for fermionic statistics. The normalization corresponds to four on-shell Dirac degrees of freedom.

We now obtain the same expression independently from the rotating-S^3 operator character. For a Dirac fermion, including particle and antiparticle states, the one-particle character at $\alpha_1 = 0$, $\alpha_2 = \theta$ is

$$f_{\rm D}(\beta_S, 0, \theta) = \frac{8x^{3/2}\cos(\theta/2)}{(1-x)(1-2x\cos\theta + x^2)}, \qquad x = e^{-\beta_S}. \tag{S78}$$

The corresponding thermal partition function is

$$\log Z^{\rm D}_{S^3} = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} f_{\rm D}(k\beta_S, 0, k\theta). \tag{S79}$$

Expanding at small β_S for each fixed winding number k, we obtain

$$\beta_S \log Z^{\rm D}_{S^3}\big|_{\rm formal} = 2\sum_{k\geq 1} \frac{(-1)^{k+1}\cos(k\theta/2)}{k^2\sin^2(k\theta/2)}. \tag{S80}$$

Thus,

$$K^{\rm formal}_{{\rm D},H}(\theta) = \frac{1}{4\pi^2}\ \beta_S \log Z^{\rm D}_{S^3}\big|_{\rm formal}\,. \tag{S81}$$

This establishes the agreement of the perturbative local coefficients on the two geometries.

Using

$$\frac{\cos(x/2)}{\sin^2(x/2)} = \frac{4}{x^2} - \frac{1}{6} - \frac{7x^2}{480} + \cdots, \tag{S82}$$

we find

$$K^{\rm pert}_{\rm D}(\theta) = \frac{7\pi^2}{360\theta^2} - \frac{1}{144} - \frac{7\theta^2}{1920\pi^2}. \tag{S83}$$

The leading term reproduces the flat-space thermal coefficient $c'_{\rm D} = 7\pi^2/180$. All higher even-power terms vanish because $\eta(-2j) = 0$ for $j \geq 1$, where $\eta(s) = (1-2^{1-s})\zeta(s)$.

The equality holds for the formal perturbative local expansion. The complete fixed-angle partition functions can contain additional winding contributions, which are not captured by this coefficient extraction.

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